PACIFIC JOURNAL OF MATHEMATICS
Vol. 36, No. 2, 1971

INVERTING OPERATORS FOR SINGULAR BOUNDARY VALUE PROBLEMS

C. L. IRWIN

Let S denote a Banach Space, B the bounded linear transformations on S, and let Q and A denote functions from $[0, \infty)$ into B with Q continuous. The objective here is to derive a Green's function K_A and hence an integral inverting operator R_A for the singular boundary value problem

$$(1) \qquad \begin{cases} Y' - QY = H \\ A(0)Y(0) + \lim_{n\to\infty} A(c_n)Y(c_n) = 0\,, \end{cases}$$

where $\{c_n\}_{n=1}^{\infty}$ is a positive, increasing, unbounded number sequence and H is a continuous function from $[0, \infty)$ into S.

The method here provides Green's functions for singular boundary value problems associated with nonself-adjoint, as well as self-adjoint, linear differential expressions. The asymptotic boundary conditions in (1) permit one to extend some of the regular two-point boundary value problem techniques suggested by [3] and [4] to the singular case without being restricted to the Hilbert Space $L_2[0, \infty)$. Similar, but different, asymptotic boundary conditions are used by Coddington and Levinson in [2, Chapter 10], and by Benzinger [1].

As noted in § 3 of [3] there exists a unique continuous function M from $[0, \infty) \times [0, \infty)$ to B so that if each of x, t, and u is in $[0, \infty)$,

(i) $M_1(x, t) = Q(x)M(x, t)$ and $M(t, t) = I$

(ii) $M(x, t)M(t, u) = M(x, u)$

(iii) if H is a continuous function from $[0, \infty)$ to S and α is in S, then the only function Y such that $Y' - QY = H$ and $Y(0) = \alpha$ is given by

$$Y(x) = M(x, 0)\alpha + \int_0^x dt\, M(x, t)H(t)$$

for all x in $[0, \infty)$.

DEFINITION. A is a determinate boundary condition function for Q on $c_1, c_2, \cdots$ means that if H is a continuous function on $[0, \infty)$ and Y is a solution of the boundary value problem (1) for the nonhomogeneous term H, then Y is unique.

NOTATION. If A is a boundary condition function for Q on c_1, $c_2, \cdots$ and n is a positive integer, let T_n denote the transformation

$[A(0) + A(c_n)M(c_n, 0)]$.

THEOREM 1. *A is a determinate boundary condition function for Q on $c_1, c_2, \cdots$ if and only if the convergence of $\{T_n\alpha\}_{n=1}^{\infty}$ to zero implies that α is the zero of S.*

Proof. The proof follows from property (iii) of the M function and the linearity of the problem.

NOTATION. Let D_0 denote the continuous functions with compact support on $[0, \infty)$.

THEOREM 2. *Suppose A is a determinate boundary condition function for Q on $c_1, c_2, \cdots$; the following two statements are equivalent.*

(i) *There is an integral inverting operator R_A with kernel K_A of the form*

$$K_A(x, t) = \begin{cases} M(x, 0)K_A(0, 0)M(0, t) & \text{if } 0 \leqq t \leqq x \\ M(x, 0)[K_A(0, 0) - I]M(0, t) & \text{if } 0 \leqq x < t \end{cases}$$

for boundary value problem (1) *so that D_0 is a subset of the domain of R_A.*

(ii) *There is a transformation π in B such that if α is in S, then $\{T_n(\pi\alpha)\}_{n=1}^{\infty}$ converges to $A(0)\alpha$.*

Proof. Assume (i) holds; if H is in D_0 and $U = R_AH$, then U is a solution of boundary value problem (1) and so

$$\lim_{n\to\infty} [A(0)U(0) + A(c_n)U(c_n)] = 0 \,.$$

Let b denote a positive number so that if $x > b$, $H(x) = 0$; then, if $c_n > b$,

$$\begin{aligned} A(0)U(0) + A(c_n)U(c_n) &= A(0)\int_0^b dt\,[K_A(0, 0) - I]M(0, t)H(t) \\ &\quad + A(c_n)\int_0^b dt\, M(c_n, 0)K_A(0, 0)M(0, t)H(t) \\ &= T_n\Big(K_A(0, 0)\int_0^b dt\, M(0, t)H(t)\Big) \\ &\quad - A(0)\int_0^b dt\, M(0, t)H(t) \end{aligned}$$

and so

$$\lim_{n\to\infty} T_n\Big(K_A(0, 0)\int_0^b dt\, M(0, t)H(t)\Big) = A(0)\int_0^b dt\, M(0, t)H(t)\ .$$

Now, if α is in S, define H as

$$H(t) = \begin{cases} M(t, 0)(2 - 2t)\alpha & \text{if } 0 \leqq t \leqq 1 \\ 0 & \text{if } t > 1\ . \end{cases}$$

H belongs to D_0 and $\int_0^b dt\, M(0, t)H(t) = \alpha$, so (ii) holds with $\pi = K_A(0, 0)$.

