PACIFIC JOURNAL OF MATHEMATICS
Vol. 35, No. 2, 1970

ISOMORPHISMS OF $C_0(Y)$ ONTO $C(X)$

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The well known Banach-Stone theorem states that if X and Y are locally compact Hausdorff spaces, then the existence of an isometry φ of $C_0(Y)$ onto $C_0(X)$ implies that X and Y are homeomorphic. This result has been generalized by showing that the same conclusion holds if the requirement that φ be an isometry is replaced by the requirement that φ be an isomorphism with $||\varphi||\,||\varphi^{-1}|| < 2$. However, the author knows of no valid examples in the literature which show that 2 is the largest number for which this generalization is true. Here such an example is provided, and it is shown that the reason for the apparent scarcity of examples is not that they need be complicated, but rather, at least in the case where X is compact and Y noncompact, that there is essentially just one way to construct them.

If X is a locally compact Hausdorff space, we denote by $C_0(X)$ the Banach space of continuous, complex-valued functions vanishing at infinity on X, provided with the usual supremum norm. If X is actually compact, so that $C_0(X)$ consists of all continuous complex functions on X, we will, whenever it is convenient to do so, represent this function space by the more customary notation $C(X)$. Now if X and Y are two locally compact Hausdorff spaces and if there exists a continuous isomorphism φ of $C_0(Y)$ onto $C_0(X)$ satisfying $||\varphi||\,||\varphi^{-1}|| = \alpha$, we write $X \overset{\alpha}{\sim} Y$. This is easily seen to be equivalent to the existence of a norm-increasing isomorphism φ of $C_0(Y)$ onto $C_0(X)$—i.e., $||g|| \leqq ||\varphi(g)||$ for $g \in C_0(Y)$—satisfying $||\varphi|| = \alpha$ and $||\varphi^{-1}|| = 1$. Employing this notation, the Banach-Stone theorem states that if $X \overset{1}{\sim} Y$, then X and Y are homeomorphic.

In [1] this theorem was strengthened by showing that if $X \overset{\alpha}{\sim} Y$, where α is any real number less than 2, then X and Y are homeomorphic. The following example, announced in [2], displays two locally compact Hausdorff spaces X and Y, with X compact and Y noncompact, such that $X^2 \overset{2}{\sim} Y$. Hence 2 is the greatest number for which the formulation of the Banach-Stone theorem given in [1] is valid.

EXAMPLE. Let $Y = \{y_k\colon k = 0, \pm 1, \pm 2, \cdots\}$ be a sequence of distinct points y_k, where $\lim_{k\to+\infty} y_{-k} = y_0$, y_0 is the only accumulation point of Y, and the set $\{y_k\colon k \geqq 1\}$ has no accumulation point in Y. Let $X = \{x_k\colon k = 0, \pm 1, \pm 2, \cdots\}$ be a sequence of distinct points x_k, where $\lim_{k\to+\infty} x_k = \lim_{k\to+\infty} x_{-k} = x_0$, and x_0 is the only accumulation point of X. Define an isomorphism φ of $C_0(Y)$ onto $C(X)$ by

$$(\varphi(g))(x_k) = g(y_k) + g(y_{-k}) \,, \qquad k > 0 \,,$$

$$(\varphi(g))(x_{-k}) = -g(y_k) + g(y_{-k}) \,, \qquad k > 0 \,,$$

$$(\varphi(g))(x_0) = g(y_0) \,, \qquad \text{for } g \in C_0(Y) \,.$$

Then $||\varphi|| = 2$ and $||\varphi^{-1}|| = 1$.

Aside from showing that the number 2 is exact for the formulation of the Banach-Stone theorem cited above, the example is of interest in that it provides a prototype of all examples which one can construct of pairs X, Y with X compact and Y noncompact, for which there exists a norm-increasing map φ of $C_0(Y)$ onto $C(X)$ with $||\varphi|| = 2$. We wish to show that, provided X and Y are first countable spaces, in any such example which can be constructed the isomorphism φ must behave very much as does that in the example given in this paper.

