PACIFIC JOURNAL OF MATHEMATICS
Vol. 34, No. 1, 1970

ON THE MEASURABILITY OF PERRON INTEGRABLE FUNCTIONS

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By means of majorants and minorants a Perron-like integral can be defined in an arbitrary topological space. Although for its definition only a finitely additive set function is used, it turns out that if the underlying topological space is Hausdorff and locally compact, then the integral itself gives rise to a regular measure. The natural question, whether every integrable function is measurable with respect to this measure, is the subject of our paper.

In § 2 some sufficient conditions for measurability of integrable functions are given and the connection of our measure with the original set function is described. The results of this section are then applied to integration with respect to the natural and monotone convergences. The natural convergence, which can be used in any topological space is discussed in § 3. In § 4 some elementary properties of the monotone convergence are derived. This convergence can be used in any locally pseudo-metrizable space and it seems to be the most important convergence for the definition of an integral over a differentiable manifold. A proof that for the monotone convergence every integrable function is measurable is given in § 5. Finally, § 6 contains a few illustrative examples.

Throughout, P is a topological space which is always assumed to be Hausdorff and locally compact. The reader can, however, easily detect those parts of the paper which remain correct in an arbitrary topological space P. By $P^{\sim} = P \cup (\infty)$ we denote a one point compactification of P. If $A \subset P$, A^- and $A^{\sim}$ stand for closure of A in P and $P^{\sim}$, respectively. The interior of a set $A \subset P$ is denoted by A^0. For $x \in P^{\sim}$, Γ_x is a local base at x in $P^{\sim}$ (see [3], p. 50). We shall always assume that $U \subset P$ and U^- is compact for all $U \in \Gamma_x$ with $x \in P$. If σ is a pre-algebra of subsets of P (see [5], 1.1) such that $\{U \cap P : U \in \Gamma_x\} \subset \sigma$ for every $x \in P^{\sim}$, we call the pair $\mathfrak{N} = \langle \sigma, \Gamma_x \rangle$ a *net structure* in P.

If $\delta \subset \sigma$ and $A \subset P^{\sim}$ we let $\delta_A = \{B \in \delta : B \subset A\}$. A system $\delta \subset \sigma$ is said to be *semihereditary* if and only if $\sigma_0 \cap \delta \neq \emptyset$ for every finite disjoint collection $\sigma_0 \subset \sigma$ whose union belongs to δ. A system $\delta \subset \sigma$ is said to be *stable* if and only if $\emptyset \notin \delta$ and for every $A \in \delta$ and every $x \in P^{\sim}$ there is a $U \in \Gamma_x$ such that $\delta_{A-U} \neq \emptyset$.

A convergence[1] in a net structure $\langle \sigma, \Gamma_x \rangle$ is a function κ which

[1] What we call a convergence is sometimes called a *derivation basis* (see [**1**], 1.1).

to every $x \in P^{\sim}$ associates a family κ_x of nets $\{B_U, U \in \Gamma, \subset\} \subset \sigma$ where Γ is a cofinal subset of Γ_x. For $\delta \subset \sigma$ and $x \in P^{\sim}$, $\kappa_x(\delta) = \{\{B_U\} \in \kappa_x : \{B_U\} \subset \delta\}$ and $\delta^* = \{x \in P^{\sim} : \kappa_x(\delta) \neq \emptyset\}$.

A convergence κ is called *admissible* if and only if the following conditions are satisfied:

$\mathscr{K}_1$. For every $x \in P^{\sim}$, $\{U \cap P, U \in \Gamma_x, \subset\} \in \kappa_x$.

$\mathscr{K}_2$. If $x \in P^{\sim}$ and $\{B_U, U \in \Gamma, \subset\} \in \kappa_x$ then for every $V \in \Gamma_x$ there is a $U_V \in \Gamma$ such that $B_U \subset V$ for all $U \in \Gamma$ for which $U \subset U_V$.

$\mathscr{K}_3$. If $x \in P^{\sim}$, $\{B_U\}_{U \in \Gamma} \in \kappa_x$, and Γ' is a cofinal subset of Γ, then also $\{B_U\}_{U \in \Gamma'} \in \kappa_x$.

$\mathscr{K}_4$. If $x \in P^{\sim}$, $\{B_U\} \in \kappa_x$, and $A \in \sigma$, then also $\{B_U \cap A\} \in \kappa_x$.

$\mathscr{K}_5$. If $\delta \subset \sigma$ is a nonempty semihereditary system, then δ^* is nonempty.

$\mathscr{K}_6$. If $\delta \subset \sigma$ is a nonempty semihereditary stable system, then δ^* is uncountable.

A triple $\mathfrak{J} = \langle \mathfrak{N}, \kappa, G \rangle$ is called an *integration base* in P if and only if $\mathfrak{N} = \langle \sigma, \Gamma_x \rangle$ is a net structure in P, κ is an admissible convergence in $\mathfrak{N}$, and G is a nonnegative finitely additive function[2] on σ such that $G(A) < +\infty$ for every $A \in \sigma$ with A^- compact.

It was shown in [6] that integration bases exist in P and that for each of them we can define a nonabsolutely convergent integral I which is closely related to the Lebesgue integral. For the reader's convenience we shall summarize the basic definitions.

Let $x \in P^{\sim}$, $A \subset P$, and let F be a function on σ_A. We call the number ${}_{\#}F(x, A) = \inf\{\liminf F(B_\alpha) : \{B_\alpha\} \in \kappa_x(\sigma_A)\}$ the *lower limit* of F at x relative to A and the number ${}_{*}F(x, A) = {}_{\#}(F/G)(x, A)$[3] the *lower derivate* of F at x relative to A and it is denoted by $I(f, A)$.

Let $A \in \sigma$ and let f be a function on A^-. A superadditive function M on σ_A is said to be a *majorant* of f on A if and only if there is a countable set $Z_M \subset A^-$ such that ${}_{\#}(-G)(x, A) \geqq 0$ for all $x \in Z_M$, ${}_{\#}M(x, A) \geqq 0$ for all $x \in Z_M \cup (\infty)$, and $-\infty \neq {}_{*}M(x, A) \geqq f(x)$ for all $x \in A^- - Z_M$. The number $I_u(f, A) = \inf M(A)$ where the infimum is taken over all majorants of f on A is called the *upper integral* of f over A. If $I_u(f, A) = -I_u(-f, A) \neq \pm\infty$ this common value is called the *integral* of f over A.

