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REAL C*-ALGEBRAS

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Several variants of the classical Gelfand-Neumark characterization of complex C^*-algebras are here extended to characterize real C^*-algebras up to isometric*-isomorphism and also up to homeomorphic *-isomorphism. The proofs depend on norming the complexification of the real algebra and applying the author's characterization of complex C^*-algebras to the result. L. Ingelstam has obtained similar but weaker results by an entirely different method.

An involution on $\mathfrak{A}$ is a map (*): $\mathfrak{A} \rightarrow \mathfrak{A}$ which is a conjugate linear involutive antiautomorphism. A generalized involution is an involution except that it may be either an automorphism or an antiautomorphism (Generalized involutions have been considered previously by B. Yood [12]. If $\mathfrak{A} = \mathfrak{A}^0 \bigoplus \mathfrak{A}^1$ is a $\mathbf{Z}_2$ graded real algebra, then $x^0 + x^1 \rightarrow x^0 - x^1$ is an automorphic generalized involution, and conversely the sets of hermitian and skew hermitian elements in a real algebra with an automorphic generalized involution give a $\mathbf{Z}_2$ grading.) An algebra $\mathfrak{A}$ with a [generalized] involution is called a [generalized] *-algebra. If $\mathfrak{A}$ is also a Banach algebra and the norm and involution satisfy $\| x^* x \| = \| x \|^2$ for all $x \in \mathfrak{A}$ then $\mathfrak{A}$ is called a [generalized] B^*-algebra.

If $\mathscr{H}$ is a real or complex Hilbert space, then $[\mathscr{H}]$, the Banach algebra of all bounded linear transformations from $\mathscr{H}$ into $\mathscr{H}$, is a B^*-algebra when the involution is defined as the map assigning to each element its Hilbert space adjoint. A subset of a generalized *-algebra is called self adjoint if it is closed under the involution. A self adjoint subalgebra is called a *-subalgebra. Obviously a norm closed *-subalgebra of $[\mathscr{H}]$ is also a B^*-algebra. A homomorphism φ from an algebra $\mathfrak{A}$ with generalized involution into $[\mathscr{H}]$ is called a *-representation if $\varphi(x^*) = \varphi(x)^*$ for all $x \in \mathfrak{A}$. A Banach generalized *-algebra $\mathfrak{A}$ will be called a C^*-algebra if there is an isometric *-representation of $\mathfrak{A}$ on some Hilbert space. In this case the generalized involution is in fact antiautomorphic. A generalized *-algebla $\mathfrak{A}$ is called hermitian if and only if $-h^2$ has a quasi-inverse in $\mathfrak{A}$ for each hermitian element h in $\mathfrak{A}$, skew hermitian if and only if j^2 has a quasi-inverse in $\mathfrak{A}$ for each skew hermitian element j in $\mathfrak{A}$. A *-algebra is called symmetric if and only if $-x^* x$ has a quasi-inverse in $\mathfrak{A}$ for each x in $\mathfrak{A}$. Complex B*-algebras are necessarily symmetric and therefore hermitian. However the complex numbers, $\mathbf{C}$ considered as a real Banach algebra with the identity map as involution are an example of a nonhermitian real B^*-algebra. The existence of an involution or generalized involution is a much weaker condition on a real algebra than on a complex algebra since the identity map is an involution on any commutative real algebra and a generalized involution on any real algebra.

It is well known that any complex B^*-algebra is a C^*-algebra. See [4] for a proof and further references (cf. [2], [11]). The analogous result for real B^*-algebras is false without further restriction. In fact we prove the following theorem which extends results of L. Ingelstam [5, 17.7, 18.6, 18.7, 18.8].

THEOREM 1. *The following are equivalent for a real Banach generalized *-algebra* $\mathfrak{A}$:

(1) $\mathfrak{A}$ *is a* C^**-algebra.*

(2) $\|x\|^2 \leqq \|x^* x + y^* y\|$ for all x, y in $\mathfrak{A}$.

(3) $\mathfrak{A}$ *is a hermitian generalized* B^**-algebra.*

A complex *-algebra $\mathfrak{A}$ with an identity is a C*-algebra if and only if $\|z^*\| \|z\| \leqq \|z^* z\|$ for all normal elements z in $\mathfrak{A}$ [3, 2.5], and any complex *-algebra $\mathfrak{A}$ is a C^*-algebra if and only if the same inequality holds for all elements x in $\mathfrak{A}$ [11]. It is not known whether these results generalize to real hermitian *-algebra.