To this end we first note that if we represent the elements of the dual spaces $C_0(Y)^*$ and $C(X)^*$ as measures via the Riesz representation theorem, and employ the customary notation for unit point masses, then the isomorphism φ of the example is completely determined by the following equations involving the adjoint mappings:

$$\varphi^{*-1}\mu_{y_k} = \tfrac{1}{2}\mu_{x_k} - \tfrac{1}{2}\mu_{x_{-k}} \,, \qquad k > 0 \,,$$

$$\varphi^{*}\mu_{x_k} = \mu_{y_k} + \mu_{y_{-k}} \,, \qquad k > 0 \,.$$

We will prove the following:

THEOREM. *Let X and Y be first countable locally compact Hausdorff spaces with X compact, Y noncompact, and $X \overset{2}{\sim} Y$. Let φ be any norm-increasing isomorphism of $C_0(Y)$ onto $C(X)$ with $||\varphi|| = 2$. Then for every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subseteqq Y$ such that for each $y \in Y - K_\varepsilon$ there are points $x, x' \in X$ and $y' \in Y$, with*

$$\varphi^{*-1}\mu_y = \beta\mu_x + \beta'\mu_{x'} + \nu \,,$$

$$\varphi^{*}\mu_x = \alpha\mu_y + \alpha'\mu_{y} + \mu \,,$$

where $\beta, \beta', \alpha, \alpha'$ are scalars with $|\beta| \geqq \frac{1}{2}$, $|\beta'| > \frac{1}{2} - \varepsilon$, $|\alpha| \geqq 1$, $|\alpha'| > 1 - \varepsilon$, and $\nu(\{x\}) = \nu(\{x'\}) = \mu(\{y\}) = \mu(\{y'\}) = 0$; hence $||\nu|| < \varepsilon$ and $||\mu|| < \varepsilon$.

Before beginning the proof of the theorem, we establish some conventions regarding notation. If $y \in Y$, we will say that a sequence $\{g_{y,n}: n = 1, 2, \cdots\} \subseteqq C_0(Y)$ is *regularly associated* with the point

y if $\|g_{y,n}\| = g_{y,n}(y) = 1$ for all n, and the support of $g_{y,n}$ is contained in U_n, where $\{U_n: n = 1, 2, \cdots\}$ is some neighborhood base at y with $U_{n+1} \subseteqq U_n$ for all n. We write $\{g_{y,n}\} \leftrightarrow y$ to denote that $\{g_{y,n}\}$ is regularly associated with y. The definition of sequences $\{f_{x,n}\} \subseteqq C(X)$ regularly associated with a point $x \in X$ is analogous, and we use the corresponding notation: $\{f_{x,n}\} \leftrightarrow x$.

If $y \in Y$, and $\{g_{y,n}\} \leftrightarrow y$, then

$$\lim_n (\varphi(g_{y,n}))(x) = \lim_n \int \varphi(g_{y,n}) d\mu_x = \lim_n \int g_{y,n} d(\varphi^* \mu_x)$$

exists for all $x \in X$, and is equal to $\varphi^*\mu_x(\{y\})$. Thus for fixed $x \in X$,

$$\lim_n (\varphi(g_{y,n}))(x) = \alpha \Leftrightarrow \varphi^*\mu_x = \alpha\mu_y + \mu \, , \tag{1}$$

where $\mu \in C_0(Y)^*$ and $\mu(\{y\}) = 0$. The scalar α is unchanged if the sequence $\{g_{y,n}\}$ is replaced by any other sequence regularly associated with y. Similarly, if $\{f_{x,n}\} \leftrightarrow x \in X$, then for fixed $y \in Y$,

$$\lim_n (\varphi^{-1}(f_{x,n}))(y) = \beta \Leftrightarrow \varphi^{*-1}\mu_y = \beta\mu_x + \nu \, , \tag{2}$$

where $\nu \in C(X)^*$ and $\nu(\{x\}) = 0$.