If $A \in \sigma$ and f is a function on A^-, we denote by $\mathfrak{M}(f, A)$ the family of all majorants of f on A. The family of all functions integrable over $A \in \sigma$ is denoted by $\mathfrak{P}(A)$.

For $A \subset P$, χ_A denotes the characteristic function of A in P. By $\mathfrak{C}$ and $\mathfrak{U}$ we denote the families of all compact and open subsets of P, respectively. Using the integral I, we shall define measure spaces

[2] Unless specified otherwise, by a function we always mean an extended real-valued function.

[3] We let $a/0 = +\infty$ for $a \geqq 0$, $a/0 = -\infty$ for $a < 0$, and $a/(\pm\infty) = 0$.

$(P, \mathfrak{T}, \tau)$ and $(P, \mathfrak{T}_0, \tau_0)$ as follows:

(i) $\mathfrak{T}$ is the family of all sets $A \subset P$ such that $\chi_{A \cap C} \in \mathfrak{P}(P)$ for every $C \in \mathfrak{C}$; and for $A \in \mathfrak{T}$, $\tau(A) = I_u(\chi_A, P)$.

(ii) For $A \subset P$,

$$\tau_0(A) = \inf \{\tau(U) : U \in \mathfrak{U} \text{ and } A \subset U\}$$

and $\mathfrak{T}_0$ is the family of all τ_0-measurable subsets of P.

These measure spaces will play an essential part in our paper. Some of their important properties can be found in [7], §'s 3 and 4; e.g., there is a proof that they actually are measure spaces. We just recall here that $\mathfrak{U} \subset \mathfrak{T}_0 \subset \mathfrak{T}$ and that the measure τ_0 is regular.

2. **Measurability in general.** In this section we shall prove a few general theorems concerning the measurability of integrable functions. Throughout we shall assume that there is given an integration base $\mathfrak{J} = \langle \sigma, \Gamma_x, \kappa, G \rangle$ in P.

PROPOSITION 2.1. *If the lower derivate of every superadditive function on σ is $\mathfrak{T}_0$-measurable, then $\mathfrak{T} = \mathfrak{T}_0$ and every function from $\mathfrak{P}(P)$ is $\mathfrak{T}_0$-measurable.*

Proof. Let $A \in \mathfrak{T}$ with $A^- \in \mathfrak{C}$. Then by [7], 2.7 there are narrow majorants $M_n \in \mathfrak{M}^\wedge(\chi_A, P)$ (see [7], 2.5) for which $M_n(P) - I(\chi_A, P) < 1/n$, $n = 1, 2, \cdots$. If $B_n = \{x \in P : {}_*M_n(x, P) \geqq 1\}$ then by our assumption $B_n \in \mathfrak{T}_0$. Letting $B = A^- \cap (\bigcap_{n=1}^\infty B_n)$, we have $B \in \mathfrak{T}_0$, $B^- \in \mathfrak{C}$, and $A \subset B$. Because

$$\tau(B_n) - \tau(A) \leqq M_n(P) - I(\chi_A, P) < 1/n$$

for $n = 1, 2, \cdots$, it follows that $\tau(B - A) = 0$. Now replacing A by $B - A$ and repeating the previous construction, we obtain a set $C \in \mathfrak{T}_0$ for which $C^- \in \mathfrak{C}$, $B - A \subset C$, and

$$\tau(C) = \tau(B - A) + \tau(C - [B - A]) = 0\,.$$

By [7], 4.7 also $\tau_0(C) = 0$ and since τ_0 is a complete measure, $A = B - (B - A)$ belongs to $\mathfrak{T}_0$.

If $A \in \mathfrak{T}$ is arbitrary, then $(A \cap C)^- \in \mathfrak{C}$ for every $C \in \mathfrak{C}$. Thus $A \cap C \in \mathfrak{T}_0$ for every $C \in \mathfrak{C}$ and it follows from [7], 4.7 that $A \in \mathfrak{T}_0$.

The last part of the proposition is now a direct consequence of [7], 4.3.

The previous proposition and Proposition 4.3 in [7] indicate the importance of the following:

PROPOSITION 2.2. *Let M be a function on σ_A where $A \in \sigma$, and* *let c be a real number. If $\sigma_A(M, c) = \{B \in \sigma_A : M(B)/G(B) < c\}$, then $\bigcap_{n=1}^{\infty} \sigma_A^*(M, c + 1/n) = \{x \in A^{\sim} : {}_*M(x, A) \leqq c\}$.*

Proof. If ${}_*M(x) \leqq c$ and n is a positive integer, then there is a net $\{B_U\}_{U \in \Gamma} \in \kappa_x(\sigma_A)$ such that

$$\liminf\,[M(B_U)/G(B_U)] < c + 1/n\,.$$

Hence there is a cofinal subset Γ' of Γ such that $\{B_U\}_{U \in \Gamma'} \subset \sigma_A(M, c + 1/n)$. It follows from $\mathscr{K}_3$ that $x \in \sigma_A^*(M, c + 1/n)$. On the other hand if $x \in \sigma_A^*(M, c + 1/n)$ then it follows that ${}_*M(x) \leqq c + 1/n$, $n = 1, 2, \cdots$.

DEFINITION 2.3. An integration base $\mathfrak{J}$ in P is said to be *measurable* if and only if every function from $\mathfrak{P}(P)$ is $\mathfrak{T}_0$-measurable. It is said to be *strongly measurable* if and only if $\delta^* - (\infty)$ belongs to $\mathfrak{T}_0$ for every $\delta \subset \sigma$.

It follows at once from 2.1 and 2.2 that every strongly measurable integration base is measurable. On the other hand, Example 6.1 shows that a measurable integration base need not be strongly measurable. From [7], 4.7 we see that if $\mathfrak{J}$ is measurable, then $\mathfrak{T} = \mathfrak{T}_0$.

