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CHARACTERIZATIONS OF RADON PARTITIONS

WILLIAM R. HARE AND JOHN W. KENELLY

A Radon partition of a subset P of R^d is a pair $\{A, B\}$ satisfying (i) $A \cup B = P$, (ii) $A \cap B = \emptyset$ and (iii) $\operatorname{conv} A \cap \operatorname{conv} B \neq \emptyset$. The sets A and B are called components of the partition. The theorem of Radon says that for any $P \subset R^d$ having at least $d+2$ elements, there exists a Radon partition. When P is in general position with exactly $d+2$ elements, the Radon partition is unique; furthermore, a pair of points of P lie in the same component if and only if they are separated by the hyperplane through the remaining d points. A generalization of this result is

Theorem 1. Let P be a set of $n \geqq d+2$ points of R^d in general position, and let $S \subset P$ have k elements. Then S is contained in a component of some Radon partition of P if and only if (i) $k \leqq n-d-1$; or, (ii) if $k \geqq n-d$, then $\operatorname{conv} S \cap \operatorname{aff}(P \sim S) \neq \emptyset$.

With the notion of a primitive partition, a useful "reduction" is obtained.

Theorem 2. Every Radon partition of P extends a primitive partition.

Finally, a new characterization of the unique Radon partition mentioned above is given by

Theorem 3. Let P be a set of $d+2$ points in general position in R^d which do not lie on a common sphere. Then a pair of points in P lie in the same component of the unique Radon partition if and only if both of them are inside (or both outside) the respective $(d-1)$-spheres determined by the other $d+1$ points.

2. A generalization of the theorem of Proskuryakov and Kosmak. In [1] and [3] it is proved that, if P is a subset of R^d having $d+2$ points in general position, then two points will lie in the same component of the (unique) Radon partition of P if and only if they are separated by the hyperplane through the remaining d points. A direct generalization of this result is the first theorem of this paper. Throughout, $|S|$ indicates the cardinality of S.

THEOREM 1. *Let P be a set of $n \geqq d+2$ points in general position in R^d; let $S \subset P$ with $|S| = k$. Then S is contained in one component of some Radon partition for P if and only if*

(i) $k \leqq n-d-1$; *or*

(ii) *if $k \geqq n-d$, then* $\operatorname{conv} S \cap \operatorname{aff}(P \sim S) \neq \emptyset$.

Proof. If $k \leqq n - d - 1$, let $x \in S$. Let $\{A', B'\}$ be a Radon partition for $(P \sim S) \cup \{x\}$; such a partition exists, by Radon's theorem, since $|(P \sim S) \cup \{x\}| = |P \sim S| + 1 \geqq d + 2$. Assuming $x \in A'$, let $A = A' \cup S$ and $B = B'$ for the desired Radon partition $\{A, B\}$ of P having S in one component.

Suppose now that $k \geqq n - d$ and $\operatorname{conv} S \cap \operatorname{aff}(P \sim S) \neq \emptyset$. There exist $\{\alpha_s \mid s \in S\}$ and $\{\beta_t \mid t \in P \sim S\}$ such that each $\alpha_s \geqq 0$, $\sum_{s \in S} \alpha_s = \sum_{t \in P \sim S} \beta_t = 1$, and $\sum_{s \in S} \alpha_s s = \sum_{t \in P \sim S} \beta_t t$. Let $R = \{t \in P \sim S \mid \beta_t < 0\}$ and $\sum_{s \in S} \alpha_s - \sum_{r \in R} \beta_r = \sum_{t \in (P \sim S) \sim R} \beta_t = \alpha \neq 0$. Then

$$\sum_{s \in S} (\alpha_s/\alpha)s - \sum_{r \in R} (\beta_r/\alpha)r = \sum_{t \in (P \sim S) \sim R} (\beta_t/\alpha)t$$

which is a point in $\operatorname{conv}(S \cup R) \cap \operatorname{conv}((P \sim S) \sim R)$. That is, S is in one component of a Radon partition of P.

Conversely, suppose $\{T, P \sim T\}$ is a Radon partition of P with $S \subset T$. In case $k \leqq n - d - 1$, the proof is complete, so assume $k \geqq n - d$. Let $R = T \sim S$, so that $T = R \cup S$. Then there exist α's and β's such that

$$\sum_{s \in S} \alpha_s s + \sum_{r \in R} \beta_r r = \sum_{t \in P \sim T} \beta_t t$$

where each coefficient is nonnegative and

$$\sum_{s \in S} \alpha_s + \sum_{r \in R} \beta_r = \sum_{t \in P \sim T} \beta_t = 1 .$$

Rearrange to give

$$\sum_{s \in S} \alpha_s s = -\sum_{r \in R} \beta_r r + \sum_{t \in P \sim T} \beta_t t .$$

If $\sum_{s \in S} \alpha_s = 0$, each α_s is 0, so

$$-\sum_{r \in R} \beta_r r + \sum_{t \in P \sim T} \beta_t t = 0 \text{ and } -\sum_{r \in R} \beta_r + \sum_{t \in P \sim T} \beta_t = 0 .$$

However, since $|P \sim T| = n - k \leqq d$, this is a contradiction to the fact $P \sim T$ is affinely independent. Thus, let $\sum_{s \in S} \alpha_s = \alpha > 0$. Then

$$\sum_{s \in S} (\alpha_s/\alpha)s = \sum_{r \in R} (-\beta_r/\alpha)r + \sum_{t \in P \sim T} (\beta_t/\alpha)t ,$$

giving a point in $\operatorname{conv} S \cap \operatorname{aff}(P \sim S)$, which completes the proof.