Now, assume (ii) holds; since A is a determinate boundary condition function for Q on $c_1, c_2, \cdots$, π must be unique. Define K_A on $[0, \infty) \times [0, \infty)$ as

$$K_A(x, t) = \begin{cases} M(x, 0)\pi M(0, t) & \text{if } 0 \leqq t \leqq x \\ M(x, 0)[\pi - I]M(0, t) & \text{if } 0 \leqq x < t \end{cases} \tag{2}$$

and let R_A denote the integral operator with kernel K_A. Let H be in D_0 and b denote a positive number so that if $x > b$, $H(x) = 0$. Define U on $[0, \infty)$ as

$$U(x) = \begin{cases} \int_0^x dt\, M(x, 0)\pi M(0, t)H(t) & \\ \quad + \int_x^b dt\, M(x, 0)[\pi - I]M(0, t)H(t) & \text{if } 0 \leqq x \leqq b \\ \int_0^b dt\, M(x, 0)\pi M(0, t)H(t) & \text{if } x > b\ . \end{cases}$$

Differentiation yields that $U'(x) - Q(x)U(x) = H(x)$ for each x in $[0, \infty)$ and if $c_n > b$,

$$\begin{aligned} &A(0)U(0) + A(c_n)U(c_n) \\ &= T_n\Big(\pi\int_0^b dt\, M(0, t)H(t)\Big) - A(0)\int_0^b dt\, M(0, t)H(t)\ . \end{aligned}$$

By the definition of π, $\lim_{n\to\infty}[A(0)U(0) + A(c_n)U(c_n)] = 0$ and so (i) holds.

For the remainder of the paper suppose that A is a determinate boundary condition function for Q on $c_1, c_2, \cdots$, condition (ii) in Theorem 2 holds and K_A is defined on $[0, \infty) \times [0, \infty)$ by (2). (Condition (ii) is implied, for example, in case the sequence $\{T_n\}_{n=1}^\infty$ converges in norm to a regular element of B.) Let D denote the set of continuous functions H on $[0, \infty)$ such that $\int_0^\infty dt\, K_A(x, t)H(t)$ exists for each x and furthermore, if U is defined as

$$U(x) = \int_0^\infty dt\, K_A(x, t)H(t) \qquad \text{for } x \text{ in } [0, \infty)\ ,$$

then U is a solution of boundary value problem (1) for the nonhomogeneous term H. Let R_A denote the integral operator with kernel K_A and domain D; i.e., if H belongs to D

$$(R_AH)(x) = \int_0^\infty dt\, K_A(x, t)H(t) .$$

Two aspects of the present development which differ from other treatments of Green's functions for singular boundary value problems are: (1) the Green's functions here are not necessarily square integrable in either place and (2) the domains of the associated integral inverting operators are not restricted to functions which are square integrable on $[0, \infty)$. However, the domain of R_A does depend upon the problem, i.e., upon the particular Q and A involved. This dependence is the subject of the following two theorems.

Two sets of continuous function on $[0, \infty)$ which are relevant to the description of D are defined as follows. Let D_1 denote the collection of continuous functions H on $[0, \infty)$ for which there exists a solution of (1) for the nonhomogeneous term H. Let D_2 denote the collection of continuous functions H on $[0, \infty)$ such that $\int_0^\infty dt(\pi - I)M(0, t)$ $H(t)$ exists.

It is clear that D is a subset of the intersection of D_1 and D_2; not so obvious is the extent to which $D_1 \cap D_2$ is contained in D.