REMARK 2.4. Let c be a real number and let $A \in \sigma$. It is easy to see that $\sigma_A(M, c)$ is semihereditary whenever M is a superadditive function on σ_A. Furthermore, it can be shown that if ${}_\#(-G)(x, A) \geqq 0$ for every $x \in A^{\sim}$, then $\sigma_A(M, c)$ is semihereditrary and stable whenever M is a majorant for some function on A^- (see [6], (4.4)). However, Example 6.1 indicates that there is no link between the semihereditariness or stability of $\delta \subset \sigma$ and the $\mathfrak{T}_0$-measurability of $\delta^* - (\infty)$.

THEOREM 2.5. *If $C \in \mathfrak{C}$, then*

$$\tau_0(C) = \inf \sum_{i=1}^{n} G(A_i)$$

where the infimum is taken over all finite families $\{A_i\}_{i=1}^n \subset \sigma$ for which $C \subset (\bigcup_{i=1}^n A_i)^0$.

Proof. Let $C \in \mathfrak{C}$ and let $A_1, \cdots, A_n$ be sets from σ for which $C \subset (\bigcup_{i=1}^n A_i)^0$. If we set $M(B) = \sum_{i=1}^n G(B \cap A_i)$ for $B \in \sigma$, then $M \in \mathfrak{M}(\chi_C, P)$ and so

$$\tau_0(C) = \tau(C) \leqq M(P) = \sum_{i=1}^{n} G(A_i)\,.$$

On the other hand, given $C \in \mathfrak{C}$ and $\varepsilon > 0$, there is a $U \in \mathfrak{U}$ such that

$C \subset U$, $U^{-} \in \mathfrak{C}$, and $\tau_0(U) < \tau_0(C) + \varepsilon$. Using [5], (1.1), we can find disjoint sets $B_1, \cdots, B_m$ from σ for which

$$C \subset \left(\bigcup_{i=1}^{m} B_i\right)^0 \subset \bigcup_{i=1}^{m} B_i^{\sim} \subset U .$$

If we set $N(B) = -\sum_{i=1}^{m} G(B \cap B_i)$ for $B \in \sigma$, then $N \in \mathfrak{M}(-\chi_U, P)$ and thus

$$\begin{aligned} \tau_0(C) + \varepsilon > \tau_0(U) = \tau(U) &= -I(-\chi_U, P) \\ &\geqq -N(P) = \sum_{i=1}^{m} G(B_i) . \end{aligned}$$

Using the regularity of τ_0 (see [7], 4.7), we obtain the following:

COROLLARY 2.6. *Let* $\langle \sigma, \Gamma_x, \kappa, G \rangle$ *and* $\langle \sigma', \Gamma'_x, \kappa', G' \rangle$ *be two integration bases in P. If* $\sigma \cap \sigma'$ *is a pre-algebra which contains a topological base of P and if* $G = G'$ *on* $\sigma \cap \sigma'$, then $(\mathfrak{T}_0, \tau_0) = (\mathfrak{T}'_0, \tau'_0)$.

Proof. Suppose $\tau'_0(C) > \tau_0(C)$ for some $C \in \mathfrak{C}$. Then by [5], (1.1) there is a disjoint finite family $\{A_i\} \subset \sigma$ such that $C \subset (\cup A_i)^0$ and $\sum G(A_i) < \tau'(C)$. Since C is compact, using again [5], (1.1), we can find a disjoint finite family $\{B_j\} \subset \sigma \cap \sigma'$ such that $C \subset \cup B_j^0 \subset \cup B_j \subset (\cup A_i)^0$. Hence

$$\begin{aligned} \tau'_0(C) &\leqq \sum_j G'(B_j) = \sum_j G(B_j) \\ &= \sum_{i,j} G(A_i \cap B_j) \leqq \sum_i G(A_i) \end{aligned}$$

which is a contradiction. By symmetry $\tau_0(C) = \tau'_0(C)$ for every $C \in \mathfrak{C}$. Now the corollary follows from [7], 4.7.

The previous corollary is the main reason why we are discussing $\mathfrak{T}_0$-measurability rather than $\mathfrak{T}$-measurability.

Let $\mathfrak{J} = \langle \sigma, \Gamma_x, \kappa, G \rangle$ and $\mathfrak{J}' = \langle \sigma', \Gamma'_x, \kappa', G' \rangle$ be two integration bases in P. If $\sigma' \subset \sigma$, G' is the restriction of G to σ', and for every $x \in P^{\sim}$, $\Gamma'_x \subset \Gamma_x$ and $\kappa'_x \subset \kappa_x$, we say that $\mathfrak{J}$ is *larger* than $\mathfrak{J}'$ and write $\mathfrak{J}' \prec \mathfrak{J}$. Obviously, the relation $\prec$ is a partial ordering in the family of all integration bases in P. Imitating the proof of Theorem 31 in [4], one can easily see that it $\mathfrak{J}' \prec \mathfrak{J}$ then $\mathfrak{P}(A) \subset \mathfrak{P}'(A)$ for every $A \in \sigma'$; here $\mathfrak{P}(A)$ and $\mathfrak{P}'(A)$ are the families associated with $\mathfrak{J}$ and $\mathfrak{J}'$, respectively. From this and Corollary 2.6 it follows that if $\mathfrak{J}$ is a measurable base in P so is $\mathfrak{J}'$ for every $\mathfrak{J}' \succ \mathfrak{J}$.

The next difinition and proposition will be used in § 5.

DEFINITION 2.7. An integration base $\mathfrak{J}$ in P is said to be *locally* *strongly measurable* if and only if for every $x \in P$ there is an integration base $\mathfrak{J}'$ in P (generally depending on x) which satisfies the following conditions:

(i) $(\mathfrak{T}_0, \tau_0) = (\mathfrak{T}'_0, \tau'_0)$.

(ii) There is a neighborhood $U \in \sigma \cap \sigma'$ of x such that $\mathfrak{P}(U) \subset \mathfrak{P}'(U)$ and $\delta^{*\prime} - (\infty) \in \mathfrak{T}'_0$ whenever $\delta \subset \sigma'_U$.

PROPOSITION 2.8. *If $\mathfrak{J}$ is locally strongly measurable then $\mathfrak{J}$ is measurable.*

Proof. Let $f \in \mathfrak{P}(P)$. We shall show that every point $x \in P$ has a neighborhood V such that f restricted to V^- is $\mathfrak{T}_0$-measurable. It will follow that f restricted to any compact subset of P is $\mathfrak{T}_0$-measurable and hence by [7], 4.7 also f itself is $\mathfrak{T}_0$-measurable.

