LOCAL RINGS WITH NOETHERIAN FILTRATIONS

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The purpose of this paper is to investigate a certain type
of complete filtered local rings, and to generalize well-known
theorems in literature which are related to the structure
theorem of complete local rings.

All rings in this article are commutative with identity element,
and all subrings possess the identity element of the containing ring.
A ring A is a local ring if it has a unique maximal ideal m, sym-
bolically we write (A, m), and by the natural topology of (A, m) we
mean the m-adic topology.

A filtration $\{q_n\}_{n=0}^{\infty}$ of a ring R is called a noetherian filtration if
it is separated and R/q_n is a noetherian ring for all n. Motivated
by I. S. Cohen's structure theorem of complete noetherian local rings
with respect to the natural topologies ([5]), extension of two theo-
rems on noetherian local rings is considered. Harrison-Kolman's
theorem [6, Th. A, Th. 1], which is annalogous to Cohen's theorem,
is generalized as follows: If R is a local ring which is complete with
respect to a noetherian filtration $\{q_n\}_{n=0}^{\infty}$ such that (i) R is equichar-
acteristic and (ii) R/q_1 is regular, then there exists a subring B
such that $B + q_1 = R$ and $B \cap q_1 = (0)$; consequently, R is a homomor-
phic image of a ring of formal power series in a finite or countably
infinite number of indeterminates over B. We also prove that if R
is a complete (resp. compact) local ring with respect to a noetherian
filtration $\{q_n\}_{n=0}^{\infty}$ such that q_1 is its maximal ideal, then R is a homo-
morphic image of a ring $k[[x_i]]_{i \in I}$ of formal power series in a finite
or countably infinite number of indeterminates over a complete (resp.
compact) noetherian local subring k for the natural topology; con-
versely, such power series ring possesses a noetherian filtration and
$k[[x_i]]_{i \in I}/D$, for any closed ideal D of $k[[x_i]]_{i \in I}$ relative to the filtra-
tion topology, is a complete (resp. compact) local ring with respect
to a noetherian filtration. This is an extension of Cohen-Warner's
theorem ([5, Th. 9, 12]; [11, Th. 14]). Finally, we apply the result
to prove that a certain class of local rings are unique factorization
domains.

1. **Natural topologies of power series rings.** Let N be the set
of all positive integers, and $N_0 = N \cup \{0\}$. Throughout the paper, we
shall denote the power series ring $R[[x_1, x_2, \cdots, x_t]]$ in t indetermi-
nates over a ring R by R_t for each $t \in N_0$, identifying R_0 with R.
Also the power series ring $R[[x_1, x_2, \cdots]] = R[[x_i]]_{i \in N}$ in a countably

infinite number of indeterminates over R will be denoted by R_ω. Let $f = f(x_1, x_2, x_3, \cdots)$ be an element of R_ω, then the power series $f(x_1, x_2, \cdots, x_t, 0, 0, 0, \cdots)$ in R_t is called the projection of f on R_t and is denoted by $(f)_t$. For any $t \in N_0$ the projection map $P_t: f \to (f)_t$ is a ring homomorphism of R_ω onto R_t.

Let R be a ring with a filtration, that is a descending sequence of ideals $R = q_0 \supseteq q_1 \supseteq \cdots \supseteq q_n \supseteq \cdots$ such that $q_i q_j \subseteq q_{i+j}$. If $\bigcap_{n \in N_0} q_n = (0)$, then R is separated for the topology induced by the filtration; furthermore, we can define an order function v on R such that

$$v(a) = \min \{n \mid a \notin q_{n+1}\} \qquad \text{if } 0 \neq a \in R$$

$$= \infty \qquad \text{if and only if } a = 0 .$$

The sets R_t, $t \in N_0$, and R_ω of formal power series over R are the sets R^{Nt} and $R^{N^{(N)}}$ respectively. The (cartesian) product topology of R_t and that of R_ω are compatible with the ring structure of the power series rings. We shall call it the *natural topology induced by the filtration* $\{q_n\}$ of R, or the (q_n)-*natural topology*. If $q_n = m^n$, $n \in N_0$, for some ideal m of R, then the induced natural topology on a power series ring is simply called the *m-natural topology*.