We call a generalized *-algebra C^*-equivalent if and only if it is homeomorphically *-isomorphic to some C^*-algebra. Thus a generalized *-algebra is C^*-equivalent if and only if it has a homeomorphic *-representation on some Hilbert space.

THEOREM 2. *The following are equivalent for a real Banach generalized *-algebra* $\mathfrak{A}$.

(1) $\mathfrak{A}$ *is* C^**-equivalent.*

(2) *There is a constant* C *such that* $\|z^*\| \|z\| \leqq C \|z^* z + w^* w\|$ *for all commuting pairs of normal elements* z, w *in* $\mathfrak{A}$.

(3) $\mathfrak{A}$ *is hermitian and there is a constant* C *such that* $\|z^*\| \|z\| \leqq C \|z^* z\|$ *for all normal elements* z *in* $\mathfrak{A}$.

(4) $\mathfrak{A}$ *is hermitian and skew hermitian and there is a constant* C *such that* $\|k\|^2 \leqq C \|k^2\|$ *for all hermitian and all skew hermitian elements* k *in* $\mathfrak{A}$.

The real group algebra of $\mathbf{Z}_2$ with ℓ^1-norm and an involution given by $(a + b\gamma)^* = a - b\gamma$ where γ is the generator of $\mathbf{Z}_2$ satisfies condition (4) except that it is not skew hermitian. Also the algebra $\mathbf{C}$ of complex numbers with the identity map as involution satisfies (3) and (4) except that it is not hermitian. The equivalence of (1) and (4) can be regarded as a real and noncommutative version of B.

Yood's result [12, 4.1(4)] or as a real version of his Theorem 2.7 in [13] as extended by a remark in [10]. Notice that condition (2), (3), (4) do not assume the continuity of the involution nor do they put any restriction on nonnormal elements of $\mathfrak{A}$. In these respects Theorem 2 significantly strengthens Theorem 17.6 of L. Ingelstam in [5].

S. Shirali and J. W. M. Ford have recently shown [10] that a complex Banach algebra with a hermitian real linear involution is symmetric. Their arguments also show that a real hermitian and skew hermitian Banach *-algebra is symmetric. Although the full force of the real version of this result could be avoided in our arguments it is noted in Lemma 1 because of its general interest.

The theorems are all proved by embedding the real algebra in a complex algebra and using a recent result of the author on complex C^*-algebras:

THEOREM A ([7]). *A complex Banach algebra $\mathfrak{A}$ with an identity element* 1 *of norm one is isometrically isomorphic to some complex C^*-algebra if and only if $\mathfrak{A}$ is the linear span of*

$$\mathfrak{A}_H = \{h \in \mathfrak{A} : \|\exp(ith)\| \leqq 1, \ \forall\, t \in \mathbf{R}\} .$$

*In this case each element of $\mathfrak{A}$ has a unique decomposition $x = h + ik$ with $h, k \in \mathfrak{A}_H$. Furthermore the map $h + ik \rightarrow h - ik$ is an involution on $\mathfrak{A}$ and any isometric isomorphism of $\mathfrak{A}$ into a C^*-algebra is a *-isomorphism relative to this involution.*

2. Embedding in a complex C*-algebra. The fundamental tool used in this paper is described in Proposition 1 at the end of this section. For convenience we establish some notation to use throughout the paper.

If $\mathfrak{A}$ is a real algebra, we shall denote the associated complex algebra by $\mathfrak{B}$. That is, $\mathfrak{B}$ is the set of formal expressions $x + iy$ with x and y in $\mathfrak{A}$ and the obvious algebraic operations. Recall that the spectrum of an element in a real algebra $\mathfrak{A}$ is defined to be its usual spectrum in $\mathfrak{B}$. Notice that with this convention a real algebra $\mathfrak{A}$ with generalized involution is hermitian if and only if each hermitian element in $\mathfrak{A}$ has real spectrum, is skew hermitian if and only if each skew hermitian element has purely imaginary spectrum, and a *-algebra is symmetric if and only if x^*x has nonnegative spectrum for each element x in $\mathfrak{A}$ [8, 4.1.7 and 4.7.6]. Clearly a complex *-algebra is skew hermitian if and only if it is hermitian. If $\mathfrak{A}$ has a generalized involution, then $\mathfrak{B}$ will be endowed with the generalized involution $(x + iy)^* = x^* - iy^*$.

