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THE TRANSCENDENTAL RANK OF A THEORY

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Morley has associated with each countable complete theory T an ordinal $\alpha_T < (2^{\aleph_0})^+$. It is shown that in fact $\alpha_T \leqq \omega_1$ and that this bound is best possible.

We shall use the notation and terminology of Morley [1], where α_T is defined to be the least ordinal α such that for all $A \in N(T)$ and all $\beta > \alpha$, $S^\alpha(A) = S^\beta(A)$. As in [1] T denotes a complete theory in a countable language L, T has an infinite model, and there is a theory Σ such that $T = \Sigma^*$. If $A \in N(T)$ and $p \in S(A)$, let $r(p) = \alpha$ if p is transcendental in rank α and let $r(p)$ be undefined otherwise. Also, if $A \in N(T)$ and $\psi \in F(A)$ define

$$r(\psi, A) = \begin{cases} -1 & \text{if} \quad U_\psi = \varnothing \\ \sup\{\alpha \mid p \in U_\psi \,\&\, r(p) = \alpha\} & \text{otherwise.} \end{cases}$$

LEMMA. *Let $A \in N(T)$, $\psi \in F(A)$, and $r(\psi, A) = \alpha$. Then for each $\beta < \alpha$ there exists $B \in N(T)$, $A \subseteqq B$, and $q \in S(B)$ such that $r(q) = \beta$ and $\psi \in q$.*

Proof. Assume the hypothesis and for contradiction that no B and q exist satisfying the conclusion. Then for every $B \in N(T)$, $A \subseteqq B$, we have $i_{AB}^{*-1}(U_\psi) \cap Tr^\beta(B) = \varnothing$. Thus for all such B, $i_{AB}^{*-1}(U_\psi) \cap (S^{\beta+1})(B) - S^\beta(B)) = \varnothing$. Suppose $q' \in Tr^{\beta+1}(B)$ then for every $C \in N(T)$, $B \subseteqq C$, $i_{BC}^{*-1}(q') \cap S^{\beta+1}(C)$ is a set of isolated points in $S^{\beta+1}(C)$. Thus, if $\psi \in q'$, $i_{BC}^{*-1}(q') \cap S^\beta(C)$ is a set of isolated points in $S^\beta(C)$ for all such C, whence $q' \in Tr^\beta(B)$. We conclude that $i_{AB}^{*-1}(U_\psi) \cap Tr^{\beta+1}(B) = \varnothing$ for all $B \in N(T)$, $A \subseteqq B$. By induction $i_{AB}^{*-1}(U_\psi) \cap Tr^\gamma(B) = \varnothing$ for all $\gamma \geqq \beta$. This contradicts the hypothesis and completes the proof of the lemma.

From 2.3(b) and 2.4 of [1] it is possible to choose B in the conclusion of the lemma such that $\kappa(B - A) = \aleph_0$; we shall make use of this fact below.

Before proceeding further we need some more definitions. A language L_1 is said to be a *simple extension* of a language L_0 if it is obtained by adjoining $\aleph_0$ individual constants to L_0. For any language L' let $F(L')$ denote the set of formulas of L' which have no free variable other than v_0. For each $n \in \omega$ let S_n denote the set of all sequences of 0's and 1's of length $\leqq n$; the empty sequence $\varnothing$ is allowed. For $s \in S_n$ and $i \leqq 1$, $s*\langle i\rangle$ denotes the member of S_{n+1}

obtained by juxtaposing i to the right of s. A map $\psi: S_n \to F(L)$ is called *admissible* if either $n = 0$, or $n > 0$ and for each $s \in S_m$, $0 \leqq m < n$ there exists $\varphi \in F(L)$ such that $\psi(s*\langle 0\rangle) = \psi(s) \& \varphi$ and $\psi(s*\langle 1\rangle) = \psi(s) \& \neg\varphi$. The main step in our proof is:

PROPOSITION. *Let $A \in N(T)$, $\kappa(A) \leqq \aleph_0$, and $n \in \omega$. Let $\psi_n: S_n \to F(L_n)$ be an admissible map, where L_n is a simple extension of $L(A)$, such that for every $\alpha < \omega_1$ there exists $B_n^\alpha \in N(T)$ with $A \subseteqq B_n^\alpha$ and $L(B_n^\alpha) = L_n$ such that for all $s \in S_n$ $r(\psi_n(s), B_n^\alpha) \geqq \alpha$. Then there exists a language L_{n+1}, which is a simple extension of L_n and an admissible map $\psi_{n+1}: S_{n+1} \to F(L_{n+1})$ extending ψ_n such that for every $\alpha < \omega_1$ there exists $B_{n+1}^\alpha \in N(T)$ with $A \subseteqq B_{n+1}^\alpha$ and $L(B_{n+1}^\alpha) = L_{n+1}$ such that for all $s \in S_{n+1}$, $r(\psi_{n+1}(s), B_{n+1}^\alpha) \geqq \alpha$.*

Proof. Form L_{n+1} by adjoining a countable number of new individual constants to L_n. Consider a fixed ordinal $\alpha < \omega_1$. By 2^{n+1} applications of the lemma we can find $C^\alpha \in N(T)$ with $B_n^{\alpha+2} \subseteqq C^\alpha$ and $L(C^\alpha) = L_{n+1}$ such that for each $s \in S_n - S_{n-1}$ there exist $p_0(s)$, $p_1(s) \in S(C^\alpha)$ both containing $\psi_n(s)$ such that $r(p_0(s)) = \alpha$ and $r(p_1(s)) = \alpha + 1$. For each $s \in S_n - S_{n-1}$ choose $\varphi^\alpha(s) \in p_0(s) - p_1(s)$. Define $\psi^\alpha: S_{n+1} \to F(L_{n+1})$ to be the extension of ψ_n such that for each $s \in S_n - S_{n-1}$, $\psi^\alpha(s*\langle 0\rangle) = \psi_n(s) \& \varphi^\alpha(s)$ and $\psi^\alpha(s*\langle 0\rangle) = \psi_n(s) \& \neg\varphi^\alpha(s)$. Letting $\psi_{n+1} = \psi^\alpha$ and $B_{n+1}^\alpha = C^\alpha$ the conclusion of the lemma holds for α. Perform the construction of ψ^α for each $\alpha < \omega_1$. Since L_{n+1} is countable the set $\{\psi^\alpha \mid \alpha < \omega_1\}$ is countable. Hence there is a cofinal subset Γ of ω_1 such that ψ^γ is independent of γ for $\gamma \in \Gamma$. Let ψ_{n+1} be the common value of ψ^γ for $\gamma \in \Gamma$. For each $\alpha < \omega_1$ let γ be the least member of Γ such that $\alpha < \gamma$ and define $B_{n+1}^\alpha = C^\gamma$. This completes the proof of the proposition.

Let S_ω denote the set of all finite sequences of 0's and 1's. A sequence $\langle s_i\rangle_{i<\omega}$ of members of S_ω is called *regular* if $s_0 = \varnothing$ and for all $i < \omega$, s_{i+1} is either $s_i*\langle 0\rangle$ or $s_i*\langle 1\rangle$. Now let $A \in N(T)$ with $\kappa(A) \leqq \aleph_0$, and let $p \in S(A)$ with $r(p) = \omega_1$. Choose $\varphi \in F(A)$ such that $U_\varphi \cap S^{\omega_1}(A) = \{p\}$. Let L_0 be $L(A)$ and define $\psi_0: S_0 \to F(L_0)$ by $\psi_0(\varnothing) = \varphi$ then φ_0 is admissible. Apply the proposition repeatedly to form $L_1, L_2, \cdots$ and $\psi_1, \psi_2, \cdots$. Let $L_\omega = U_{n<\omega} L_n$ and let $\psi = \lim_{n<\omega} \psi_n$ where ψ maps S_ω into $F(L_\omega)$. By the compactness theorem there exists $B \in N(T)$ such that $A \subseteqq B$, $\kappa(B) = \aleph_0$, $L(B) = L_\omega$, and such that if $\langle s_i\rangle_{i<\omega}$ is a regular sequence in S_ω then $\{\psi(s_i) \mid i < \omega\} \subseteqq q$ for some $q \in S(B)$. Let $s \in S_\omega$ then it is clear that the basic open set $U_{\psi(s)}$ of $S(B)$ has power $2^{\aleph_0}$. Also, since $\kappa(B) = \aleph_0$, for every α $S^{\alpha+1}(B) - S^\alpha(B)$ is countable. Thus $U_{\psi(s)} \cap S^\alpha(B) \neq \varnothing$ for all $\alpha < \omega_1$. Since $S^\alpha(B)$ is closed and decreasing with α, $U_{\psi(s)} \cap S^{\omega_1}(B) \neq \varnothing$. It follows immediately that $\kappa(U'_\varphi \cap S^{\omega_1}(B)) \geqq \aleph_0$ where U'_φ denotes the basic open set of $S(B)$ determined by φ. From 2.3(b) of [1] $i^*_{AB}(S^{\omega_1}(B)) = S^{\omega_1}(A)$. Since $i^*_{AB}(U'_\varphi) = U_\varphi$ it follows that $i^{*-1}_{AB}(p) = U'_\varphi \cap S^{\omega_1}(B)$. But this contradicts $r(p) = \omega_1$ because $U'_\varphi \cap S^{\omega_1}(B)$ having power $\geqq \aleph_0$ is not a set of isolated points.