The Proskuryakov and Kosmak Theorem is the case of $n = d + 2$ and $k = 2$.

3. Primitive partitions. In case $P \subset R^d$ has a Radon partition, it may happen that certain subsets of P also have this property. We say that $\{A, B\}$ is a Radon partition *in* P provided $\{A, B\}$ is a Radon partition of $A \cup B$ and $A \cup B \subset P$. We say that the Radon partition $\{A', B'\}$ *extends* the Radon partition $\{A, B\}$ provided $A \subset A'$ and $B \subset B'$. Finally, $\{A, B\}$ is called a *primitive partition* in P provided it is a Radon partition in P and $\{A, B\}$ extends the Radon partition $\{A', B'\}$ only if $\{A, B\} = \{A', B'\}$. It should be observed that for P in general position, any primitive partition $\{A, B\}$ in P will have $|A \cup B| = d + 2$.

THEOREM 2. *Let $P \subset R^d$ with $|P| = n$, and let $\{A', B'\}$ be a Radon partition of P. Then there exists a primitive partition $\{A, B\}$ in P such that $\{A', B'\}$ extends $\{A, B\}$. Furthermore, $|A \cup B| \leqq d + 2$.*

Proof. Let $\{A, B\}$ be a minimal Radon partition in $\{A', B'\}$ in the sense that: $A \subset A'$, $B \subset B'$ and if $\{A, B\}$ extends the Radon partition $\{A'', B''\}$ in P, then $\{A, B\} = \{A'', B''\}$. Let us suppose, to the contrary, that $|A| = k$, $|B| = m$ and $k + m \geqq d + 3$. The minimality assumption tells us that $\sigma_A = \operatorname{conv} A$ and $\sigma_B = \operatorname{conv} B$ are both simplices and that the point in their intersection is relatively interior to each. Without loss of generality, suppose this point is the origin 0. Thus, we have the existence of positive numbers $\{\alpha_a \mid a \in A\}$ and $\{\beta_b \mid b \in B\}$ such that $\sum_{a \in A} \alpha_a = \sum_{b \in B} \beta_b$ and $\sum_{a \in A} \alpha_a a = \sum_{b \in B} \beta_b b$.

Now

$$\begin{aligned}\dim(\operatorname{aff} \sigma_A \cap \operatorname{aff} \sigma_B) &= \dim(\operatorname{aff} \sigma_A) + \dim(\operatorname{aff} \sigma_B) \\ &\quad - \dim(\operatorname{aff} \sigma_A + \operatorname{aff} \sigma_B) \\ &= (k-1) + (m-1) - d \\ &= k + m - (d+2) \\ &\equiv N \geqq 1 .\end{aligned}$$

Let $\{x_1, \cdots, x_{N+1}\} \equiv C$ be an affine basis for this intersection. Then we can write

$$x_i = \sum_{a \in A} \gamma_{ia} a = \sum_{b \in B} \delta_{ib} b ,$$

where $\sum_{a \in A} \gamma_{ia} = \sum_{b \in B} \delta_{ib} = 1$, for each $1 \leqq i \leqq N + 1$. Furthermore, since $0 \in \text{rel. int.}\,(\sigma_A \cap \sigma_B)$, we may as well assume each coefficient to be nonnegative. For some i, suppose (say) $\beta_{b'}/\delta_{ib'}$ is the smallest of all positive ratios β_b/δ_{ib} and α_a/γ_{ia}.

Now solve $x_i = \sum_{b \in B} \delta_{ib} b$ for b' and substitute the resulting expression into $\sum_{a \in A} \alpha_a a = \sum_{b \in B} \beta_b b$. Next, replace x_i in this new equation by $\sum_{a \in A} \gamma_{ia} a$ and regroup A-terms on one side, B-terms on the other. This yields

$$(*) \qquad \sum_{a \in A} (\alpha_a - (\beta_{b'}/\delta_{ib'})\gamma_{ia})a = \sum_{b \in B \sim \{b'\}} (\beta_b - (\beta_{b'}/\delta_{ib'})\delta_{ib})b .$$

Routine calculations show that all coefficients are nonnegative and that those on each side add up to $1 - (\beta_{b'}/\delta_{ib'})$. Provided this common sum is positive, simply divide both sides of (*) by this number to obtain a convex combination of points in A equal to a convex combination of points in $B \sim \{b'\}$. This is a contradiction to the minimality assumption, so $|A \cup B| \leqq d + 2$.