Proof of the Theorem. We may suppose that $\varepsilon < 1/6$. Denote by $\mathbf{1}$ that function which is identically equal to 1 on X and let $K_\varepsilon = \{y \in Y: |(\varphi^{-1}(\mathbf{1}))(y)| \geqq \varepsilon/8\}$. Suppose that $y \in Y - K_\varepsilon$. Then the conclusion of the theorem will follow if we establish the validity of the following statements (I)–(IV):

(I) If $x \in X$ is such that $|\varphi^*\mu_x(\{y\})| > 1/(1+\varepsilon)$, then $|\varphi^{*-1}\mu_y(\{x\})| > \frac{1}{2} - \varepsilon$.

(II) There exists at least one $x \in X$ such that $|\varphi^*\mu_x(\{y\})| \geqq 1$, and this point x can be chosen so that $|\varphi^{*-1}\mu_y(\{x\})| \geqq \frac{1}{2}$.

(III) There exists a second point $x' \in X$, $x' \neq x$, such that $|\varphi^{*-1}\mu_y(\{x'\})| > \frac{1}{2} - \varepsilon$.

(IV) If x is a point associated with y by (II), there exists a second point $y' \in Y$ such that $|\varphi^*\mu_x(\{y'\})| > 1 - \varepsilon$.

Now if $\{g_{y,n}\} \leftrightarrow y$ then, by (1), a point $x \in X$ will be such that

$$\varphi^*\mu_x = \alpha\mu_y + \mu, \quad |\alpha| > 1/(1+\varepsilon) \, , \qquad \mu(\{y\}) = 0 \, , \tag{3}$$

if and only if x belongs to the set

$$\{x \in X: |\lim_n (\varphi(g_{y,n}))(x)| > 1/(1+\varepsilon)\} \, .$$

This set is nonvoid. For (using the fact that $\|\varphi^{*-1}\| = \|\varphi^{-1}\| = 1$)

we have

$$1 = \lim_n \int g_{y,n} d\mu_y = \int \Big(\lim_n \varphi(g_{y,n})\Big) d(\varphi^{*-1}\mu_y) \tag{4}$$
$$\leqq \sup_{x \in X} |\lim_n (\varphi(g_{y,n}))(x)| \, .$$

We next note that for all $x \in X$, we have

$$|\lim_n (\varphi(g_{y,n}))(x)| \leqq 1 + \frac{\varepsilon}{4} \, . \tag{5}$$

For if not, we could find a point $x' \in X$ and an integer m such that $|(\varphi(g_{y,m}))(x')| > 1 + \varepsilon/4$, and such that the support of $g_{y,m}$ is contained in $\{y' \in Y: |(\varphi^{-1}(1))(y')| < \varepsilon/8\}$. Define a scalar λ by $|\lambda| = 1$, and $\arg \lambda = -\arg (\varphi(g_{y,m}))(x')$. Then $\varphi^{-1}(1) + \lambda g_{y,m}$ is an element of $C_0(Y)$ with norm less than $1 + \varepsilon/8$, while

$$||\varphi(\varphi^{-1}(1) + \lambda g_{y,m})|| \geqq 1 + \lambda(\varphi(g_{y,m}))(x') > 2 + \frac{\varepsilon}{4} \, ,$$

contradicting the fact that $||\varphi|| = 2$.

Now suppose that x is any point of X which satisfies (3). We then write $\varphi^{*-1}\mu_y = \beta\mu_x + \nu$, where $\nu(\{x\}) = 0$, and claim that $|\beta| > \frac{1}{2} - \varepsilon$. For if $\{g_{y,n}\} \leftrightarrow y$, we have

$$1 = \lim_n \int g_{y,n} d\mu_y = \lim_n \int \varphi(g_{y,n}) d(\varphi^{*-1}\mu_y)$$
$$\leqq \sup_{x' \in X} |\lim_n (\varphi(g_{y,n}))(x')| \, ||\varphi^{*-1}\mu_y|| < (1 + \varepsilon)(|\beta| + ||\nu||) \, .$$

Now if we suppose that $|\beta| \leqq \frac{1}{2} - \varepsilon$, the previous inequality gives $||\nu|| > \frac{1}{2}$.