If $\mathfrak{A}$ is an algebra without an identity then $\mathfrak{A}^1$ will represent the algebra (under the obvious operation) of all formal expressions $x + t$ with x in $\mathfrak{A}$ and t a scalar. If $\mathfrak{A}$ is normed $\mathfrak{A}^1$ is given the norm $\| x + t \| = \| x \| + | t |$ unless $\mathfrak{A}$ is assumed to be a generalized B^*-algebra in which case the norm

$$\| x + t \| = \sup \{ \| xu + tu \| : u \in \mathfrak{A}, \| u \| = 1 \}$$

is used instead. If $\mathfrak{A}$ is a Banach algebra the first norm on $\mathfrak{A}^1$ is complete, and if $\mathfrak{A}$ is a B^*-algebra so is $\mathfrak{A}^1$ with the second norm [8, 4.1.13].

It is also convenient to introduce once and for all the following notation for the sets of hermitian, skew hermitian, unitary, normal and positive elements in a generalized *-algebra:

$$\begin{aligned} &\mathfrak{A}_H = \{ h \in \mathfrak{A} : h = h^* \}, \ \mathfrak{A}_J = \{ j \in \mathfrak{A} : -j = j^* \} , \\ &\mathfrak{A}_U = \{ u \in \mathfrak{A} : uu^* = u^*u = 1 \}, \ \mathfrak{A}_N = \{ z \in \mathfrak{A} : z^* z = z^* z \} , \\ &\mathfrak{A}_+ = \{ h \in \mathfrak{A}_H : h \text{ has nonnegative real spectrum} \} . \end{aligned}$$

Notice that this is only one of several possible notions of positivity. It will be convenient to use $\mathfrak{A}_G$ to denote $\mathfrak{A}_H \cup \mathfrak{A}_J$ in a (real or complex) generalized *-algebra. Denote the spectrum and spectral radius of an element x in a Banach algebra by $\sigma(x)$ and $\nu(x)$, respectively. Note that $\sigma(x^*) = \{ \bar{\lambda} : \lambda \in \sigma(x) \}$ so that $\nu(x) = \nu(x^*)$ for all x in $\mathfrak{A}$.

LEMMA 1. (*Shirali and Ford* [10].) *A real hermitian and skew hermitian Banach *-algebra is symmetric.*

Proof. Ford's square root lemma [1] is proved for a real Banach *-algebra $\mathfrak{A}$ by applying the original proof to the complexification $\mathfrak{C}$ of a closed maximal commutative *-subalgebra of $\mathfrak{A}$ which contains h, and noting that $u = \lim h_n$ lies in the natural image of $\mathfrak{A}$ in $\mathfrak{C}$. Lemmas 1 through 5 of [10] now follow for real *-algebras without essential change. The proof is completed by constructing the real commutative *-subalgebra $\mathfrak{C}$ as in [10] and noting that θ is defined on the complexification of $\mathfrak{C}$.

We note that the proof of Ford's square root lemma holds even for real Banach generalized *-algebras.

LEMMA 2. *Let $\mathfrak{A}$ be a (real or complex) Banach generalized *-algebra. Let there be a constant C such that $\| k \|^2 \leqq C \| k^2 \|$ for all $k \in \mathfrak{A}_G$. Then*

(a) *$\| k \| \leqq C \nu(k)$ for all $k \in \mathfrak{A}_G$.*

(b) *The involution is continuous.*

(c) *If* $\mathfrak{A}$ *is hermitian and lacks an identity then* $\|k+t\|^2 \leqq 9C^2 \|(k+t)^2\|$ *for all* $k+t \in (\mathfrak{A}^1)_G$.

(d) *Let* $\mathfrak{A}$ *be hermitian and if the involution is antiautomorphic let* $\mathfrak{A}$ *be skew hermitian. Then* $\mathfrak{A}_+$ *is closed under addition.*

Proof. (a) $\|k\| \leqq (CC^2 \cdots C^{2^{n-1}})^{2^{-n}} \|k^{2^n}\|^{2^{-n}}$

(b) This follows from Theorem 3.4 in [12].