LEMMA. *Suppose H belongs to $D_1 \cap D_2$; let Y denote the solution of* (1) *for H and let $X(x) = \int_0^\infty dt\, K_A(x, t)H(t)$ for all x in* $[0, \infty)$, *then*

$$\begin{aligned} T_n[Y(0) - X(0)] = {} & [A(0)Y(0) + A(c_n)Y(c_n)] \\ & + [A(0) - T_n\pi] \int_0^{c_n} dt\, M(0, t)H(t) \\ & - T_n \int_{c_n}^\infty dt(\pi - I)M(0, t)H(t) \end{aligned}$$

for each positive integer n.

Proof. Let n denote a positive integer; property (iii) of the M function provides that

$$Y(c_n) = M(c_n, 0)Y(0) + M(c_n, 0)\int_0^{c_n} dtM(0, t)H(t) ,$$

so

$$\begin{aligned} T_nY(0) &= A(0)Y(0) + A(c_n)M(c_n, 0)Y(0) \\ &= A(0)Y(0) + A(c_n)Y(c_n) - A(c_n)M(c_n, 0)\int_0^{c_n} dt\, M(0, t)H(t) . \end{aligned}$$

Also,

$$T_n X(0) = [A(0) + A(c_n)M(c_n, 0)] \int_0^\infty dt(\pi - I)M(0, t)H(t) .$$

A straightforward computation provides the result of the lemma.

The domain D of the inverting operator R_A may be studied for the following three cases.

Case 1. There is an increasing sequence of positive integers n_1, $n_2, \cdots$ such that $T_{n_i}^{-1}$ exists for all i and the transformation sequence $\{T_{n_i}^{-1}\}_{i=1}^\infty$ is uniformly norm bounded.

Case 2. There is an increasing sequence of positive integers n_1, $n_2, \cdots$ such that $T_{n_i}^{-1}$ exists for all i, but no subsequence of inverses is uniformly norm bounded.

Case 3. There is a positive integer N such that if $n > N$, then T_n^{-1} does not exist.

Note. Case 1 above is a sufficient condition for a function A from $[0, \infty)$ into B to be a determinate boundary condition function for Q on $c_1, c_2, \cdots$.

THEOREM 3. *Suppose Case* 1 *above holds; if H is in $D_1 \cap D_2$ then H is in D if*

$$\lim_{i\to\infty} [A(0) - T_{n_i}\pi] \int_0^{c_{n_i}} dt\, M(0, t)H(t) = 0 .$$

Proof. By the lemma and existence of $T_{n_i}^{-1}$ for all i, we obtain in the notation of the lemma that

$$\begin{aligned} Y(0) - X(0) = {} & T_{n_i}^{-1}[A(0)Y(0) + A(c_{n_i})Y(c_{n_i})] \\ & + T_{n_i}^{-1}[A(0) - T_{n_i}\pi] \int_0^{c_{n_i}} dt\, M(0, t)H(t) \\ & - \int_{c_{n_i}}^\infty dt(\pi - I)M(0, t)H(t) \qquad \text{for each } i . \end{aligned}$$

H in D_2 provides that $\lim_{i\to\infty} \int_{c_{n_i}}^\infty dt(\pi - I)M(0, t)H(t) = 0$ and Y satisfies the asymptotic boundary condition so

$$Y(0) - X(0) = \lim_{i\to\infty} T_{n_i}^{-1}[A(0) - T_{n_i}\pi] \int_0^{c_{n_i}} dt\, M(0, t)H(t) .$$

Now, if $\lim_{i\to\infty} [A(0) - T_{n_i}\pi] \int_0^{c_{n_i}} dt\, M(0, t)H(t) = 0$, then $Y(0) - X(0) = 0$

and so $Y = X$, i.e., X is the unique solution of (1) for H and so H is in the domain of R_A.

A subcase of Cases 1, 2, and 3 above is that the transformation sequence $\{T_n\}_{n=1}^{\infty}$ be uniformly norm bounded, which occurs, for example, with $S = E_2$ and $T_n = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ n odd, $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ if n is even.