Next from (3), and the fact that $||\varphi^*|| = 2$, we obtain $\mu = \varphi^*\mu_x - \alpha\mu_y$, and $||\mu|| \leqq 2 - |\alpha|$. Thus $\varphi^{*-1}\mu = (1 - \alpha\beta)\mu_x - \alpha\nu$, and since φ^{*-1} is norm-decreasing, it follows that

$$2 - |\alpha| \geqq ||\varphi^{*-1}\mu|| \geqq 1 - |\alpha||\beta| + |\alpha| \, ||\nu|| \, , \tag{6}$$

a quantity which, since $|\beta| < \frac{1}{2} - \varepsilon$ and $||\nu|| > \frac{1}{2}$, is strictly greater than $1 + \varepsilon|\alpha|$. But this implies that $1/(1 + \varepsilon) > |\alpha|$, contradicting our choice of the point x. This proves (I).

We next note that if $\{g_{y,n}\} \leftrightarrow y$, then by what has been proven, $\{x \in X: |\lim_n (\varphi(g_{y,n}))(x)| > 1/(1 + \varepsilon)\} \subseteqq \{x \in X: |\varphi^{*-1}\mu_y(\{x\})| > \frac{1}{2} - \varepsilon\}$, a finite set. It thus follows that the function $|\lim_n \varphi(g_{y,n})|$ has a maximum on X. And since we may now replace "sup" by "max" in (4), this maximum is necessarily greater than or equal to one. Thus if we choose a point $x \in X$ at which the function attains its supremum, then by (1) we have $\varphi^*\mu_x = \alpha\mu_y + \mu$, where $|\alpha| \geqq 1$ and $\mu(\{y\}) = 0$.

For this point x write $\varphi^{*-1}\mu_y = \beta\mu_x + \nu$, where β is a scalar and $\nu(\{x\}) = 0$. We then have

$$1 = \lim_n \int g_{y,n} d\mu_y = \lim_n \int \varphi(g_{y,n}) d(\varphi^{*-1}\mu_y) = \alpha\beta + \lim_n \int \varphi(g_{y,n}) d\nu .$$

Hence

$$\lim_n \int \varphi(g_{y,n}) d\nu = 1 - \alpha\beta . \tag{7}$$

We thus have

$$|\alpha| \, ||\nu|| = \max_{x' \in X} |\lim_n (\varphi(g_{y,n}))(x')| \, ||\nu|| \geqq \left|\lim_n \int \varphi(g_{y,n}) d\nu\right| \geqq 1 - |\alpha||\beta| ,$$

so that

$$||\nu|| \geqq (1 - |\alpha||\beta|)/|\alpha| . \tag{8}$$

Combining (6) and (8) we obtain $2 - |\alpha| \geqq 2 - 2|\alpha||\beta|$, which gives $|\beta| \geqq \frac{1}{2}$ and completes the proof of (II).

Now let x be a fixed point of X whose existence is guaranteed by (II)—i.e., $\varphi^*\mu_x = \alpha\mu_y + \mu$, where $|\alpha| \geqq 1$ and $\mu(\{y\}) = 0$, and $\varphi^{*-1}\mu_y = \beta\mu_x + \nu$, where $|\beta| \geqq \frac{1}{2}$ and $\nu(\{x\}) = 0$. If $\{g_{y,n}\} \leftrightarrow y$, then by (1) $\alpha = \lim_n (\varphi(g_{y,n}))(x)$, so (5) provides an upper bound for $|\alpha|$: $|\alpha| < (4 + \varepsilon)/4$.

We wish to find an upper bound for $|\beta|$.