(c) If $\mathfrak{A}$ is real $(\mathfrak{A}^1)_J = \mathfrak{A}_J$ and if $\mathfrak{A}$ is complex the inequality for elements in $(\mathfrak{A}^1)_J$ follows from the inequality for elements in $(\mathfrak{A}^1)_H$. Thus let $h \in \mathfrak{A}_H$ and $t \in \mathbf{R}$. By replacing h by $-h$ if necessary we can assume that $\nu(h)$ is the greatest real number in $\sigma(h)$. Let the convex hull of $\sigma(h)$ be $[-r, s]$. Then r and $s = \nu(h)$ are nonnegative since $\mathfrak{A}$ lacks an identity, and $\sigma(h+t) \subseteqq [-r+t, s+t]$.

Case 1. $t \geqq 0$. Then $C\nu(h+t) = C(s+t) \geqq \|h\| + |t| = \|h+t\|$.

Case 2. $0 > t \geqq r-s/2$. Then $3C\nu(h+t) = 3C(s+t) \geqq 3C(s+(r-s/2)) \geqq 3C(S/2) \geqq C(s-(r-s/2) \geqq C(s+|t|) \geqq \|h+t\|$.

Case 3. $r-s/2 > t$. Then $3C\nu(h+t) = 3C(r-t) \geqq 3C(r-(2/3)(r-s/2) - 1/3\, t) \geqq C(s-t) \geqq \|h+t\|$. Thus in any case $3C\nu(h+t) \geqq \|h+t\|$ so that $\|h+t\|^2 \leqq 9C^2 \nu(h+t)^2 = 9C^2 \nu(h+t))^2) \leqq 9C^2 \|(h+t)^2\|$.

(d) If the involution is antiautomorphic this follows from Lemma 1 and [8, 4.7.10] and in any case is an intermediate step in the proof of Lemma 1. If the involution is automorphic then $\mathfrak{A}_H$ is a *-subalgebra of $\mathfrak{A}$ in which every element satisfies $\|h\|^2 \leqq C\|h^2\|$ and has real spectrum. Then $\mathfrak{A}_H$ is semisimple by [12, 3.5] and thus is commutative by [6, Th. 4.8]. Thus $\mathfrak{A}_+ \subseteqq \mathfrak{A}_H$ is closed under addition since the spectrum is subadditive in a commutative algebra.

The existence of C such that $\|k\|^2 \leqq C \|k^2\|$ for all $k \in \mathfrak{A}_G$ is equivalent to the existence of B or D such that $\|k\| \leqq B\nu(k)$ for all $k \in \mathfrak{A}_G$ or $\|z\| \leqq D\nu(z)$ for all $z \in \mathfrak{A}_N$, since $\|z\| \leqq \|(z+z^*)/2\| + \|(z-z^*)/2\| \leqq C(\nu(z) + \nu(z^*)) = 2C\nu(z)$.

PROPOSITION 1. *Let* $\mathfrak{A}$ *be a real hermitian and skew hermitian Banach generalized *-algebra. Let there be a constant* C *such that* $\|k\|^2 \leqq C\|k^2\|$ *for each* $k \in \mathfrak{A}_G$. *Then there is a complex* C^**-algebra* $\mathfrak{B}$ *and a homeomorphic *-isomorphism of* $\mathfrak{A}$ *into* $\mathfrak{B}$.

Proof. $\mathfrak{A}^1$ is hermitian and skew hermitian. Thus using Lemma 2(c) we may assume $\mathfrak{A}$ has an identity element. We will define a norm on $\mathfrak{B}$ which makes it a complex Banach algebra satisfying the hypotheses of Theorem A. The norm $\|\cdot\|_U$ for $\mathfrak{B}$ is defined to be the Minkowski functional of the convex hull of $\mathfrak{B}_U$, or directly:

$$\|x+iy\|_U = \inf\{\textstyle\sum_{j=1}^n t_j : x+iy = \sum_{j=1}^n t_j u_j;\ t_j \in \mathbf{R},\ t_j \geqq 0;\ u_j \in \mathfrak{B}_U\}.$$

(This norm has been used previously by Russo and Dye [9]).