Choose $x \in P$ and let $\mathfrak{J}'$ and U have the same meaning as in Definition 2.7. Then by [6], 6.8, $f \subset \mathfrak{P}'(U)$ and we can choose majorants $M_n \in \mathfrak{M}'(f, U)$ such that $M_n(U) - I'(f, U) < 1/n$, $n = 1, 2, \cdots$. By the definition of majorant (see [6], 3.2) with each M_n there is associated a certain countable set $Z_{M_n} \subset U^-$. For $x \in P$ let $h_n(x) = {}_{*'}M_n(x, U)$ if $x \in U^- - Z_{M_n}$ and $h_n(x) = +\infty$ otherwise; here, of course, ${}_{*'}M_n(x, U)$ denotes the lower derivate computed in $\mathfrak{J}'$. By 2.2, the h_n are $\mathfrak{T}_0$-measurable, and so is $h = \inf h_n$. Set $r(x) = h(x) - f(x)$ if this difference has meaning and $r(x) = 0$ otherwise. Since $h \geqq f$, $r \geqq 0$ and

$$\begin{aligned} 0 &\leqq I'_u(r, U) \leqq \inf I'_u(h_n, U) - I'(f, U) \\ &\leqq \inf [M_n(U) - I'(f, U)] = 0 \end{aligned}$$

(see [6], 6.4). Now choose $V \in \Gamma'_x$ such that $V^- \subset U^0$ and let $r_1 = r\chi_{V^-}$ (we define $(\pm\infty) .0 = 0$). Since $0 \leqq r_1 \leqq r$, $I'_u(r_1, U) = 0$ and $r_1 \in \mathfrak{P}'(U)$. Exactly as before we can define a $\mathfrak{T}_0$-measurable function $g \geqq r_1$ such that if we set $s(x) = g(x) - r_1(x)$ whenever this difference has meaning and $s(x) = 0$ otherwise, then $I'(s, U) = 0$. Letting $g_1 = g\chi_{V^-}$ and $s_1 = s\chi_{V^-}$, we obtain $I'(g_1, P) = I'(g_1, U) = I'(s_1, U) + I'(r_1, U) = 0$; for $g_1 = 0$ on $(P - U)^-$ and $0 \leqq s_1 \leqq s$. Since g_1 is nonnegative, $\mathfrak{T}_0$-measurable, and has a compact support it follows from [7] 4.2 and 4.7 that $\int_P g_1 d\tau_0 = 0$. Because τ_0 is a complete measure and $g_1 \geqq r_1 \geqq 0$, also r_1 is $\mathfrak{T}_0$-measurable. Therefore f restricted to V^-, which is equal to $h + r_1$ restricted to V^-, is $\mathfrak{T}_0$-measurable too.

Let $\mathfrak{J}$ be measurable or strongly measurable and let $A \in \sigma$ be different from P. Then, in general, we do not know whether the functions from $\mathfrak{P}(A)$ are $\mathfrak{T}_0$-measurable over A^-. This fact, e.g., caused the main difficulty in proving Proposition 2.8. The following proposition is a contribution to this problem.

PROPOSITION 2.9. *Let* $\mathfrak{J}$ *be measurable, let* $\sigma \subset \mathfrak{T}_0$, *and let* $G(A) = \tau_0(A)$ *for every* $A \in \sigma$ *for which* A^- *is compact. If* $A \in \sigma$ *is such that* $A \cap (P - A)^-$ *is* τ_0-σ-*finite, then every function from* $\mathfrak{P}(A)$ *is* $\mathfrak{T}_0$-*measurable.*

Proof. If $f \in \mathfrak{P}(A)$ let $f^\wedge(x) = f(x)$ for $x \in A^-$ and $f^\wedge(x) = 0$ for $x \in P - A^-$. According to [7], 4.14, $f^\wedge \in \mathfrak{P}(P)$ and the proposition follows.

3. Some remarks on the natural convergence. Let $\mathfrak{N} = \langle \sigma, \Gamma_x \rangle$ be a net structure in P and let κ be a convergence in $\mathfrak{N}$. If for every $x \in P^\sim$, κ_x consists of all nets $\{B_U\}$ which satisfy the condition $\mathscr{K}_2$, then κ is called the *natural convergence* and it is denoted by κ^0. According to [6], 4.3 the natural convergence κ^0 is admissible.

Hence assume that there is given an integration base $\mathfrak{J} = \langle \sigma, \Gamma_x, \kappa^0, G \rangle$ where κ^0 is the natural convergence. It is easy to see that for $\delta \subset \sigma$, δ^* is closed in $P^\sim$ (see [5], 2.1) and so $\mathfrak{J}$ is strongly measurable. In fact we have more precise information.

LEMMA 3.1. *Let* $A \in \sigma$ *and let* M *be a function on* σ_A. *If* ${}_*M(x, A) > -\infty$ *for all* $x \in P$, *then the function* ${}_*M(\cdot, A)$ *is lower semicontinuous.*[4)]

Proof. If c is a real number, then

$$\{x \in P : {}_*M(x, A) > c\} = (P - A^-) \cup \{x \in A^- : {}_*M(x, A) > c\} = P - \{x \in A^\sim : {}_*M(x, A) \leqq c\}.$$

By 2.2, $\{x \in A^\sim : {}_*M(x, A) \leqq c\}$ is closed in $P^\sim$ and the lemma follows.

THEOREM 3.2. *The measure* τ *is regular.*

Proof. Since we already know that τ is inner regular on $\mathfrak{U}$ and finite on $\mathfrak{C}$ (see [7], 3.13), it remains to show that τ is outer regular on $\mathfrak{T}$. Hence choose $A \in \mathfrak{T}$ and $\varepsilon \in (0, 1)$. Since everything is trivial if $\tau(A) = +\infty$, we may assume that $\tau(A) < +\infty$. By [7], 3.10, $\chi_A \in \mathfrak{P}_0(P)$ and so there is a narrow majorant $M \in \mathfrak{M}(\chi_A, P)$ such that $M(P) - \tau(A) < \varepsilon$ (see [7], 2.5, 2.7). By Lemma 3.1, ${}_*M(\cdot, P)$ is lower semicontinuous and hence the set $U = \{x \in P : {}_*M(x, P) > 1 - \varepsilon\}$ is open. Clearly $A \subset U$ and $M/(1 - \varepsilon)$ is a majorant of χ_U. Therefore

4) See [3], Chapter 3, Problem F, p. 101.

$$\tau(A) \leqq \tau(U) \leqq M(P)/(1-\varepsilon) < [\tau(A)+\varepsilon]/(1-\varepsilon)$$

and the outer regularity of τ at A follows from the arbitrariness of ε.