Since $Tr^\alpha(A) \neq \varnothing$ for some finite $A \in N(T)$ if for any $A \in N(T)$, we have shown that $Tr^{\omega_1}(A) = \varnothing$ for every $A \in N(T)$. It follows easily that $S^\beta(A) = S^{\omega_1}(A)$ for every $\beta > \omega_1$ and every $A \in N(T)$. Thus $\alpha_T \leqq \omega_1$ and our main theorem is proved.

We shall now construct a theory T such that $\alpha_T = \omega_1$.[1] In Example III of § 2 of [1] Morley showed how to construct a theory T_β for any $\beta < \omega_1$ such that $\alpha_{T_\beta} = \beta + 1$ and such that $L(T_\beta) = \{R_n \mid n < \omega\}$ where each R_n is a unary relation symbol. For $\beta < \omega_1$ let A_β be a model of T_β. Suppose without loss that the sets $|A_\beta|$, $\beta < \omega_1$, are pairwise disjoint and each disjoint from ω_1. Now let A be the relational system such that $|A| = \omega_1 \cup_{\beta<\omega_1} |A_\beta|$ and define relations R^A, R^A_0, R^A_1, $\cdots$ as follows: for all $x, y \in |A|$

$$R^A(x, y) \Longleftrightarrow x \in \omega_1 \,\&\, y \in |A_x|$$

and

$$R^A_n(y) \Longleftrightarrow \bigvee x(x \in \omega_1 \,\&\, y \in R^{A_x}_n)\,.$$

If T is the theory of the system A then it is easy to see that $\alpha_T = \omega_1$.

In fact α_T can have as its value any ordinal $\leqq \omega_1$ other than 0. From the examples to be found above it is sufficient to treat the case in which β is a limit ordinal $< \omega_1$. Let $\langle \beta_n \rangle_{n<\omega}$ be a strictly increasing sequence with limit β. Let $T^\#$ be the theory with the same language as T_β above such that if A is any model of $T^\#$ and F, G are disjoint finite subsets of ω then

$$\bigcap \{R^A_n \mid n \in F\} \cap \bigcap \{|A| - R^A_n \mid n \in G\} \neq \varnothing\,.$$

Choose axioms $\psi_0, \psi_1, \cdots$ for $T^\#$ which are all existential, this is easy to do. For each n modify the theory T_{β_n} to obtain a theory T'_n whose transcendental rank is $\beta_n + 1$ and which has $\psi_0, \psi_1, \cdots, \psi_{n-1}$ amongst its theorems. For each $n < \omega$ let A_n be a model of T_n. Suppose that the sets $|A_n|$, $n < \omega$, are pairwise disjoint and disjoint from ω. Now let A be the relational system such that $|A| = \omega \cup \bigcup_{n<\omega} |A_n|$ with relations R^A, R^A_0, R^A_1, $\cdots$ defined by

$$R^A(x, y) \Longleftrightarrow x \in \omega \,\&\, y \in |A_x|$$

and

$$R^A_n(y) \Longleftrightarrow \bigvee x(x \in \omega \,\&\, y \in R^{A_x}_n)\,.$$

[1] The referee informs me that similar examples have been found independently by several people.

If T is the theory of the system A then it is easy to see that $\alpha_T = \beta$.

REFERENCES

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