Let $\{n_i\}$ be any sequence of positive integers n_i such that $n_i \to \infty$ as $i \to \infty$, and $A_r = \{f \in R_\omega \mid$ each nonzero monomial $bx_1^{\lambda_1} x_2^{\lambda_2} \cdots x_t^{\lambda_t}$ of f satisfies that $v(b) + \sum_{1 \leq i \leq t} n_i \lambda_i \geq r\}$ for each $r \in N_0$. Then $\{A_r\}_{r \in N_0}$ forms a filtration of R_ω and therefore induces a linear topology on R_ω. We note that the linear topology is independent of the choice of the sequence $\{n_i\}$ of positive integers n_i such that $n_i \to \infty$ as $i \to \infty$. In particular, we take the sequence $\{n_i\}$ such that $n_i = i$ for every i, and for each $r \in N_0$ let $B_r = \{f \in R_\omega \mid$ each nonzero monomial $bx_1^{\lambda_1} x_2^{\lambda_2} \cdots x_t^{\lambda_t}$ of f satisfies that $v(b) + \sum_{i \leq i \leq t} i \lambda_i \geq r\}$. $\cdots$ (*). Then it is not difficult to verify that the natural topology of R_ω induced by the filtration of R is equivalent to the (B_r)-topology; hence to the (A_r)-topology for any sequence $\{n_i\}$ of positive integers n_i tending to ∞. Consequently, the (q_n)-natural topology of each R_t could be defined by the basis of neighborhoods $\{A_{t, r}\}$ of 0, where $A_{t, r} = R_t \cap A_r$ for each $r \in N_0$. If m is an ideal of R such that $\bigcap_{n \in N_0} m^n = (0)$, then the m-natural topology of R_t is equivalent to the $(m, x_1, x_2, \cdots, x_t)$-adic topology, because the latter is obviously the $(A_{t, r})$-topology with respect to any sequence $\{n_i\}$ of positive integers n_i tending to ∞ such that $n_1 = n_2 = \cdots = n_t = 1$. In addition, if (R, m) is a local ring, then the m-natural topology of R_t is identical with the natural topology of the local ring (R_t, M), where $M = (m, x_1, x_2, \cdots, x_t)$.

Since each R_t can be canonically identified with a subset $\prod_{\alpha \in N^{(N)}} S_\alpha$

of R_ω, wehre $S_\alpha = R$ for some α's and $S_\alpha = (0)$ for all other α, the following proposition is trivially true.

PROPOSITION 1. *If R is a ring with a separated filtration $\{q_n\}$, then every R_t, $t \in N_0$, is closed in R_ω which is equipped with the (q_n)-natural topology.*

THEOREM 1. *Let R be a ring with a separated filtration $\{q_n\}$, and let R_ω and each R_t be equipped with the (q_n)-natural topologies respectively. Then the following statements are equivalent: (1) R is compact [a Hausdorff space, complete, linearly compact (cf. [4]), precompact]. (2) R_ω is compact [the same nature as (1)]. (3) R_t is compact [the same nature as (1)] for all $t \geq 1$.*

Proof. The theorem follows from Proposition 1 and well known properties of products of topological rings (cf. [2], [3], [4]).

2. Local rings with noetherian filtrations.

In this section we assume that every filtration $\{q_n\}$ of rings R is separated.

Let $\mathrm{gr}(R)$ be the associated graded ring of a filtered ring R with respect to a filtration $\{q_n\}$, k a subring of R, ρ the natural homomorphism of R to $\mathrm{gr}_0(R) = q_0/q_1$ and σ the restriction of ρ to k. Then $\mathrm{gr}(R)$ has a structure of k-algebra if the product $c \cdot x$ for each $c \in k$ and $x \in \mathrm{gr}(R)$ is defined as $\sigma(c)x$. Given $0 \neq a \in R$, the class of a in $\mathrm{gr}_i(R)$ where $i = v(a)$ is called the initial form of a and is denoted by a^*.

