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THE SUSLIN-KLEENE THEOREM FOR V_κ WITH COFINALITY(κ) = ω

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It is easy to extend to arbitrary structures $\mathfrak{A} = \langle A, R_1, \cdots, R_l f_1, \cdots, f_m\rangle$ the concepts of $\prod_1^1$ and inductively definable relations, which are familiar for the structure of the integers. The second author showed in a recent paper that these two concepts coincide for countable $\mathfrak{A}$ that satisfy certain mild definability conditions—this is a generalization of the classical Suslin-Kleene theorem. Here we generalize the Suslin-Kleene theorem in a different direction.

MAIN RESULT. Let V_κ be the set of sets of rank less than κ, i.e., $V_0 = \phi$, $V_{\xi+1}$ = power of V_ξ, $V_\kappa = \bigcup_{\xi<\kappa} V_\xi$, if κ is limit. The classes of inductively definable and $\prod_1^1$ relations on the structure $\mathscr{V}_\kappa = \langle V_\kappa, \in \restriction V_\kappa\rangle$ ($\kappa \geq \omega$) coincide if and only if κ is a limit ordinal with cofinality ω.

This implies several corollaries about the class of $\prod_1^1$ relations on V_κ, when cofinality(κ) = ω, e.g., that it has the reduction property.

The nontrivial part of the theorem is the implication $\prod_1^1 \Rightarrow$ *inductively definable* for $\mathscr{V}_\kappa$ with *cofinality*(κ) = ω.

1. Proof of the main result. We assume familiarity with [7], whose notation we shall use.

Notice first that for each $\kappa \geq \omega$, $\mathscr{V}_\kappa$ is an *acceptable* structure, in the sense of [7]. This is immediate for limit κ, by taking the ordinary set-theoretic pair and the standard ω for the integers within V_κ. For successor κ the proof is by induction; let $\kappa = \lambda + 1$, let $(\ ,\ )_\lambda$ be a definable pair in $\mathscr{V}_\lambda$, for $x_1, \cdots, x_n$ in V_κ put

$$\langle x_1, \cdots, x_n\rangle = \{(1, u)_\lambda : u \in x_1\} \cup \cdots \cup \{(n, u)_\lambda : u \in x_n\} .$$

These n-tuple functions are definable in $\mathscr{V}_\kappa$ and using them one can easily define a pair for $\mathscr{V}_\kappa$ and also show that first-order definability on $\mathscr{V}_\kappa$ is preserved under inductive definitions.

Since $\mathscr{V}_\kappa$ is acceptable, the inductively definable relations on $\mathscr{V}_\kappa$ are $\prod_1^1$ by the argument given in §3 of [7]. Also, if $\kappa \geq \omega$ and κ is a successor or cofinality(κ) > ω, then the relation

$$S \in WF \Leftrightarrow \textit{there is no sequence } u_0, u_1, \cdots, \textit{ so that}$$
$$(n)[(u_n, u_{n+1}) \in S]$$

is first-order definable on $\mathscr{V}_\kappa$, so that by the usual analysis of transfinite inductions "from within", each inductively definable relation is $\sum_1^1$, and hence these relations do not exhaust $\prod_1^1$. To complete the proof we must show that if cofinality $(\kappa) = \omega$, then each $\prod_1^1$ relation on $\mathscr{V}_\kappa$ is inductively definable.

Let $P(x) \Leftrightarrow (\alpha)(Ey)Q(\alpha, y, x)$ be a typical $\prod_1^1$ relation, where $Q(\alpha, y, x)$ is defined by the simple, quantifier-free formula $Q(\alpha, y, x)$ of $\mathfrak{L}^2$, let $\mathrm{t}_1(x, y), \cdots, \mathrm{t}_d(x, y)$ be the finitely many terms s such that the term $\alpha(\mathrm{s})$ occurs in $Q(\alpha, y, x)$, let $t_1(x, y), \cdots, t_d(x, y)$ be the functions on V_κ that these terms define, choose $Q^*(z, y, x)$ as in §4 of [7] so that

$$(1) \qquad Q(\alpha, y, x) \Leftrightarrow Q^*(z, y, x)$$

whenever

$$(2) \qquad \mathrm{Seq}\,(z)\ \&\ K(z) = d\ \&\ (i)_{1 \leq i \leq d}[(z)_i = \alpha(t_i(x, y))]\,.$$

We shall define for each $x \in V_\kappa$ a game $\mathscr{G}(x)$ so that when cofinality $(\kappa) = \omega$,

$$P(x) \Leftrightarrow \mathrm{I}\ \textit{has a winning strategy in}\ \mathscr{G}(x)\,.$$

In the game $\mathscr{G}(x)$, player I chooses $a_1 (a_1 \in V_\kappa)$, player II chooses a pair $b_1, c_1 (b_1, c_1 \in V_\kappa)$, then I chooses a_2, then II chooses b_2, c_2, etc. We say that *the outlook is good for player* II *at step* k, when $a_1 \cdots, a_k, b_1, c_1, \cdots, b_k, c_k$ have been played, if the following conditions are satisfied.