In order to prove that this expression is always finite and in fact a complete norm, it is easiest to introduce another norm $|||\cdot|||$ on $\mathfrak{B}$ which is obviously finite and complete and then compare $\|\cdot\|_U$ and $|||\cdot|||$. Let $|||x+iy||| = \|x\| + \|y\|$ for all $x, y \in \mathfrak{A}$. With respect to this norm $\mathfrak{B}$ is a real Banach generalized *-algebra.

By Lemma 2(b) the involution in $\mathfrak{A}$ is continuous. Let [illegible] a constant such that $\|x^*\| \leqq B\|x\|$ for all $x \in \mathfrak{A}$. If $x \in \mathfrak{A}$ then $x = h + j$ where $h = (x+x^*)/2 \in \mathfrak{A}_H$ and $j = (x - x^*)/2 \in \mathfrak{A}_J$. Clearly $\|h\|$ and $\|j\|$ are bounded by $(1 + \mathrm{B})\ \|x\|/2 \leqq B\ \|x\|$.

Let s be a real number greater than $B\ \|x\|$. Then the power series for $V = \cos^{-1}(h/s)$ and $w = \sinh^{-1}(j/s)$ converge and $h = s[\exp(iv) + \exp(-iv)]/2$, $j = s[\exp(w) + (-\exp(-w))]/2$ with each exponential in $\mathfrak{B}_U$. Similarly iy can be expressed as a positive real linear combination of elements in $\mathfrak{B}_U$. Thus $\|x+iy\|_U$ is always finite and in fact $\|x+iy\|_U \leqq 2B(\|x\| + \|y\|) = 2B|||x+iy|||$ for all $x, y \in \mathfrak{A}$.

It is obvious from the definition that $\|\cdot\|_U$ is a norm for a real linear space. However $\mathfrak{B}$ is also a complex normed algebra with respect to $\|\cdot\|_U$ since $\mathfrak{B}_U$ is a multiplicative group closed under multiplication by complex numbers of norm one. Furthermore the involution is an isometry.

Any element $\mathrm{u} \in \mathfrak{B}_U$ can be written as $u = h + j + i(k + g)$ with $h, k \in \mathfrak{A}_H$ and $j, g \in \mathfrak{A}_J$. Taking the real part of the equations $u^* u = 1$ and $uu^* = 1$ we get

$$h^2 - j^2 + k^2 - g^2 + hj - jh + ky - gk = 1$$

$$h^2 - j^2 + k^2 - g^2 + jh - hj + gk - kg = 1\,.$$

Thus $h^2 - j^2 + k^2 - g^2 = 1$. Since $\mathfrak{A}$ is hermitian and skew hermitian, h^2, k^2, $-j^2$ and $-g^2$ all belong to $\mathfrak{A}_+$. Thus by Lemma 2(d) $-j^2 + k^2 - g^2 \in \mathfrak{A}_+$. Therefore $\sigma(h^2) \leqq \sigma(1-(-j^2+k^2-g^2)) \leqq [0, 1]$ and $\nu(h) \leqq 1$. Similarly $\nu(j) \leqq 1$, $\nu(k) \leqq 1$ and $\nu(g) \leqq 1$. Thus

$$|||u||| = \|h+j\| + \|k+g\| \leqq \|h\| + \|j\| + \|k\| + \|g\| \leqq 4C$$

for all $u \in \mathfrak{B}_U$. Thus if $x + iy = \sum_{j=1}^n t_j u_j$ with $t_j \geqq 0$ and $u_j \in \mathfrak{B}_U$ then $|||x+iy||| \leqq (\sum_{j=1}^n t_j)\ |||u_j||| \leqq 4C\sum_{j=1}^n t_j$. Therefore $|||x+iy||| \leqq 4C\|x+iy\|_U$ for all $x + iy$ in $\mathfrak{B}$.

Since $\|\cdot\|_U$ is equivalent to a complete norm it is a complete norm. Thus $\mathfrak{B}$ is a complex Banach algebra with an identity element of norm one. Furthermore $\mathfrak{B}$ is the linear span of $\mathfrak{B}_H$. For each h in $\mathfrak{B}_H$, $\exp(ith)$ is in $\mathfrak{B}_U$ and hence $\|\exp(ith)\|_U \leqq 1$. Therefore $(\mathfrak{B}, \|\cdot\|_U)$ satisfies the hypotheses of Theorem A and is a complex C^*-algebra with respect to its involution.