PROPOSITION 2. *Let R be a ring with a filtration $\{q_n\}$, k its subring and $I = \{1, 2, \cdots, j\}$ for some $j \in N$(resp. $I = N$). Let $S = \{a_i\}_{i \in I}$ be a set of elements of R such that $v(a_i) = n_i > 0$ for each i (resp. $v(a_i) = n_i > 0$ and $n_i \to \infty$ as $i \to \infty$). (1) If R is complete and $S \cap k$ has at most one element, then there exists a unique k-homomorphism u of the k-algebra k_j(resp. k_ω) into R such that $u(x_i) = a_i$ for every $i \in I$, which is continuous for the $(k \cap q_n)$-natural topology of k_j(resp. k_ω). (2) If the family $\{a_i^*\}_{i \in I}$ of the initial forms of a_i's is a system of generators of the k-algebra $\mathrm{gr}(R)$, then the homomorphism u in (1) is suvjective.*

Proof. First we assume that $I = N$. For the sequence of positive integers $\{n_i\}$ and the filtration $\{q_n'\}$ of k, where $q_n' = (0)$ for every n, let A_r for each $r \in N_0$ be the ideal of k_ω which was defined in §1. Then k_ω is separated and complete for the (A_r)-topology, namely, the (0)-natural topology by Theorem 1 and has $\bar{k} = U_{t \in N} k[x_1, x_2, \cdots, x_t]$

as a dense subset. Regarding $\bar{k}$ as a subspace of k_ω, let η be the k-homomorphism of the k-algebra $\bar{k}$ to R such that $\eta(x_i) = a_i$ for every i, then η is a continuous ring homomorphism and can be extended uniquely to a continuous homomorphism u of k_ω to R with respect to the (0)-natural topology of k_ω. The proofs of this and (2) are completely analogous to that of Proposition 11 of [4, p. 40]; hence we omit them. Note that if $S \cap k = \{a_1\}$, then $u: k_\omega \to R$ is actually a homomorphism of $k[[x_2, x_3, x_4, \cdots]]$ to R. The continuity of u for the $(k \cap q_n)$-natural topology of k_ω can be proved very easily by applying the following fact: If $v(b) + \sum_{1 \le i \le t} n_i \lambda_i \ge r$, then

$$u(bx_1^{\lambda_1} x_2^{\lambda_2} \cdots x_t^{\lambda_t}) = ba_1^{\lambda_1} a_2^{\lambda_2} \cdots a_t^{\lambda_t}$$

and

$$v(ba_1^{\lambda_1} a_2^{\lambda_2} \cdots a_t^{\lambda_t}) \ge v(b) + \sum_{1 \le i \le t} n_i \lambda_i \ge r \ .$$

When $I = \{1, 2, \cdots, j\}$ the proof of the proposition is similar to that of the above case.

DEFINITION 1. A filtration $\{q_n\}_{n \in N_0}$ of a ring R is called a *noetherian filtration* if $\bigcap_{n \in N_0} q_n = (0)$ and R/q_n is a noetherian ring for all n.

If R is a noether ring with a filtration $\{q_n\}$, then the induced filtration $\{B_n\}$ on R_ω, where B_n is defined by (*) in § 1, is a noetherian filtration. For if we let P_n, $n \in N_0$, be the projection map of R_ω to R_n such that $P_n(f) = (f)_n$ for each $f \in R_\omega$ and η_n the natural homomorphism of R_n to $R_n/(R_n \cap B_{n+1})$, then $\ker(\eta_n \circ P_n) = B_{n+1}$; therefore $R_\omega/B_{n+1} \cong R_n/(R_n \cap B_{n+1})$, which is a noetherian ring for every n. Moreover $\bigcap_{n \in N_0} B_n = (0)$ by Theorem 1.

DEFINITION 2. A set $\{a_i\}_{i \in I}$ of elements of a filtered ring R is called a *formal system of generators for R*, if the set $\{a_i^*\}_{i \in I}$ of the initial forms of a_i's forms a minimal basis for the homogeneous ideal $\bigoplus_{i \ge 1} \mathrm{gr}_i(R)$ of $\mathrm{gr}(R)$.