(i) For each $i \leq k$, b_i is a function with domain ω and range power (a_i), so that

$$\bigcup_{j \in \omega} b_i(j) = a_i\,.$$

(Thus II decomposes a_i into an ω-sequence of sets.)

(ii) c_k is a function with domain $\{(i, j): i, j \leq k\}$ which assigns to each pair (i, j) a function $f^k_{i,j}$ with domain $(f^k_{i,j}) = b_i(j)$.

(iii) The union

$$f^k = \bigcup_{i,j \leq k} f^k_{i,j}$$

is a function.

(iv) There is no element $y \in V_\kappa$ such that all $t_1(x, y), \cdots, t_d(x, y)$ are in the domain of f^k and such that (1) holds when we choose z so that (2) holds with $f^k(t_i(x, y))$ substituted for $\alpha(t_i(x, y))$, $i = 1, \cdots, d$.

At the end of the game, player II wins if the outlook is good for him at every step k, otherwise player I wins.

LEMMA 1. *If cofinality* $(\kappa) = \omega$ *and player* I *has a winning strategy in* $\mathscr{G}(x)$, *then* $(\alpha)(Ey)Q(\alpha, y, x)$.

Proof. Given a function α on V_κ to V_κ, consider the game where I plays $a_1, a_2, \cdots$ following his winning strategy and II plays as follows. Since cofinality $(\kappa) = \omega$, we can choose a countable sequence $v_1, v_2, \cdots$ of elements of V_κ such that

$$V_\kappa = \bigcup_{j \in \omega} v_j .$$

For each k, the function $\alpha \restriction a_k$ is a subset of V_κ and it can be decomposed into a countable union of subfunctions which are elements of V_κ,

$$\alpha \restriction a_k = \bigcup_{j \in \omega} ((\alpha \restriction a_k) \cap v_j) .$$

At step k, II chooses a b_k so that

$$b_k(j) = \text{domain}\,((\alpha \restriction a_k) \cap v_j) \qquad (j \leqq k)$$

and a c_k so that

$$f^k_{i,j} = (a \restriction \alpha_i) \cap v_j \qquad (i, j, \leq k) .$$

It is now clear that at each k, conditions (i), (ii), (iii) above are satisfied. Since I wins the game, there must be a k at which condition (iv) fails. For that k we have $Q^*(z, y, x)$ for some y and some z that codes a subfunction of α, so that by (1) we have $Q(\alpha, y, x)$ and the proof is complete.

LEMMA 2. *If cofinality* $(\kappa) = \omega$ *and* $(\alpha)(Ey)Q(\alpha, y, x)$, *then* I *has a winning strategy in* $\mathscr{G}(x)$.

Proof. Let I simply play $a_k = v_k$, where the v_j are elements of V_κ such that $V_\kappa = \bigcup_{j \in \omega} v_j$. Any winning sequence of plays for II determines a completely defined function α on V_κ to V_κ, so by hypothesis there is some y, so that $Q(\alpha, y, x)$. Now $y \in a_i$, for some i, and for large enough j, all $t_1(x, y), \cdots, t_d(x, y)$ must be elements of $b_i(1) \cup \cdots \cup b_i(j)$. It is then clear that the outlook is not good for II at step $k = \max(i, j)$, since condition (iv) will fail at that k.

Proof of the main result from these two lemmas is just like the proof in §5 of [7] and we shall omit it. The key points are that the game $\mathscr{G}(x)$ is *open* (i.e., if I wins, then he knows it at some point k of the game) and that conditions (i)-(iv) are first-order definable on $\mathscr{V}_\kappa$.

The result can be easily relativized to relations on *functions* on V_κ to V_κ as in §6 of [7]. One can also imitate the argument of §7 of [7] to show that the result cannot be proved by the classical method of representing $\prod^1_1$ relations via the property of well-foundedness.