Thus, it remains to show that, for some i, $1 - (\beta_{b'}/\delta_{ib'}) > 0$. A straight-forward computation shows that, if it is 0, then necessarily $x_i = 0$. But since there are at least two points in C, and these points are affinely independent, some $x_j \neq 0$. Thus, an appropriate choice of i is possible. The proof is complete.

This leads directly to another characterization of Radon partitions:

COROLLARY. *Let $P \subset R^d$ with $|P| = n \geqq d + 2$. Then a subset S of P is one component of a Radon partition of P if and only if there exists a primitive partition $\{A, B\}$ in P with (say) $S \cap A = \emptyset$ and $B \subset S$. The set S is contained in a component of a Radon partition of P if and only if there exists a primitive partition $\{A, B\}$ in P with $S \cap A = \emptyset$.*

4. Another characterization of Radon partitions of $d + 2$ points in general position. Here we seek a condition for determining whether two points are in the same, or opposite, component of a Radon partition in case $n = d + 2$. While the theorem is stated and proved for the case of general position in R^d, it is easily applied to any primitive partition in some set P. The criterion given here is in the spirit of that of Proskuryakov and Kosmak except that a type of spherical "separation" is used in place of their hyperplane.

THEOREM 3. *Let $P \subset R^d$ be a set of $d + 2$ points in general position and not all points lying on a common d-sphere. Suppose that $\{A, B\}$ is the unique Radon partition of P. Then two points of P lie in the same component if and only if they are each inside (or each outside) the respective $(d - 1)$-spheres determined by the remaining $d + 1$ points.*

Proof. Let u and v be two points of P. The remaining d points determine a hyperplane H and some $(d - 2)$-sphere in H. Let us suppose that the center of this sphere is the origin and the d-th coordinate axis is normal to H. Thus, the $(d - 2)$-sphere is $\{(\xi_1, \cdots, \xi_d) \in R^d \mid \sum_{i \neq d} \xi_i^2 = \rho^2, \xi_d = 0\}$, for some number $\rho > 0$. Now the point u, along with the points of P lying in H give a sphere

$$S_u = \left\{(\xi_1, \cdots, \xi_d) \in R^d \mid \sum_{i \neq d} \xi_i^2 + (\xi_d - \psi)^2 = \rho^2 + \psi^2\right\}$$

for some number $\psi > 0$. Similarly v and the d points of P in H give a $(d-1)$-sphere

$$S_v = \left\{(\xi_1, \cdots, \xi_d) \in R^d \mid \sum_{i \neq d} \xi_i^2 + (\xi_d - \omega)^2 = \rho^2 + \omega^2\right\}$$

for some ω.

In case $u = (\mu_1, \cdots, \mu_d)$ is inside S_v, we have

$$\sum_{i \neq d} \mu_i^2 + (\mu_d - \omega)^2 < \rho^2 + \omega^2 .$$

Since u is on S_u, we have

$$\sum_{i \neq d} \mu_i^2 + (\mu_d - \psi)^2 = \rho^2 + \omega^2 .$$

Thus, $2\mu_d(\psi - \omega) < 0$ if and only if u is inside S_v. The condition $2\mu_d(\psi - \omega) > 0$ characterizes the case that u is outsider S_v. Similarly, $v = (\nu_1, \cdots, \nu_d)$ is inside S_u if and only if $2\nu_d(\omega - \psi) < 0$, and outside if and only if $2\nu_d(\omega - \psi) > 0$.

Now apply the characterization in [1] and [3] to the preceding: Points u and v are in the same component A or B if and only if they are separated by $H = \{(\xi_1, \cdots, \xi_d) \in R^d \mid \xi_d = 0\}$, using the previous coordinatization. Thus, μ_d and ν_d carry opposite signs, and

$$2\mu_d(\psi - \omega) < 0 \quad \text{and} \quad 2\nu_d(\omega - \psi) < 0$$

(or, else, $2\mu_d(\psi - \omega) > 0$ and $2\nu_d(\omega - \psi) > 0$). That is, the points u and v are both inside (or outside) the respective $(d-1)$-spheres determined by the other d + 1 points of P.

In case u and v are not in the same component A or B, they are not separated by H and consequently μ_d, ν_d have the same signs and opposite inside-outside classifications.

This theorem yields a nice generalization of a problem stated and proved in [2].

COROLLARY. *Given a set P of $d + 2$ points in general position in R^d and not all the points on a common $(d-1)$-sphere, then* (i) *at most $d + 1$ of these points can each be outside the sphere through the other $d + 1$, and* (ii) *at most d of the points can each be inside the sphere through the other $d + 1$ points.*

Proof. Let $\{A, B\}$ be the unique Radon partition of P. Then, in particular, we have $A \neq \emptyset$ and $B \neq \emptyset$ so (i) is valid. Part (ii) follows, since an assumption to the contrary would result in an outside class of one point and an inside class of $d + 1$ points. This type of partition is only possible when one point is in the convex hull of the other $d + 1$ points. But we have an obvious contradiction, namely an interior point of a simplex outside the circumsphere of the simplex.

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CLEMSON UNIVERSITY