COROLLARY 3.3. $(\mathfrak{T}, \tau) = (\mathfrak{T}_0, \tau_0)$.

Proof. By a rather standard procedure it follows from [7], 4.7 that the measure τ_0 has no proper regular extension. Hence $\mathfrak{T} = \mathfrak{T}_0$ and because both τ and τ_0 are regular, also $\tau = \tau_0$.

4. **The monotone convergence.** Let $\mathfrak{N} = \langle \sigma, \Gamma_x \rangle$ be a given net structure in P and let κ be a convergence in $\mathfrak{N}$. If for every $x \in P^{\sim}$, κ_x consists of all nets $\{B_U, U \in \Gamma, \subset\} \in \kappa_x^0$ such that $B_U \subset B_V$ whenever $U \subset V$, then κ is called the *monotone convergence* and it is denoted by κ^1. The following proposition indicates the essential difference between κ^0 and κ^1.

PROPOSITION 4.1. *If* $x \in P^{\sim}$ and $\{B_U, U \in \Gamma, \subset\} \in \kappa_x^1$, *then either* $x \in \bigcap_{U \in \Gamma} B_U^{\sim}$ *or there is a* $V \in \Gamma$ *such that* $B_U = \emptyset$ *for all* $U \in \Gamma$ *for which* $U \subset V$.

Proof. If $x \notin \bigcap_{U \in \Gamma} B_U^{\sim}$ then $x \notin B_{U_0}^{\sim}$ for some $U_0 \in \Gamma$ and hence there is a $U_1 \in \Gamma_x$ such that $U_1 \cap B_{U_0} = \emptyset$. To U_1 we can assign a $U_2 \in \Gamma$ such that $U \in \Gamma$ and $U \subset U_2$ implies $B_U \subset U_1$. On the other hand, $U \in \Gamma$ and $U \subset U_0$ implies $B_U \subset B_{U_0}$ and thus V can be any element of Γ for which $V \subset U_0 \cap U_2$.

REMARK 4.2. Let P be the set of all real numbers with the usual topology. Let $\mathfrak{J} = \langle \sigma, \Gamma_x, \kappa, G \rangle$ be an integration base in P defined as follows: σ is the pre-algebra generated by all one-side-open intervals, for $x \in P^{\sim}$, $\Gamma_x \subset \sigma$ is an arbitrary local base at x in $P^{\sim}$, and G is the Lebesgue measure on σ. Using the previous proposition, we see rather easily that if $\kappa = \kappa^1$ is the monotone convergence, then $\mathfrak{J}$ gives precisely the classical *Perron integral* (see [10], Chapter VI, § 8).

We also note that a *singularization* of a monotone convergence is again a monotone convergence (see [8], § 2).

PROPOSITION 4.3. *Let* $\mathfrak{N} = \langle \sigma, \Gamma_x \rangle$ *be a net structure in P. If the space P is locally metrizable, then the monotone convergence in* $\mathfrak{N}$ *is admissible.*

Proof. Conditions $\mathscr{K}_1 - \mathscr{K}_4$ are satisfied obviously. To show that also $\mathscr{K}_5$ and $\mathscr{K}_6$ are satisfied we can repeat verbatim the proofs of Proposition 3.1 and Theorem 3.2 in [5], respectively.

We note that in an arbitrary topological space the monotone convergence still satisfies conditions $\mathscr{K}_1 - \mathscr{K}_4$; however, we do no know whether it also satisfies conditions $\mathscr{K}_5$ and $\mathscr{K}_6$. An example of a net structure in a nonlocally metrizable space in which the monotone convergence is still admissible will be given in 6.2.

We shall close this section with a proposition which will show how conditions $\mathscr{K}_1 - \mathscr{K}_6$ are related to each other.

PROPOSITION 4.4. *Conditions $\mathscr{K}_1 - \mathscr{K}_5$ are independent and they do not imply $\mathscr{K}_6$. Conditions $\mathscr{K}_1 - \mathscr{K}_4$ and $\mathscr{K}_6$ are independent and they imply $\mathscr{K}_5$.*

Proof. Examples 6.3 and 6.4 show that $\mathscr{K}_1 - \mathscr{K}_4$ do not imply $\mathscr{K}_5$ and that $\mathscr{K}_1 - \mathscr{K}_5$ do not imply $\mathscr{K}_6$, respectively. The remaining examples which are needed to prove the independence are quite simple and their construction will be left to the reader. We shall complete the proof by showing that $\mathscr{K}_1$, $\mathscr{K}_4$, and $\mathscr{K}_6$ imply $\mathscr{K}_5$.

Let $\mathfrak{N} = \langle \sigma, \Gamma_x \rangle$ be a net structure in P, let κ be a convergence in $\mathfrak{N}$ satisfying conditions $\mathscr{K}_1$, $\mathscr{K}_4$, and $\mathscr{K}_6$, and let $\delta \subset \sigma$ be a nonempty semi-hereditary system. If δ is stable, then by $\mathscr{K}_6$, δ^* is uncountable and so nonempty. Hence suppose that δ is not stable. Then either $\emptyset \in \delta^* = P^\sim$ [see [6], (4.1)] or there is an $A \in \delta$ and an $x \in P^\sim$ such that $\delta_{A-U} = \emptyset$ for all $U \in \Gamma_x$. Choose $U \in \Gamma_x$. Since $U \cap A \in \sigma$, $A = (U \cap A) \cup (\bigcup_{i=1}^n B_i)$ where $B_1, \cdots, B_n$ are disjoint sets from σ_{A-U}. Therefore $B_1, \cdots, B_n$ do not belong to δ and because δ is semi-hereditary, we conclude that $A \cap U \in \delta$. Now it follows from $\mathscr{K}_1$ and $\mathscr{K}_4$ that $x \in \delta^*$ and thus again δ^* is nonempty.