We must still show that the natural map of $\mathfrak{A}$ into $\mathfrak{B}$ is a homeomorphism. This is true since, for all x in $\mathfrak{A}$, $\|x\|_U \leqq 2B \||x\|| = 2B\|x\| \leqq 8BC\|x\|_U$.

COROLLARY 1. *Any generalized *-algebra satisfying the hypotheses of Proposition 1 has an antiautomorphic involution.*

COROLLARY 2. *Let $\mathfrak{A}$ be a real hermitian and skew hermitian generalized B^*-algebra. Then there is a complex C^*-algebra and a real isometric *-isomorphism of $\mathfrak{A}$ into $\mathfrak{B}$.*

Proof. Consider $\mathfrak{A}$ as embedded in $(\mathfrak{B}, \|\cdot\|_U)$ as described in Proposition 1. Using Lemma 2(a), Corollary 1 and the fact that a C^*-algebra is a B^*-algebra we get

$$\|x\|^2 = \|x^* x\| = \nu(x^* x) = \|x^* x\|_U = \|x\|_U^2 \text{ for all } x \in \mathfrak{A} .$$

Thus the embedding is an isometry.

3. **Proofs of Theorems 1 and 2.** We need three more lemmas. The first one records the connection between real and complex *-representations.

LEMMA 3. *Let φ be an isometric *-representation of the [real, respectively, complex] B^*-algebra $\mathfrak{A}$ on the [real, respectively, complex] Hilbert space $\mathscr{h}$. Then there is a natural isometric *-representation ψ of the [complex, respectively, real] algebra $\mathfrak{B}$ associated with $\mathfrak{A}$ on the complex, respectively, real] Hilbert space $\mathscr{K}$ associated with $\mathscr{h}$.*

Proof. If $\mathscr{h}$ is real let $\mathscr{K}$ be the set of formal expressions $\xi + i\eta$ where ξ and η belong to $\mathscr{h}$. The inner product in $\mathscr{K}$ is given by

$$(\xi + i\eta, \zeta + i\mu) = (\xi, \zeta) + i(\eta, \zeta) - i(\xi, \mu) + (\eta, \mu)$$

and thus the norm in $\mathscr{K}$ is given by $\|\xi + i\eta\|^2 = \|\xi\|^2 + \|\eta\|^2$. The complex B^*-algebra $\mathfrak{B}$ associated to the real B^*-algebra $\mathfrak{A}$ is that defined in the proof of Proposition 1. The typical element of $\mathfrak{B}$ is of the form $x + iy$ with x and y elements of $\mathfrak{A}$. Define ψ by

$$\psi(x + iy)(\xi + i\eta) = \varphi(x)\xi + i\varphi(x)\eta + i\varphi(y)\xi - \varphi(y)\eta .$$

It is easy to check that this is a *-isomorphism, and that the image is closed in the norm of $[\mathscr{K}]$. Thus the complex *-algebra $\mathfrak{A}$ can be provided with a B^*-norm pulled back through ψ. This norm must agree with the B^*-norm defined in the proof of Proposition 1. Thus ψ is an isometry.

Now consider the case where $\mathfrak{A}$ and $\mathscr{h}$ are complex. The associated real algebra and vector space are obtained by merely restricting scalar multiplication to the real numbers. The inner product and norm in $\mathscr{K}$ are $(\xi, \eta)_{\mathscr{K}} = \mathrm{Re}(\xi, \eta)_{\mathscr{h}}$, $\|\xi\|_{\mathscr{K}} = \|\xi\|_{\mathscr{h}}$. Thus φ considered as a *-representation of a real algebra coincides with ψ.