We start with some λ with cofinality $(\lambda) > \omega$ and then use the Montague-Vaught method of [4] to find a $\kappa < \lambda$, with cofinality $(\kappa) = \omega$ and such that for some $C' \subseteq$ power (V_λ), the structure $\langle V_\kappa, C', \varepsilon \rangle$ is an elementary substructure of $\langle V_\lambda, \text{power}(V_\lambda), \varepsilon \rangle$. It is then easy to show that some $\prod_1^1$ relation $P(x)$ on $\mathscr{V}_\kappa$ is not of the form

$$P(x) \Leftrightarrow \lambda uv\, Q(x, u, v) \quad \textit{is well-founded}\,,$$

with first-order definable $Q(x, u, v)$.

2. Corollaries and comments. Let Γ be a class of relations on some acceptable structure $\mathfrak{A}$. We say that Γ is *parametrized* if there is a binary relation $G(z, x)$ in Γ, so that each unary relation $P(x)$ in Γ is of the form

$$P(x) \Leftrightarrow G(z_0, x)$$

for some fixed z_0 in the domain of the structure. It is easy to verify that the classes of $\prod_1^1$, $\sum_1^1$ and inductively definable relations on an acceptable structure are parametrized.

Suppose Γ is parametrized by $G(z, x)$. Put

Prewellordering $(\Gamma) \Leftrightarrow$ *there is a function Ψ on (the extension of) G into some ordinal κ and relations $\underset{\bullet}{\leq}$ and $\overset{\bullet}{\leq}$ in Γ and $\neg\Gamma$ (= the class of negations of relations in Γ) respectively, so that*

$$\begin{aligned} G(z, x) \Rightarrow (u)(v)[(u, v) \overset{\bullet}{\leq} (z, x) &\Leftrightarrow (u, v) \underset{\bullet}{\leq} (z, x) \\ &\Leftrightarrow [G(u, v) \;\&\; \Psi(u, v) \leq \Psi(z, x)]]\,. \end{aligned}$$

It is well-known that if Γ satisfies reasonable closure conditions, then *Prewellordering* (Γ) implies that Γ satisfies many interesting structure properties -e.g., see [5], [3], [6]. One of them is

Reduction (Γ). Given relations $P(x)$, $Q(x)$ in Γ, there exist relations $P_1(x)$, $Q_1(x)$ in Γ such that

$$\begin{aligned} &P_1(x) \Rightarrow P(x),\ Q_1(x) \Rightarrow Q(x)\,, \\ &P(x) \vee Q(x) \Rightarrow P_1(x) \vee Q_1(x)\,, \\ &(x)\ \neg\,[P_1(x) \;\&\; Q_1(x)]\,. \end{aligned}$$

Others include the existence of a hierarchy on $\Gamma \cap \neg\Gamma$, where Γ now must satisfy fairly strong closure properties.

Our main result here together with the results in [5] gives

$$\textit{cofinality}\ (\kappa) = \omega \Rightarrow \textit{Prewellordering}\ (\textstyle\prod_1^1(\mathscr{V}_\kappa))\,,$$

where $\prod_1^1(\mathscr{V}_\kappa)$ is the class of $\prod_1^1$ relations on $\mathscr{V}_\kappa$. Since $\prod_1^1(\mathscr{V}_\kappa)$

satisfies all the required closure properties, this further gives *Reduction* $(\prod_1^1(\mathscr{Y}_\kappa))$ and the existence of a hierarchy on $\Delta_1^1(\mathscr{Y}_\kappa) = \prod_1^1(\mathscr{Y}_\kappa) \cap \sum_1^1(\mathscr{Y}_\kappa)$.

The classical arguments of Gödel and Addison [1], [2] suffice to show

$$[\textit{Axiom of Constructibility} \ \& \ \kappa \textit{ a successor or } \textit{cofinality}\,(\kappa) > \omega] \Rightarrow \textit{Prewellordering}\,(\textstyle\sum_1^1(\mathscr{Y}_\kappa)).$$

However we do not know how to settle *Prewellordering* $(\prod_1^1(\mathscr{Y}_\kappa))$ or *Prewellordering* $(\sum_1^1(\mathscr{Y}_\kappa))$ when κ is a successor or cofinality $(\kappa) > \omega$ in Zermelo-Fraenkel set theory or in extensions of that theory by strong axioms which do not restrict our conception of arbitrary set. The problem has been attacked without success by some people for the case $\kappa = \omega + 1$, corresponding to the class of $\prod_1^2$ or $\sum_1^2$ relations on the continuum in type-theoretic notation. We suspect that it may be easier to settle for limit κ with cofinality $(\kappa) > \omega$, perhaps for κ satisfying strong axioms of infinity. An optimist would hope that for each κ, one of $\prod_1^1(\mathscr{Y}_\kappa)$ or $\sum_1^1(\mathscr{Y}_\kappa)$ must satisfy the prewellordering property.

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