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PRODUCTS OF FINITELY ADDITIVE SET FUNCTIONS FROM ORLICZ SPACES

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This note establishes two results on products of finitely additive vector-valued set functions from Orlicz spaces. A triple (Ω, Σ, μ) is called a charge space if Σ is a ring of subsets of a set Ω and the charge μ is a finitely additive, nonnegative, finite-valued function with domain Σ.

THEOREM. **For $(\Omega_i, \Sigma_i, \mu_i)$ $(i = 1, \cdots, n)$ a family of charge spaces and (Ω, Σ, μ) the corresponding product charge space, for u an n-linear continuous operator from the product of the Banach spaces $Z_1, \cdots, Z_n$ into a Banach space W, the function v defined by $v(A) = u(v_1(A), \cdots, v_n(A))$ for $A \in \Sigma$ and v_i from the Orlicz space $A^\Phi(\Omega_i, \Sigma_i, \mu_i, Z_i)$ belongs to the Orlicz space $A^\Phi(\Omega, \Sigma, \mu, W)$.**

For the infinite product case the following result holds:

THEOREM. **For $(\Omega_t, \Sigma_t, \mu_t)$ $(t \in T)$ a family of probability charge spaces and (Ω, Σ, μ) the product probability charge space, for u an infinitely linear bounded operator on the multiplicative product space $P_T(A^\Phi(\Omega_t, \Sigma_t, \mu_t, Z_t), v_t')$ the function v^0 defined by $v^0(A) = u(v(A))$ for $A \in \Sigma$ belongs to the Orlicz space $A^\Phi(\Omega, \Sigma, \mu, W)$.**

These results allow one to develop an integral determined by a product of charges from Orlicz spaces.

In a recent paper by Uhl [5] the Orlicz space A^Φ of vector-valued finitely additive set functions is investigated. The present paper presents results concerning finite and infinite products of finitely additive vector-valued set functions from the Orlicz spaces

$$A^\Phi(\Omega_r, \Sigma_r, \mu_r, X_r)$$

for r ranging through an index set. The results for a finite product of set functions resemble a generalization of a result by Bogdanowicz [2, Th. 1] for the L_p-spaces of Lebesgue-Bochner summable functions; and the results for an infinite product of set functions resemble a generalization of a result for the Lebesgue space L_1 by Bogdanowicz and Zander [3, Proposition 5], but the techniques used for the results in the present paper are different.

We shall assume throughout that Φ is a convex, nondecreasing function defined on the positive real line such that $\Phi(0) = 0$ and Φ is continuous except for at most one point, after which the function must be identically infinite. We shall also assume throughout that the function Φ satisfies the following growth condition:

$$\Phi(xy) \leqq M\Phi(x)\Phi(y)$$

for all $x, y \geqq 0$, where M is a positive constant. (This growth condition is called the Δ'-condition (see [4, p. 29]).)

REMARK 1. Each of the functions $f(x) = x^q$ with $q \geqq 1$ and

$$g(x) = x^\alpha(\log^+ x + 1)$$

with $\alpha > 1$ satisfies the Δ'-condition with constant 1, and each is a candidate for the function Φ.

Let Ω be any set, Σ a ring of subsets of Ω, and μ a nonnegative, real-valued, finitely additive function with domain Σ. The associated triple (Ω, Σ, μ) shall be called a charge space. If, further, $\Omega \in \Sigma$ and $u(\Omega) = 1$ then the triple (Ω, Σ, μ) shall be called a probability charge space.

For (Ω, Σ, μ) a charge space and Z a Banach space denote by $A^\Phi(\mu, Z)$ the space of all finitely additive functions v from the ring Σ into the space Z which satisfy the following conditions: (i) v vanishes on the μ-null sets, (ii) $I\Phi(v/k) \leqq 1$ for some positive k, where the function I_Φ is defined by the following expression:

$$I_\Phi(v) = \sup \Sigma\{\Phi(|v(A)|/\mu(A))\mu(A) \colon A \in \mathscr{F}\}$$

where the supremum is taken over all finite families $\mathscr{F}$ of disjoint sets from the ring Σ. This definition is recently given by Uhl [5, p. 24], and is due originally to Bochner [1, p. 778]. By Theorem 11 of [5] the functional N_Φ defined by

$$N_\Phi(v) = \inf\{k > 0 \colon I_\Phi(v/k) \leqq 1\}$$

is a norm under which the space $A^\Phi(\mu, Z)$ is Banach space.