Let $\{a_i\}_{i \in I}$ be a formal system of generators for a ring R with a filtration $\{q_n\}$. According to Proposition 1 of [4, p. 9], the set $\{a_i^*\}_{i \in I}$ is also a minimal set of generators for the $\mathrm{gr}_0(R)$-algebra $\mathrm{gr}(R)$. If k is a subring of R such that $k + q_1 = R$, then the above conclusion is also true for $\mathrm{gr}(R)$ as a k-algebra.

We remark that if one formal system of generators for R is of the finite type (resp. countably infinite type), then so are all other formal systems of generators by virtue of the minimal condition imposed on the set of the initial forms of generators.

The following proposition is easy to see.

PROPOSITION 3. *Let R be a ring with a noetherian filtration. Then* (1) *R satisfies the ascending chain condition for open ideals,* (2) *a homomorphic image and the completion of R possess noetherian filtrations respectively, and* (3) *if R is complete and possesses a formal system of generators of the finite type, then R is noetherian* (cf. [4, p. 42, Corollary 2 to Proposition 12]; [4, p. 45, Corollary 4 to Theorem 2]).

LEMMA. *Every ring R with a noetherian filtration $\{q_n\}$ possesses a formal system of generators, and each formal system $\{a_i\}_{i \in I}$ consists of a countable (finite or infinite) number of elements such that $v(a_i) \to \infty$ as $i \to \infty$ when $I = N$.*

Proof. The first part is evident from [12, p. 151, Th. 7]. Since each finite basis for the ideal $gr_i(R)$ of the noetherian ring R/q_{i+1} is also a set of generators for $\mathrm{gr}_i(R)$ as a $\mathrm{gr}_0(R)$-module, $\bigoplus_{i \geq 1} \mathrm{gr}_i(R)$ is a module generated by a countable number of homogeneous elements over $\mathrm{gr}_0(R)$; hence over $\mathrm{gr}(R)$. This yields that all other formal systems of generators for R must be countable as we remarked previously. To verify the remaining part of the lemma, assume the contrary and let $S = \{a_i\}_{i \in N}$ be a formal system of generators such that $v(a_i) \neq \infty$ as $i \to \infty$, then there exists a $t \in N_0$ and an infinite subset $\{a_{j_i} \in S \mid i \in N$ and $v(a_{j_i}) = t\}$ of S. Since $\{a_i^*\}_{i \in N}$ is a minimal basis for the ideal $\bigoplus_{i \geq 1} \mathrm{gr}_i(R)$, the chain of ideals

$$(a_{j_1}^*) \subset (a_{j_1}^*, a_{j_2}^*) \subset (a_{j_1}^*, a_{j_2}^*, a_{j_3}^*) \subset \cdots$$

in the noetherian ring R/q_{t+1} does not terminate, forming a contradiction.

DEFINITION 3. A local ring R with a filtration $\{q_n\}$ is called a *special local ring* with respect to the (q_n)-topology (or the filtration $\{q_n\}$) if it satisfies the following properties: (1) R is equicharacteristic, (2) R/q_1 is a regular noetherian local ring, (3) R is complete for the (q_n)-topology.

The combination of Harrison's and Kolman's theorems in [6] (Theorem A and Theorem 1) can be stated as follows:

THEOREM α. *Let R be a special noetherian local ring with respect to the A-adic Hausdorff topology for some ideal A of R, then there exists a subring B of R such that $B + A = R$ and $B \cap A = (0)$.*

Now we consider an extension of this theorem.

THEOREM 2. *Let R be a special local ring with respect to a noetherian filtration $\{q_n\}$, then there exists a subring B of R such that $B + q_1 = R$ and $B \cap q_1 = (0)$.*

Proof. If $q_2 = (0)$, then $q_1^2 = (0)$ and $R/q_2 \cong R$. Hence, R is a special noetherian local ring for the q_1-adic topology which is discrete, and the theorem is identical with Theorem α. For the case $q_2 \neq (0)$, the proof is similar to that of Theorem 1 of [6], in which we only need to replace each A^i with q_i and the A-adic topology with the (q_n)-topology.