THEOREM 4. *Suppose $\{T_n\}_{n=1}^{\infty}$ is uniformly norm bounded; if H is in $D_1 \cap D_2$, then H is in D if and only if*

$$\lim_{n\to\infty} [A(0) - T_n\pi] \int_0^{c_n} dt\, M(0, t)H(t) = 0\ .$$

Proof. Let H denote a function in $D_1 \cap D_2$; by the lemma

$$\begin{aligned} T_n [Y(0) - X(0)] = {} & [A(0)Y(0) + A(c_n)Y(c_n)] \\ & + [A(0) - T_n\pi] \int_0^{c} dt\, M(0, t)H(t) \\ & - T_n \int_{c_n}^{\infty} dt\, (\pi - I)M(0, t)H(t) \end{aligned}$$

for each positive integer n. Where Y denotes the solution of (1) for H and X is defined on $(0, \infty)$ by

$$X(x) = \int_0^{\infty} dt\, K_A(x, t)H(t) \qquad x \text{ in } [0, \infty)\ .$$

Y satisfies the asymptotic boundary condition so

$$A(0)Y(0) + \lim_{n\to\infty} A(c_n)Y(c_n) = 0\ .$$

The transformation sequence $\{T_n\}_{n=1}^{\infty}$ is uniformly norm bounded and H is in D_2 so

$$\lim_{n\to\infty} T_n \int_{c_n}^{\infty} dt(\pi - I)M(0, t)H(t) = 0\ .$$

So

$$\lim_{n\to\infty} T_n [Y(0) - X(0)] = \lim_{n\to\infty} [A(0) - T_n\pi] \int_0^{c_n} dt\, M(0, t)H(t)\ .$$

The result of the theorem follows from A being a determinate boundary condition function.

The following example illustrates the subcase for a Case 1 problem and shows that the domain of R_A may be a proper subset of $D_1 \cap D_2$.

EXAMPLE. Let $c_1, c_2, \cdots$ denote a positive, increasing, unbounded number sequence. Consider the singular boundary value problem associated with the differential expression $Ly = y''$ and the boundary condition function A defined as

$$A(x) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} & \text{if } x = 0\,, \\ \begin{bmatrix} -1/[1+\log(1+x)] & x/[1+\log(1+x)] \\ 0 & -1 \end{bmatrix} & \text{if } x > 0\,, \end{cases}$$

We have $Q(x) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ for $x \geqq 0$, $M(x, t) = \begin{bmatrix} 1 & x - t \\ 0 & 1 \end{bmatrix}$ for all numbers x and t and if n is a positive integer,

$$T_n = \begin{bmatrix} \log(1+c_n)/[1+\log(1+c_n)] & 0 \\ 0 & 1 \end{bmatrix}.$$

So, $\lim_{n\to\infty} T_n = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\pi = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. A is a determinate boundary condition function for Q on $c_1, c_2, \cdots$ and K_A is calculated by equation (2).

Let $H(x) = \begin{bmatrix} 0 \\ 1/(1+x^2) \end{bmatrix}$ $x \geqq 0$. H is in D_1 since the function Y defined by

$$Y(x) = \begin{bmatrix} x \arctan x - \log + (1x^2)^{1/2} + (\pi/2)x - 1 \\ \arctan x + \pi/2 \end{bmatrix}$$

for $x \geqq 0$ is a solution of the singular boundary value problem with nonhomogeneous term H. Also, $\int_0^\infty dt(\pi - I)M(0, t)H(t)$ exists so H is in D_2 and $\int_0^\infty dt\, K_A(x, t)H(t)$ exists for each $x \geqq 0$. The function X defined by $X(x) = \int_0^\infty dt\, K_A(x, t)H(t)$ for $x \geqq 0$ does not satisfy the asymptotic boundary condition and so H is in $D_1 \cap D_2$ but not in the domain of R_A.

It remains to more completely describe how the domain of R_A depends upon the problem and to investigate the complex numbers λ for which one obtains an inverting operator $R(A, \theta, \lambda)$ for the singular boundary value problem

$$\begin{cases} Y' - (Q + \lambda\theta)Y = H \\ A(0)Y(0) + \lim\limits_{n\to\infty} A(c_n)Y(c_n) = 0 \end{cases}$$

where θ denotes a function from $[0, \infty)$ into B.

References

1. H. E. Benzinger, *Green's functions on large intervals.* Preliminary Report, Notices Amer. Math. Soc. **16** (1969), 114.
2. E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw Hill Book Co., Inc., 1955.
3. J. W. Neuberger, *Concerning boundary value problems*, Pacific J. Math. **10** (1960), 1385-1392.
4. A. J. Zettl, *Quasi-Differential Operators in Hilbert Space*, M. R. C. Technical Summary Report No. 845, 1968.

Received February 11, 1970. This paper is part of the author's doctoral dissertation which was directed by Professor John W. Neuberger.

WEST VIRGINIA UNIVERSITY