COROLLARY 4.5. *Let $\mathfrak{N} = \langle \sigma, \Gamma_x \rangle$ be a net structure in P and let σ contain no nonempty semihereditary stable system. Then every convergence in $\mathfrak{N}$ which satisfies conditions $\mathscr{K}_1 - \mathscr{K}_4$ is admissible.*

The assumption of this corollary is always satisfied if P is countable. It is also satisfied if P is the set of all ordinals less than a given ordinal α topologized by the order topology (see [5], 1.4).

COROLLARY 4.6. *Let $\mathfrak{N}$ be a net structure in P and let κ and κ' be two convergences in $\mathfrak{N}$ satisfying conditions $\mathscr{K}_1 - \mathscr{K}_4$. If $\kappa_x = \kappa'_x$ for all but countably many $x \in P^\sim$, then κ is admissible if and only if κ' is admissible.*

Proof. Let S be the countable set of those $x \in P^\sim$ for which

$\kappa'_x \neq \kappa_x$. Since $\delta^* - S = \delta^{*\prime} - S$ for every $\delta \subset \sigma$, it follows that κ satisfies condition $\mathscr{K}_6$ if and only if κ' does.

According to Proposition 4.4 condition $\mathscr{K}_5$ is superfluous for the admissibility of a convergence. Nevertheless, for a given convergence, establishing $\mathscr{K}_5$ is usually the first step in establishing $\mathscr{K}_6$ (see [5] and [9]). It should be also noted that a convergence which satisfies only conditions $\mathscr{K}_1 - \mathscr{K}_5$ is still adequate for the definition of the narrow integral (see [7], 2.5).

5. Measurability with respect to the monotone convergence.

Throughout this section we shall assume that the space P, in addition to Hausdorff and locally compact, is also locally metrizable. We shall assume that there is given an integration base $\mathfrak{J} = \langle \sigma, \Gamma_x, \kappa^1, G \rangle$ where κ^1 is the monotone convergence and we shall prove that $\mathfrak{J}$ is measurable. We begin with a simple but useful remark.

REMARK 5.1. Let $x \in P$ and let $\{B_U, U \in \Gamma, \subset\} \in \kappa^1_x$. Since P, being locally metrizable, is first countable, Γ has a linearly ordered countable cofinal subset $\Gamma' = \{U_n\}$. Hence there is a sequence $\{C_n\} \in \kappa^1_x(\{B_U\})$ such that $C_{n+1} \subset C_n$ for $n = 1, 2, \cdots$; for it suffices to set $C_n = B_{U_n}$. The sequence $\{C_n\}$ may consist only of a single element if x is an isolated point of P.

LEMMA 5.2. *Let $A \in \sigma$ and let $\delta \subset \sigma_A$. If σ_A is countable then $\delta^* - (\infty)$ is $\mathfrak{T}_0$-measurable.*

Proof. Since $\Gamma_x \subset \sigma$ for all $x \in P$, it follows from the countability of σ_A that A is paracompact and hence metrizable by [2], Th. 2–28, p. 81. Choose a metric on A and if $B \subset A$ denote by $d(B)$ the diameter of B with respect to this metric. Because $\emptyset \in \delta$ implies $\delta^* - (\infty) = P$ which is $\mathfrak{T}_0$-measurable, we shall assume that $\emptyset \notin \delta$. Let $\{B_k\}_k$ be an enumeration of the family $\{B \in \delta : d(B) < 1\}$. If $B_{k_1 \cdots k_n}$, where n and $k_1, \cdots, k_n$ are positive integers, has been already defined we let $\{B_{k_1 \cdots k_n k}\}_k$ be an enumeration of the family $\{B \in \delta_{B_{k_1 \cdots k_n}} : d(B) < 1/(n+1)\}$. Setting $B_{k_1 \cdots k_n} = \emptyset$ for those groups $(k_1 \cdots k_n)$ of positive integers for which $B_{k_1 \cdots k_n}$ was not previously defined, we obtain a determining system $\{B^-_{k_1 \cdots k_n}\}$ of $\mathfrak{T}_0$-measurable sets (see [10], Chapter II, § 5). By [10], Chapter II, Th. (5.5), p. 50, its uncleus

$$N = \bigcup_{k_1 k_2 \cdots} \bigcap_{n=1}^{\infty} B^-_{k_1 \cdots k_n}$$

is also $\mathfrak{T}_0$-measurable. On the other hand, using 5.1 it is easy to see that $N = \delta^* - (\infty)$.

COROLLARY 5.3. *If σ is countable then $\mathfrak{J}$ is strongly measurable.*

LEMMA 5.4. *Let σ be a pre-algebra of subsets of P, $\delta \subset \sigma$, and let $A \in \delta$ be such that $\{A \cap B : B \in \delta\}$ is countable. Then there is a pre-algebra $\sigma' \subset \sigma$ containing δ and for which σ'_A is countable.*

Proof. Let δ^0 consist of all finite intersections of elements from δ. For $B, B' \in \delta^0_A$ we let $(B, B') = \{C_1, \cdots, C_n\}$ where $C_1, \cdots, C_n$ are disjoint sets from σ for which $B - B' = \bigcup_{i=1}^n C_i$. For $B, B' \in \delta^0$ we let $[B, B'] = \{D_1, \cdots, D_m\}$ where $D_1, \cdots, D_m$ are disjoint sets from σ for which $(B - B') - A = \bigcup_{j=1}^m D_j$. Set $\alpha = \cup \{(B, B') : B, B' \in \delta^0_A\}$, $\beta = \cup \{[B, B'] : B, B' \in \delta^0\}$, and $\delta^1 = \delta^0 \cup \alpha \cup \beta$. Then $\delta \subset \delta^1 \subset \sigma$, $A \in \delta^1$, and $\{A \cap B : B \in \delta^1\} = \{A \cap B : B \in \delta^0 \cup \alpha\}$ is countable. If $B, B' \in \delta$, then $B \cap B \in \delta^1$ and

$$\begin{aligned} B - B' &= (B \cap A - B' \cap A) \cup [(B - B') - A] \\ &= \left(\bigcup_{i=1}^{n} C_1\right) \cup \left(\bigcup_{j=1}^{m} D_j\right) \end{aligned}$$

where the last term is a disjoint union of sets from δ^1. Note also that $\varnothing \in \delta^1$ and $P \in \delta^1$, for $\varnothing = A - A$ and P is the empty intersection of sets from δ. Let $\sigma_1 = \delta$ and assuming that σ_n has been already defined let $\sigma_{n+1} = \sigma_n^1$, $n = 1, 2, \cdots$. The system $\sigma' = \bigcup_{n=1}^\infty \sigma_n$ has now all the desired properties.