We shall use the following notation: Let T_i be a family of subsets from an abstract set X_i, for $i = 1, \cdots, n$. By $T_1 \times \cdots \times T_n$ we shall mean the family of all sets of the form $A_1 \times \cdots \times A_n$ where $A_i \in T_i$ for $i = 1, \cdots, n$.

For charge spaces $(\Omega_i, \Sigma_i, \mu_i)$ $(i = 1, \cdots, n)$ define the triple (Ω, Σ_0, μ) by $\Omega = \Omega_1 \times \cdots \times \Omega_n$, $\Sigma = \Sigma_1 \times \cdots \times \Sigma_n$, Σ_0 is the ring generated from the prering Σ, and $\mu(A) = \mu_1(A_1) \cdots \mu_n(A_n)$ for $A \in \Sigma$. It is easy to see that (Ω, Σ_0, μ) is a charge space.

DEFINITION. For $\mathscr{F}$ a finite family of disjoint sets from the prering Σ and for i between 1 and n let $\mathscr{F}_i$ denote the family of all i-th coordinate sets A_i, where $A_1 \times \cdots \times A_i \times \cdots \times A_n \in \mathscr{F}$, and let

$\mathscr{F}_i^r$ denote the refinement of $\mathscr{F}_i$ consisting of the finite family of disjoint sets from Σ_i whose union is the same as that of all sets from $\mathscr{F}_i$. That is, if $\mathscr{F}_i = \{B_1, \cdots, B_m\}$, and if $B_j' = \Omega_i \backslash B_j$ then $\mathscr{F}_i^r$ is the set of all intersections $C_1 \cap \cdots \cap C_m$ where each C_j is B_j or B_j' and at least one C_j is B_j. By the $\mathscr{F}$-product we shall mean the corresponding finite family of disjoint sets $\mathscr{F}_1^r \times \cdots \times \mathscr{F}_n^r$ from the prering Σ. We shall denote this family by $(\mathscr{F})$.

LEMMA 1. *For $v \in A^\Phi(\mu, Z)$ we have the relation*

$$I_\Phi(v) = \sup \Sigma\{\Phi(|v(A)|/\mu(A))\mu(A): A \in (\mathscr{F})\}$$

where the supremum is taken over all finite families $\mathscr{F}$ of disjoint sets from Σ.

Proof. Let a denote the right side of the above relation. Then $I_\Phi(v) \geqq a$ since an $\mathscr{F}$-product is a particular finite disjoint family from Σ. Let B, C be disjoint sets from Σ, and put $A = B \cup C$. Then the relation $I_\Phi(v) \leqq a$ follows from the positivity of values of the function Φ and from the relation

$$\Phi(|v(A)|/\mu(A))\mu(A) \leqq \Phi(|v(\beta)|/\mu(B))\mu(B) + \Phi(|v(C)|/\mu(C))\mu(C)$$

which in turn follows from the convexity of the function Φ.

Let Z_i, W $(i = 1, \cdots, n)$ be Banach spaces and let u be an n-linear continuous operator from the product of the spaces $Z_1, \cdots, Z_n$ into the space W. Denote the norms in the above spaces by $|\ |$.