COROLLARY. *Let R be a special local ring with respect to a noetherian filtration $\{q_n\}$. Then R, as a topological ring, is a homomorphic image of a power series ring in a finite or countably infinite number of indeterminates over a subring B of R such that $B + q_1 = R$ and $B \cap q_1 = (0)$, where the power series ring is equipped with the $(B \cap q_n)$-natural topology.*

Proof. The corollary can be proved by successive applications of Theorem 2, Lemma, the remark following Definition 2 and Proposition 2.

Generalizing I. S. Cohen's structure theorem of complete noetherian local rings for the natural topologies ([5]), Nagata proved that embedding of coefficient rings in separated complete nonnetherian local rings for the natural topologies is also possible ([8]). A similar attempt to extend the embedding theorem to a wider class of complete local rings was made by Roquette ([10]). He proved that a complete local ring (R, M) equipped with a Hausdorff linear topology, which is weaker than the natural topology, possesses a complete local subring (k, m) such that $R = k + M$. k is either a field or a v-ring according as R is equicharacteristic or not, and in the latter case the natural topology of k is equivalent to the topology induced on k as a subset of R ([10, p. 76, Satz 1]; [10, p. 86, § 1.4. (b)]). Applying Roquette's theorem and our results so far, we can extend Theorem 9 and Theorem 12, both of [5], and Theorem 14 of [11] as follows.

THEOREM 3. *Let (R, M) be a complete (resp. compact) local ring with respect to a noetherian filtration $\{q_n\}$ such that $q_1 = M$. Then (1) R, as a topological ring, is a homomorphic image of a power series ring $k[[x_i]]_{i \in I}$ in a finite or countably infinite number of indeterminates over a complete (resp. compact) local ring (k, m) for the natural topology; where k is either R/M (resp. the finite field R/M)*

or a v-ring whose residue field is R/M according as R is equicharacteristic or not, and the power series ring is equipped with the m-natural topology. (2) *Conversely, if D is any closed ideal of such power series ring, then $k[[x_i]]_{i \in I}/D$ is a complete* (resp. *compact*) *local ring with respect to a noetherian filtration.*

Proof. Since $q_1 = M$, the (q_n)-topology of R is weaker than the natural topology; hence there exists a coefficient ring (k, m) which is either a field or a v-ring such that $m = (p)$ and $k/m \cong R/M$ due to Roquette's theorem, where p is the characteristic of R/M. According to Lemma, R possesses a formal system of generators, say, $S = \{a_i\}_{i \in I}$ for some $I \subseteq N$. If k is a field, then $S \cap k = \varnothing$ as $v(a_i) > 0$ for all i. If k is a v-ring, then $S \cap k$ has at most one element. To see this, let a_i and a_j be two distinct elements of $S \cap k$. Then $a_i, a_j \in m$, so that $a_i = p^e \alpha$ and $a_j = p^\lambda \beta$ for some $e, \lambda \in N$ and some units α and β of k. Assume that $e \geq \lambda$, then $a_i = p^e \alpha = (p^\lambda \beta)(p^{e-\lambda} \alpha \beta^{-1}) = a_j y$, where $y = p^{e-\lambda} \alpha \beta^{-1} \in k$; consequently, $a_i^* = a_j^* y^*$, which is a contradiction because $\{a_i^*\}_{i \in I}$ is a minimal basis for $\bigoplus_{i \geq 1} \mathrm{gr}_i(R)$. Now the theorem can be proved by the same method as in the proof of corollary to Theorem 2. Note that the $(k \cap q_n)$-natural topology of $k[[x_i]]_{i \in I}$ and the m-natural topology are equivalent since the $(k \cap q_n)$-topology and the natural topology of (k, m) are equivalent. When R is compact, the compactness or the finiteness of k follows from Lemma 3 and Theorem 7, both of [11]. (2) is trivially true for the case that I is finite. If $I = N$, then $k[[x_i]]_{i \in I} = k_\omega$ and $\{(B_n + D)/D\}_{n \in N_0}$ is a noetherian filtration of the local ring k_ω/D due to the remark following Definition 1. The completeness or the compactness of k_ω/D with respect to the noetherian filtration can be proved by applying Theorem 1 and the fact that k_ω is a Hausdorff metric space for the m-natural topology (cf. [3, p. 40, p. 48]).