To this end note that

$$\frac{\varepsilon}{8} > |(\varphi^{-1}(1))(y)| = \left|\int \varphi^{-1}(1) d\mu_y\right| = \left|\int 1 d(\varphi^{*-1}\mu_y)\right| \geqq |\beta| - ||\nu||$$

Combining this with $1 \geqq ||\varphi^{*-1}\mu_y|| = |\beta| + ||\nu||$ gives $|\beta| < (8 + \varepsilon)/16$. Now using (7) we obtain

$$\begin{aligned} \sup_{x' \in X - \{x\}} |\lim_n (\varphi(g_{y,n}))(x')| \, ||\nu|| &\geqq \left|\lim_n \int \varphi(g_{y,n}) d\nu\right| \\ &\geqq 1 - |\alpha||\beta| > (32 - 12\varepsilon - \varepsilon^2)/64 . \end{aligned} \tag{9}$$

But we have

$$(32 - 12\varepsilon - \varepsilon^2)/64 > 1/(2 + \varepsilon) > 1/2(1 + \varepsilon) , \tag{10}$$

the first inequality holding since $0 < \varepsilon < 1/6$, while the second is valid for all positive values of ε. Thus since $1 \geqq |\beta| + ||\nu||$ and $|\beta| \geqq \frac{1}{2}$ imply together that $||\nu|| \leqq \frac{1}{2}$, it follows from (9) and (10) that there exists a point $x' \in X - \{x\}$ such that $|\lim_n (\varphi(g_{y,n}))(x')| >$

$1/(1+\varepsilon)$. Hence by (I) we must have $\varphi^{*-1}\mu_y = \beta'\mu_{x'} + \nu'$, where $|\beta'| > \frac{1}{2} - \varepsilon$ and $\nu'(\{x'\}) = 0$. Thus (III) is proved.

Next, for this point $x \in X$, if $\{f_{x,n}\} \leftrightarrow x$ then (2), and a computation exactly analogous to that preceding (7), (with the $f_{x,n}$ replacing the $g_{y,n}$ and μ_x replacing μ_y), yield

$$\lim_n \int \varphi^{-1}(f_{x,n})d\mu = 1 - \alpha\beta .$$

Thus, noting that $||\mu|| = ||\varphi^*\mu_x|| - |\alpha| \leqq 1$, an argument paralleling that of (9), and an application of the first inequality in (10) provide the existence of a point $y' \in Y - \{y\}$ such that

$$|\lim_n (\varphi^{-1}(f_{x,n}))(y')| > 1/(2+\varepsilon) .$$

Hence by (2), we have

$$\varphi^{*-1}\mu_{y'} = \gamma\mu_x + \tilde{\nu}, \quad |\gamma| > 1/(2+\varepsilon), \quad \tilde{\nu}(\{x\}) = 0 . \tag{11}$$

Now write $\varphi^*\mu_x = \alpha\mu_y + \alpha'\mu_{y'} + \hat{\mu}$, where $\hat{\mu}(\{y\}) = \hat{\mu}(\{y'\}) = 0$. We show that $|\alpha'| > 1 - \varepsilon$. First note that $\tilde{\nu} = \varphi^{*-1}\mu_{y'} - \gamma\mu_x$, so that $\varphi^*\tilde{\nu} = (1 - \alpha'\gamma)\mu_{y'} - \gamma(\alpha\mu_y + \hat{\mu})$. Next observing that

$$||\alpha\mu_y + \hat{\mu}|| \geqq |\alpha| \geqq 1 ,$$

and that (by (11)) $||\tilde{\nu}|| \leqq 1 - |\gamma|$, we obtain

$$2 - 2|\gamma| \geqq ||\varphi^*\tilde{\nu}|| \geqq 1 - |\alpha'||\gamma| + |\gamma| .$$

This is equivalent to $|\alpha'||\gamma| \geqq 3|\gamma| - 1$ which, together with (11), gives $|\alpha'| > 1 - \varepsilon$. This completes the proof of (IV).

References

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Received June 12, 1969.

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