THEOREM 5.5. *The integration base $\mathfrak{J}$ is locally strongly measurable.*

Proof. Choose $x_0 \in P$ and $U \in \Gamma_{x_0}$ whose closure U^- is compact and contained in some open metrizable neighborhood of x_0. Then for each $x \in P^\sim$ we can define a local base $\Gamma'_x \subset \Gamma_x$ such that $\bigcup_{x \in U^-} \Gamma'_x$ is countable and $U \cap V = \varnothing$ for every $V \in \Gamma'_x$ with $x \in P^\sim - U^-$. Setting $\delta = \{U\} \cup (\bigcup_{x \in P^\sim}\{V \cap P : V \in \Gamma'_x\})$, we have $\delta \subset \sigma$, $U \in \delta$, and $\{U \cap V : V \in \delta\}$ is countable. Let σ' be a prealgebra from Lemma 5.4 and let G' be the restriction of G to σ'. Then $\mathfrak{J}' = \langle \sigma', \Gamma'_x, \kappa^1, G' \rangle$ is an integration base and by 2.6, $(\mathfrak{T}_0, \tau_0) = (\mathfrak{T}'_0, \tau'_0)$. Since $\mathfrak{J}' \prec \mathfrak{J}$ (see § 2), $\mathfrak{P}(U) \subset \mathfrak{P}'(U)$ and the theorem follows from 5.2.

COROLLARY 5.6. *If P is metrizable then $(\mathfrak{T}, \tau) = (\mathfrak{T}_0, \tau_0)$.*

This corollary follows from [3], Chapter V, Corollary 35, p. 160 and [7], 4.9.

6. Examples. Four examples illustrating the previous sections will be given here.

EXAMPLE[5] 6.1. For $i = 1, 2, \cdots$ let $P_i = \{0, 1\}$ be the two point set with the discrete topology and let μ_i be the measure in P_i defined by $\mu_i(\{0\}) = \mu_i(\{1\}) = 1/2$. We set $P = \prod_{i=1}^{\infty} P_i$, $\mu = \prod_{i=1}^{\infty} \mu_i$, and define σ as the family of all μ-measurable subsets of P. Then P is a compact metrizable space whose points are sequences $\{x_i\}_{i=1}^{\infty}$ of zeroes and ones, μ is a regular measure in P, and σ is a σ-algebra containing all rectangles. If $x = \{x_i\} \in P$ we let $\Gamma_x = \{U_n\}_{n=1}^{\infty}$ where $U_n = \{\{y_i\} \in P : y_i = x_i,\ i = 1, 2, \cdots, n\}$, $n = 1, 2, \cdots$. It follows from 5.5 and 2.8 that $\mathfrak{J} = \langle \sigma, \Gamma_x, \kappa^1, \mu \rangle$, where κ^1 is the monotone convergence in $\langle \sigma, \Gamma_x \rangle$, is a measurable integration base. We shall show, however, that there is a nonempty semihereditary stable system $\delta \subset \sigma$ for which δ^* is not τ_0-measurable. Thus, in particular, the integration base $\mathfrak{J}$ is not strongly measurable.

For $x = \{x_i\}$ let $f(x) = \{f_i(x)\}$ where $f_{2i}(x) = x_{2i}$ and $f_{2i-1}(x) = 0$, $i = 1, 2, \cdots$. Then $f : P \to P$ is a continuous map and we denote by Q its image. The sets $Q_x = f^{-1}(x)$ with $x \in Q$ are disjoint, nonempty and prefect, and their union is equal to P. If $x = \{x_i\} \in Q$ and $n \geqq 1$ is an integer, let $Q_x^n = \{\{y_i\} \in P : y_{2i} = x_{2i},\ i = 1, 2, \cdots, n\}$. Then $\mu(Q_x^n) = 2^{-n}$ and $\bigcap_{n=1}^{\infty} Q_x^n = Q_x$. Hence $\mu(Q_x) = 0$ for all $x \in Q$. Let $A \subset P$ be closed and let $\mu(A) > 0$. By the compactness of P, $f(A)$ is also closed and so it is either countable or its cardinality is the continuum. Since $A \subset \bigcup_{x \in A} Q_{f(x)} = \bigcup_{y \in f(A)} Q_y$ and $\mu(A) > 0$, it follows that the cardinality of $f(A)$ is the continuum. Plainly $Q_y \cap A \neq \emptyset$ for all $y \in f(A)$.

Let γ be the least ordinal whose cardinality is the continuum and let $\{A_\alpha : 0 \leqq \alpha < \gamma\}$ be a well-ordering of all closed subsets of P with positive measure. By the previous paragraph there are $x_0, x_0' \in Q$, $x_0 \neq x_0'$, such that $Q_{x_0} \cap A_0 \neq \emptyset$ and $Q_{x_0'} \cap A_0 \neq \emptyset$. Let β be an ordinal less than γ and assume that for all ordinals α less than β we have already defined distinct elements $x_\alpha, x_\alpha' \in Q$ such that $Q_{x_\alpha} \cap A_\alpha \neq \emptyset$ and $Q_{x_\alpha'} \cap A_\alpha \neq \emptyset$. Since the cardinality of $Q' = \{x_\alpha, x_\alpha' : 0 \leqq \alpha < \beta\}$ is less than the continuum and the cardinality of $\{x \in Q : Q_x \cap A_\beta \neq \emptyset\}$ is equal to the continuum, we can choose $x_\beta, x_\beta' \in Q - Q'$, $x_\beta \neq x_\beta'$, such that $Q_{x_\beta} \cap A_\beta \neq \emptyset$ and $Q_{x_\beta'} \cap A_\beta \neq \emptyset$. Letting $B = \cup \{Q_{x_\alpha} : 0 \leqq \alpha < \gamma\}$ and $B' = \cup \{Q_{x_\alpha'} : 0 \leqq \alpha < \gamma\}$, we have $B \cap B' = \emptyset$ and $A \cap B \neq \emptyset$, $A \cap B' \neq \emptyset$ for every closed set $A \subset P$ for which $\mu(A) > 0$. Therefore every closed subset of B or $P - B$ has measure equal to zero. If $B \in \sigma$, then by the regularity of μ, $\mu(B) = \mu(P - B) = 0$ which is impossible for $\mu(P) = 1$. Hence B and similarly B' are not μ-measurable.