LEMMA 4. *Let $\mathfrak{A}$ be a Banach generalized *-algebra. Let there be a constant C such that $\|z^*\|\,\|z\| \leqq C\|z^*z + w^*w\|$ for all commuting elements z and w in $\mathfrak{A}_N$. Then $\mathfrak{A}$ is hermitian and skew hermitian.*

Proof. Any $k \in \mathfrak{A}_G$ lies in some closed maximal commutative *-subalgebra $\mathfrak{C}$ [8, 4.1.3] where it has the same spectrum as in $\mathfrak{A}$. By Lemma 2(b) there is a constant B such that $\|z\|^2 \leqq B\|z^*\|\,\|z\| \leqq BC\|z^*z + w^*w\|$ when z and w lie in $\mathfrak{C}$. Thus $\mathfrak{C}$ satisfies Theorem 4.2.3 in [8] so that it is hermitian and skew hermitian. Thus $\mathfrak{A}$ is also.

LEMMA 5. *Let $\mathfrak{A}$ be a Banach generalized *-algebra satisfying $\|z^*\|\,\|z\| \leqq C\|z^*z\|$ for all $z \in \mathfrak{A}_N$. Then $\mathfrak{A}$ is skew hermitian.*

Proof. Let B be the bound for the generalized involution guaranteed by Lemma 2(b). Then the involution in $\mathfrak{A}^1$ is also bounded by B. For an arbitrary skew hermitian element j of $\mathfrak{A}$, $e^j(e^j)^* = e^je^{-j} = 1 = (e^j)^*(e^j)$ is $\mathfrak{A}^1$. *If* $z + t$ is in $(\mathfrak{A}^1)_U$, then $z^*z + tz^* + tz = 0$ and $t^2 = 1$. Thus $\|z\|^2 \leqq B\|z^*\|\,\|z\| \leqq BC\|z^*z\| \leqq BC(1 + \mathrm{B})\|z\|$, so $\|z + t\| \leqq BC(1 + B) + 1$. Applying this to e^{nj} for $n \in \mathbf{Z}$ gives $\nu(e^j) = \nu(e^{-j}) = 1$. Therefore the spectrum of e^j lies on the unit circle and the spectrum of j is purely imaginary.

Proof of Theorem 1. (1) $\Rightarrow$ (2): Consider $\mathfrak{A}$ as embedded in $[\mathscr{h}]$ for a suitable Hilbert space $\mathscr{h}$. Then for x and y in $[\mathscr{h}]$.

$$\begin{aligned}\|x\|^2 &= \sup\{\|x\xi\|^2\} \leqq \sup\{\|x\xi\|^2 + \|y\xi\|^2\} \\ &= \sup\{(x^*x\xi, \xi) + (y^*y\xi, \xi)\} = \sup\{((x^*x + y^*y)\xi, \xi)\} \\ &\leqq \|x^*x + y^*y\|\end{aligned}$$

where each supremum is over all $\xi \in \mathscr{h}$ with $\|\xi\| \leqq 1$.

$(2) \Rightarrow (3)$: Lemma 4.

$(3) \Rightarrow (1)$: Lemma 5, Corollary 2 and Lemma 3.

Note that without changing this proof, condition (2) of Theorem 1 can be weakened to: $\|x\|^2 \leqq \|x^* x\|$ for all $x \in \mathfrak{A}$ and there exists a constant C such that $\|z^*\| \|z\| \leqq C \|z^*z + w^* w\|$ for all commuting pairs z and w in $\mathfrak{A}_N$. This is essentially the condition Dc^* in [5, 18.6].

Proof of Theorem 2. $(1) \Rightarrow (2)$: Theorem 1 and Lemma 2b.
$(2) \Rightarrow (3)$: Lemma 4.
$(3) \Rightarrow (4)$: Lemma 5.
$(4) \Rightarrow (1)$: Proposition 1 and Lemma 3.

The following corollary bears the same relationship to Theorem 2 that [5, 18.7] bears to Theorem 1 or [5, 18.6].

COROLLARY 3. *Let $\mathfrak{A}$ be a real normed generalized *-algebra. Let there be a constant C such that $\|x\|^2 \leqq C \|x^* x + y^* y\|$ for all x and y in $\mathfrak{A}$. Then $\mathfrak{A}$ has a homeomorphic *-representation on some Hilbert space.*

Proof. The generalized involution is continuous since $\|x\|^2 \leqq C \|x^* x\| \leqq C \|x^*\| \|x\|$. Thus the completion of $\mathfrak{A}$ is a generalized *-algebra which satisfies the same inequality and hence satisfies Theorem 2.

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