THEOREM 2. *If $v_i \in A^\Phi(\mu_i, Z_i)$ for $i = 1, \cdots, n$ then $v \in A^\Phi(\mu, W)$, where*

$$v(A_1 \times \cdots \times A_n) = u(v_1(A_1), \cdots, v_n(A_n)) \textit{ for } A \in \Sigma\ .$$

Proof. We shall establish that $I_\Phi(v/k) \leqq 1$ for some positive k. Assume throughout that the Δ'-constant $M \geqq 1$. Put

$$w_j(\cdot) = |v_j(\cdot)|/N_\Phi(v_j)$$

and $b = M^n|u|N_\Phi(v_1) \cdots N_\Phi(v_n)$, and let $\mathscr{F}$ be a finite family of disjoint sets from Σ. Then we have the following estimates:

$$\begin{aligned}
&\Sigma\{\Phi(|v(A)|/b\mu(A))\mu(A): A \in (\mathscr{F})\}\\
&\leqq \Sigma\left\{\Phi\Big(\prod_{k=1}^n w_k(A_k)/M\mu_k(A_k)\Big)\mu(A): A \in (\mathscr{F})\right\}\\
&\leqq \Sigma\left\{\Big[\prod_{k=1}^n M\Phi(w_k(A_k)/M\mu_k(A_k))\Big]\mu(A): A \in (\mathscr{F})\right\}.
\end{aligned}$$

Noting that $\Phi(a/M) \leqq \Phi(a)/M$ for $a \geqq 0$ since Φ is convex and $M \geqq 1$, we conclude that

$$\begin{aligned} &\Sigma\{\Phi(|v(A)|/b\mu(A))\mu(A): A \in (\mathscr{F})\} \\ &\leqq \prod_{k=1}^{n} \{\Sigma\Phi(w_k(A)/\mu_k(A))\mu_k(A): A \in \mathscr{F}_k{}^r\} \\ &\leqq I_\Phi(w_1) \cdots I_\Phi(w_n) \leqq 1 . \end{aligned}$$

By taking the appropriate supremum, we see that $I_\Phi(v/b) \leqq 1$. From this it can easily be seen that $v \in A^\Phi(\mu, W)$.

We shall now consider infinite products of vector-valued finitely additive set functions.

Let T be an arbitrary index set, let $(\Omega_t, \Sigma_t, \mu_t)$ $(t \in T)$ be a family of probability charge spaces and let (Ω, Σ, μ) be the product space (i.e., $\Omega = X_T\Omega_t$, Σ is the family of all finite disjoint unions of sets of the form $A = X_TA_t$ where $A_t \in \Sigma_t$ and $A_t \neq \Omega_t$ for at most a finite number of indices $t \in T$, and $\mu(A) = \Pi_T\mu_t(A_t)$ for all $A \in \Sigma$.) It is clear that the triple (Ω, Σ, μ) is a probability charge space.

Let Y, Z_t $(t \in T)$, W be Banach spaces, let $z'_t \in Z_t$ each be of unit norm, and let $Z' = P_T(Z_t, z'_t)$ be the corresponding multiplicative product space. That is, $P_T(Z_t, z'_t)$ is the set of all $z = (z_t)_T$ such that $z_t \in Z_t$ for $t \in T$ and $\Sigma_T|z_t - z'_t| < \infty$. Define a functional d by

$$d(y, z) = \Sigma_T|y_t - z_t|$$

for all $y, z \in Z'$. In [3] it is shown that the space (Z', d) is a complete metric space. It is easy to see that the space Z' is also an affine subspace of the linear space X_TZ_t, and if T is uncountable the metric d cannot be extended to a metric on the space X_TZ_t.

An operator u mapping the space Z' into the space W shall be called infinitely linear and bounded if u is linear on each coordinate space Z_t separately, for all $t \in T$, and if there is a positive constant c such that $|u(z)| \leqq c\Pi_T|z_t|$ for all $z \in Z'$. Let $||u||$ denote the smallest of all such constants. Let $L(Z'; W)$ denote the space of all such bounded infinitely linear operators. In [3] it is shown that the space $L(Z'; W)$ under the functional $||\ ||$ is a Banach space.