COROLLARY 1. *A strictly linearly compact local ring (R, M), with respect to a filtration $\{q_n\}_{n \in N_0}$ such that $q_1 = M$, is a homomorphic image of a formal power series ring in a finite or a countably infinite number of indeterminates over a strictly linearly compact subring k of R which is either field or a v-ring.*

Proof. Evidently $\{q_n\}$ is a noetherian filtration because R/q_n is an artinian ring for each n; hence the corollary follows immediately from Theorem 3 (cf. [4, p. 110, excer. 19, excer. 22, (b)]).

The above corollary is identical with Theorem 1 of [1] relative to our particular linear topology.

COROLLARY 2. *Let (R, M) be a complete local ring with respect to a noetherian filtration $\{q_n\}$ such that $q_1 = M$. If R possesses a formal system $S = \{a_i\}_{i \in I}$ of generators of the finite type, then it is a complete noetherian local ring for the natural topology, and M is generated by S. Consequently, the (q_n)-topology is equivalent to the natural topology.*

Proof. Assume that $I = \{1, 2, \cdots, j\}$, and let u be the homomorphism of k_j onto R stated in (1) of Theorem 3; where k is either a field or a v-ring with the maximal ideal m. Then evidently k_j is a complete noetherian local ring for the natural topology. According to [4, p. 41, Proposition 12], M is generated by S; hence it is the image of the maximal ideal of k_j under the k-homomorphism u. Consequently, u is continuous for the natural topology of k_j and that of R, and $R = u(k_j)$ is a noetherian local ring which is complete for the natural topology. It follows that the (q_n)-topology and the natural topology of R are equivalent by [12, p. 270, Th. 13].

The next theorem is similar to [5, p. 88, Th. 15].

THEOREM 4. *Let (R, M) be a complete local ring with respect to a noetherian filtration $\{q_n\}$ such that $q_1 = M$, $S = \{a_i\}_{i \in I}$ a formal system of generators for R, and p the characteristic of R/M. (1) Suppose that R is equicharacteristic (resp. R is unequicharacteristic and $p \in S$). If elements of each finite subset of S are analytically independent, then as a topological ring, R is isomorphic to a power series ring $k[[x_i]]_{i \in I}$ (resp. $k[[x_i]]_{i \in I - \{1\}}$) in a finite or countably infinite number of indeterminates over a field k (resp. a v-ring (k, m) whose residue field is R/M), where the power series ring is equipped with the (0)-natural topology (resp. the m-natural topology). Consequently, R is a unique factorization domain. (2) Conversely, such power series ring is a complete local ring with respect to a noetherian filtration having $T = \{x_i\}_{i \in I}$ (resp. $T = \{p\} \cup \{x_i\}_{i \in I}$) as a formal system of generators, and T satisfies the condition that elements of each finite subset are analytically independent.*

Proof. Let us put $p = a_1$ if $p \in S$. Assume that I is finite. Then R is a complete noetherian local ring for the natural topology, and M is generated by S due to Corollary 2 to Theorem 3. Moreover, S is a minimal basis for M and R is regular, because the elements of S are analytically independent (cf. [9, p. 72, Th. 4]). If R is unequicharacteristic and $p \in S$, then $p \notin M^2$ by [9, p. 69, Proposition 6]; thus R is an unramified complete regular noetherian local ring. Now the theorem is well known by [5, p. 88, Th. 15]). Note that