Now let δ consist of all uncountable subsets of Q_{x_α}, $0 \leqq \alpha < \gamma$. Then δ is a nonempty semihereditary stable subsystem of σ and δ^*

[5] This example is due to K. Prikry.

computed by κ^1 is equal to B. Since the measure μ has no regular extension, it follows from [7], 4.12 that $\mu = \tau_0$ and so δ^* is not $\mathfrak{T}_0$-measurable.

Note that the hypothesis of the continuum was not used in this example.

EXAMPLE 6.2. Let P be a compact Boolean space (see [3], Chapter 5, Problem S, p. 168), let σ be the algebra of all compact-open subsets of P, and let $\Gamma_x = \{U \in \sigma : x \in U\}$ for all $x \in P$. If κ^1 is the monotone convergence in $\langle \sigma, \Gamma_x \rangle$, then by 4.1, every net from κ^1_x has the form $\{U, U \in \Gamma, \subset\}$ where Γ is a cofinal subset of Γ_x. It follows from [9], 4.3 that κ^1 is admissible. Since, e.g., the Tychonoff product of any family of finite discrete spaces is a compact Boolean space, we see that the space P need not be locally metrizable.

EXAMPLE 6.3. Let $P = [0, 1)$ together with the usual topology and let σ be the pre-algebra consisting of all half-open intervals $[a, b) \subset P$. We shall identity $P^\sim$ with $[0, 1]$ and for every $x \in P^\sim$ we shall let $\Gamma_x = \{[x - 1/n, x + 1/n) \cap [P \cup (x)]\}_{n=1}^\infty$. If $x \in P^\sim$ then let κ_x consist of all sequences $\{[x - 1/n_k, x + 1/n_k) \cap B\}_{k=1}^\infty$ where $B \in \sigma$ and $\{n_k\}$ is an increasing sequence of positive integers. Thus defined the convergence $\kappa = \{\kappa_x : x \in P^\sim\}$ clearly satisfies conditions $\mathscr{K}_1 - \mathscr{K}_4$ and if $\{B_n\} \in \kappa_x$, then for all sufficiently large n, B_n is a half-open interval of rational length (which may be zero). Hence if δ consists of all intervals of irrational length, then $\delta^* = \emptyset$. However, it is easy to see that $\delta \subset \sigma$ is a nonempty semihereditary stable system (see [9], 4.2) and so κ does not satisfy conditions $\mathscr{K}_5$ and $\mathscr{K}_6$.

EXAMPLE 6.4. Let P, $P^\sim$, and σ be the same as in Example 6.3. For $x \in P^\sim$, $\Gamma_x = \{[x - 1/n, x_n) \cap [P \cap (x)]\}_{n=1}^\infty$ where $\{x_n\}_{n=1}^\infty$ is a decreasing sequence of irrational numbers converging to x. Denote by Q the set of all rational numbers in $[0, 1]$. If $x \in P^\sim$ then let κ_x consist of all sequences $\{[a_n, b_n)\}_{n=1}^\infty \subset \sigma$ such that $\lim a_n = \lim b_n = x$ and for all sufficiently large n, either $b_n \in P - Q$ or $b_n = b_{n+1}$. It is easy to see that thus defined the convergence $\kappa = \{\kappa_x : x \in P^\sim\}$ satisfies conditions $\mathscr{K}_1 - \mathscr{K}_4$. Let $\delta \subset \sigma$ be a nonempty semihereditary system and let $[a, b) \in \delta$. If $a = b$ then $\delta^* = P^\sim$. Hence assume that $a < b$ and choose an $x_1 \in (a, b) - Q$ such that $\max(x_1 - a, b - x_1) \leqq \frac{2}{3}(b - a)$. By the semihereditariness of δ, e.g., $[a, x_1) \in \delta$. Now choose $x_2 \in (a, x_1) - Q$ such that $\max(x_2 - a, x_1 - x_2) \leqq \frac{2}{3}(x_1 - a)$ and select an interval from $[a, x_2)$, $[x_2, x_1)$ which belongs to δ. Inductively, we obtain a decreasing sequence $\{B_n\}_{n=1} \subset \delta$ for which $\bigcap_{n=1}^\infty B_n^\sim = (x)$. Obviously, $x \in \delta^*$ and so κ satisfies also condition $\mathscr{K}_5$. However, κ does not satisfy condition $\mathscr{K}_6$. To see this, let δ_0 consist of all intervals $[a, b) \in \sigma$ such that $b - a > 0$ and $b \in Q$. Then $\delta_0 \subset \sigma$ is a nonempty semihereditary stable system and $\delta_0^* = Q$ is countable.

References

1. C. A. Hayes, and C. Y. Pauc, *Full individual and class differentiation theorems in their relations to halo and Vitali properties*, Canad. J. Math. **7** (1955) 221-274.
2. J. G. Hocking and G. S. Young, *Topology*, Addison-Wesley, London, 1961.
3. J. L. Kelley, *General topology*, Van Nostrand, New York, 1955.
4. W. F. Pfeffer, *The Perron integral in topological spaces*, Časopis pro pěst. mat. **88** (1963), 322-348.
5. ———, *On the lower derivate of a set function*, Canad. J. Math. **20** (1968), 1489-1498.
6. ———, *An integral in topological spaces I*, J. Math. Mech. **18** (1969), 953-972.
7. ———, *An integral in topological spaces II*, Mathematica Scandinavica (to appear)
8. ———, *Singular integrals are Perron integrals of a certain type*, Canad. J. Math. (to appear)
9. W. F. Pfeffer and W. J. Wilbur, *A note on cluster points of a semihereditary stable system of sets*, Proc. Amer. Math. Soc. **21** (1969), 121-125.
10. S. Saks, *Theory of the integral*, Hafner, New York, 1937.

Received May 1, 1969.

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