In order to make the next definition we must further restrict the function Φ so that $\Phi(1) = 1$ and the Δ'-constant $M = 1$. These conditions are the results of applying a version of Lemma 13 of [5], modified to assume the Δ'-condition. Since the functions g and f from Remark 1 are nontrivial examples of functions which satisfy these conditions, the additional conditions are not as restrictive as they originally appeared to be.

Select $v'_t \in A^\Phi(\mu_t, Z_t)$ such that $v'_t(\Omega_t) = z'_t$ and $N_\Phi(v'_t) = 1$, and let

$P_T(A^\Phi(\mu_t, Z_t), v'_t)$ be the corresponding multiplicative product space. We remark that from Lemma 13 of [5] it follows that $v(\Omega) \in Z'$ for all $v \in P_T(A^\Phi(\mu_t, Z_t), v'_t)$.

For the infinite product ring Σ we need some notation similar to the $\mathscr{F}$-product concept for the finite case. With this in mind let $\mathscr{F}$ be a finite family of disjoint sets from the infinite product ring Σ, let the families $\mathscr{F}_t$ and $\mathscr{F}_t{}^r$ be defined as in the finite product case, and define the $\mathscr{F}$-product as the corresponding finite family of disjoint sets

$$X_T \mathscr{F}_t{}^r = \{A = X_T A_t \colon A_t \in \mathscr{F}_t{}^r \text{ for all } t \in T\}$$

from the ring Σ. Again we shall denote the $\mathscr{F}$-product by $(\mathscr{F})$. (It follows from the definition of Σ and properties of a ring that $(\mathscr{F})$ consists of a finite number of disjoint sets from Σ.)

THEOREM 3. *If $v \in P_T(A^\Phi(\mu_t, Z_t), v'_t)$ and $u \in L(Z'; W)$ then the function v^0 defined by $v^0(A) = u(v(A))$ for all $A \in \Sigma$ belongs to the space $A^\Phi(\mu, W)$.*

Proof. It is clear that the function v^0 is finitely additive on Σ, and from the estimate $|v^0(A)| \leqq ||u|| \Pi_T |v_t(A_t)|$ for all $A \in \Sigma$ it is clear that $v(A) = 0$ when $\mu(A) = 0$. To establish that $I_\Phi(v/k) \leqq 1$ for some constant k, put $w_t(\cdot) = |v_t(\cdot)|/N_\Phi(v_t)$ and $b = ||u|| \Pi_T N_\Phi(v_t)$ and let $\mathscr{F}$ be a finite family of disjoint sets from Σ. Applying Lemma 1, which is clearly valid here, we get the following estimates:

$$\begin{aligned}
&\Sigma\{\Phi(|v^0(A)|/b\mu(A))\mu(A) \colon A \in (\mathscr{F})\} \\
&\leqq \Sigma\{\Phi(\Pi_{S'} w_t(\Omega_t) \Pi_S (w_t(A_t)/\mu_t(A_t)))\mu(A) \colon A \in (\mathscr{F}) \\
&\leqq \Sigma\{\Pi_S \Phi(w_t(A_t)/\mu_t(A_t))\mu(A) \colon A \in (\mathscr{F})\} \\
&\leqq \Pi_S\{\Sigma\{\Phi(w_t(A)/\mu_t(A))\mu_t(A) \colon A \in \mathscr{F}_t{}^r\}\} \\
&\leqq \Pi_S I_\Phi(w_t) \leqq 1 .
\end{aligned}$$

Hence $v^0 \in A^\Phi(\mu, W)$ and $N_\Phi(v^0) \leqq ||u|| \Pi_T N_\Phi(v_t)$.

REMARK 2. The results in this note remain valid if all rings are relaxed to prerings, with no changes needed in the above proofs to establish this.

REMARK 3. Forthcoming papers will discuss the other formulations of the Fubini theorems for Orlicz spaces in terms of a countably additive nonnegative finite valued set function (called a volume) defined on a prering. The prering, volume combination appears to be the most natural context in which to discuss the Fubini theorems; that it is also a valid context follows from the work of Bogdanowicz, which develops an integration theory based upon the prering and volume. See [2] for a suitable bibliography.

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