the (0)-natural topology (resp. the m-natural topology) of the power series ring, which is a local ring, is equivalent to its natural topology due to a remark in §1. Next, we assume that $I = N$. We shall prove that the k-homomorphism u of k_ω to R considered in (1) of Theorem 3 is an isomorphism; we do this only for the case that k is a v-ring, since the proof for the other case is similar. Because $p = a_1 \in S$, we may consider u as a k-homomorphism such that $u(x_i) = a_{i+1}$ for all $i \geqq 1$. Put $k' = U_{i \in N} k_i$, $u' = u \mid k'$ and $u_i = u \mid k_i$ for each i, then each $u_i(k_i) = R^{(i)}$ is a noetherian local ring with the maximal ideal $m^{(i)}$ generated by $S_i = \{a_1, a_2, \cdots, a_{i+1}\} \subset M$; hence $m^{(i)} = M \cap R^{(i)}$ and the elements of S_i are analytically independent in $(R^{(i)}, m^{(i)})$. Consequently, S_i is a minimal basis for $m^{(i)}$. Thus each $R^{(i)}$ is an unramified complete regular noetherian local ring for the natural topology which has k as a coefficient ring. By Theorem 15 of [5], each u_i is bijective and so is $u' \colon k' \to R' = U_{i \in N} R^{(i)}$. Let k_ω be equipped with the m-natural topology and $\{B_r\}$ the basis of neighborhoods of 0 which induces the topology, where B_r is defined by (*) in §1. Regarding R' and k' as subspaces of R and k_ω respectively, we assert that u' is a homeomorphism. Evidently u' is continuous by Proposition 2. For each $n \in N$ let us consider the complete noetherian local ring $(R^{(n)}, m^{(n)})$ for the natural topology. By virtue of [12, p. 270, Th. 13], there must exist a $t \in N_0$ such that $R^{(n)} \cap q_t \subseteqq m^{(n)\,n}$; hence $u_n^{-1}(R^{(n)} \cap q_t) \subseteqq u_n^{-1}(m^{(n)\,n}) = (p, x_1, x_2, \cdots, x_n)^n k_n \subseteqq k_n \cap B_n$. If $i > n$, then $u_i^{-1}(R^{(i)} \cap q_t) \subseteqq k_i \cap B_n$ because $R^{(n)} \subset R^{(i)}$ and $u_i^{-1}(a_j) = x_{j-1} \in k_i \cap B_n$ for all j such that $i \geqq j > n$. It follows that $u'^{-1}(R' \cap q_t) \subseteqq k' \cap B_n$. Since $u'^{-1}(R' \cap q_t) = U_{i \in N_0} u_i^{-1}(R^{(i)} \cap q_t) = U_{i \geqq n} u_i^{-1}(R^{(i)} \cap q_t)$. Thus u'^{-1} is continuous; therefore u' is a homeomorphism. Moreover, $R' = u'(k')$ is dense in R, for k' is dense in k_ω and u is a surjective continuous homomorphism by Proposition 2. We can conclude that u is an isomorphism from [3, p. 41, Proposition 5]; therefore R is a unique factorization domain by]7]. (2) We also prove only for the case that $I = N$ and k is a v-ring. The first half of the statement is evident from (2) of Theorem 3, where we take $D = (0)$. Since each element c of $m = (p)$ can be written in the form that $c = \alpha q^\lambda$ for some unit α of k and $\lambda \in N$, it is easy to see that $\mathrm{gr}(k_\omega) = \bigoplus_{n \in N_0} B_n / B_{n+1} = k/m[p^*, x_1^*, x_2^*, \cdots]$. Hence $T = \{p\} U \{x_i\}_{i \in N}$ is a formal system of generators for k_ω by [4, p. 9, Proposition 1]. Let E be an arbitrary finite subset of T, say $E = \{p, x_1, x_2, \cdots, x_j\}$. Let $\psi(p, x) = \sum_i f^{(i)} p^{\lambda_{i0}} x_1^{\lambda_{i1}} \cdots x_j^{\lambda_{ij}}$ be a form of degree n in the elements of E over k_ω, nemely, $\lambda_{i0} + \lambda_{i1} + \cdots + \lambda_{ij} = n$ and $f^{(i)} \in k_\omega$, for each i in the finite sum $\sum_i$. If $\psi(p, x) = 0$, then $0 = P_j(\psi(p, x)) = \sum_i (f^{(i)})_j p^{\lambda_{i0}} x_1^{\lambda_{i1}} \cdots x_j^{\lambda_{ij}}$ is a form of degree n in k_j which is a regular noetherian local ring with the maximal ideal generated by E.

Hence, each $(f^{(i)})_j$ is a nonunit of k_j, and so is $f^{(i)}$ in k_ω. Thus the elements of E are analytically independent in k_ω. This completes the proof of the theorem.

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