PACIFIC JOURNAL OF MATHEMATICS
Vol. 37, No. 1, 1971

ON EPIMORPHISMS TO FINITELY GENERATED MODULES

E. GRAHAM EVANS, JR.

Serre's theorem on projective modules says roughly that if a projective R module is big enough it can map onto R. Forster and Swan discuss how big a free module is needed to map onto a given finitely generated module. This note examines a common generalization of these results and extends Swan's technique.

This paper follows Swan [5]. The reader is urged to refer to Swan for a more complete exposition of some of the ideas. The author is also in debt to Professor Kaplansky whose unpublished exposition of Swan's result [2] isolated one of the ideas for this paper.

Throughout the paper R will be a commutative ring with identity whose maximal ideal spectrum is a noetherian space and Λ is an R algebra which is a finitely generated R module. Following Swan we define J-Spec(R) to be the set of all prime ideals of R which are intersection of maximal ideas with topology the subspace topology inherited from Spec(R). If M is a finitely generated R module, then for each $p \in J$-Spec(R) we define $b(p, M) = 0$ if $M_p = 0$ and

$$b(p, M) = r + d$$

where $r = \dim_{(R/p)_p} (M/pM)_p$ and $d = \dim J\text{-Spec}(R/p)$ otherwise. We also call an element $x \in M_p$ *basic* if it will serve as part of a set of generators with the minimal number of elements, i.e., if $M_p/R_p x$ requires fewer generators than M_p.

THEOREM 1. R *a commutative ring with* J-Spec(R) *a noetherian space* M *a finitely generated* R *module and* P *a finitely generated projective* R *module with rank* $(P) \geqq \max b(p, M)$. *Then there exists an epimorphism from* P *to* M.

Proof. We might as well assume that M is faithful. For if $\mathfrak{a} = \mathrm{ann}(M)$, then we pass to $P/\mathfrak{a}P$ which is projective over $R/\mathfrak{a}$ with rank at least as large. Then if p is a minimal prime in J-Spec(R) such that a chain of maximal length of primes in J-Spec(R) passes through p, $M_p \neq 0$ since otherwise there would exist an $s \in R - p$ with $sM = 0$ contrary to M being faithful. Hence $\dim_{(R/p)_p} \geqq 1$. Thus rank $(P) \geqq d + 1$ where $d = \dim J\text{-Spec}(R) = \dim M\text{-spec}(R)$. Hence by Serre's theorem $P = R \oplus P'$. We define an epimorphism f from P to M by $f((1, 0)) = m$ where m is an element of M which is basic at all p' such that $b(p', M)$ is maximal. (See Swan [5, p 320] or below for details.) Then $\mathrm{Max}(b(p, M/(m)))$ is one less. Hence P' maps onto $M/(m)$ by induction. P' is projective. Hence that map lifts to $g: P' \to M$. Let $f((0, x)) = g(x)$. Then f is clearly an epimorphism as desired.

REMARKS. This, of course, extends to the case of P and M being finitely generated Λ modules since both Serre's and Swan's theorems are true in that case also. See [4, Theorem 11. 2 p. 171] and [5, Theorem 2, p. 320].

COROLLARY 2. *R as above P a projective R module of rank $\geqq r + d$ where $d = \dim J\text{-Spec}(R)$. Q any projective of rank r. Then P is isomorphic to $P' \oplus Q$.*

Proof. Clear.

THEOREM 3. *R commutative with J-Spec(R) a noetherian space. M a finitely generated R module. N any R module such that a direct sum of some number of copies (finite will of course suffice) of N maps onto M. Then if $n = \max b(p, M)$ a direct sum of n copies of N will map onto M.*

Proof. The key result needed from Swan is [5, p. 320 remark after Proposition 3] which states that the number of primes where $b(p, M)$ is maximum is finite.

We proceed to construct $f: \sum_{i=1}^{n} N \to M$ on each component in such a way that $\mathrm{Max}\ b(p, M/\mathrm{image}\ (\sum_{i=1}^{j} N \to M)) \leqq \max b(p, M) - j$ until $\max b(p, M/\mathrm{image}\ (\sum_{i=1}^{j} N \to M)) = 0$ in which case image $(\sum_{i=i}^{j} N \to M) = M$. Then we finish the epimorphism by sending the remaining components anywhere.

Suppose f has been contructed on the first j components ($j = 0$ is allowed). Let the image of $\sum_{i=1}^{j} N = I_j$. Then if $\max b(p, M/I_j) = 0$ we are done. Otherwise $\max b(p, M/I_j) > 0$. Let $p_1, \cdots, p_m$ be all the primes where $b(p, M/I_j)$ is maximal. Consider the submodules $p_i * I_j = \{m \mid \exists s \in R - p_i \text{ with } sm \in p_i M + I_j\}$. $p_i * I_j \neq M$ since $(M/I_j)_{p_i} \neq 0$. Furthermore an element $m \in M$ is part of a minimal generating set for $(M/I_j)_{p_i}$ if and only if $m \notin p_i * I_j$. (This is an easy consequence of Nakayama's lemma.) Since a direct sum of copies of N maps onto M there is some map $f_{i,j}; N \to M$ such that image $f_{i,j} \not\subset p_i * I_j$. We will achieve our objective if we can find an $f_j: N \to M$ with image $f_j \not\subset p_i * I_j \cup \cdots \cup p_m * I_j$. We prove this by induction on m. The case $m = 1$ is already done. We arrange the primes $p_1, \cdots, p_m$ so that p_i is minimal among $p_1, \cdots, p_i$. We assume we have an f_j

that works for $p_1, \cdots, p_s$. Then we want one working for $p_1, \cdots, p_{s+1}$. If f_j does, fine. Otherwise image $f_j \subset p_{s+1} * I_j$. Pick

$$r \in p_1 \cap \cdots \cap p_s - p_{s+1}$$

which exists since p_{s+1} cannot contain p_i if $i < s + 1$. Then I claim $f_j + rf_{s+1,j}$ works. It works at p_{s+1} since image $f_j \subset p_{s+1} * I_j$ while $rf_{s+1} \not\supset p_{s+1} * I_j$. Hence image $f_j + rf_{s+1,m} \not\supset p_{s+1} * I_j$. On the other hand at p_i for $i < s + 1$ we have image $rf_{s+1,j} \subset p_i * I_j$ while image $f_j \not\subset p_i * I_j$. Hence image $f_j + rf_{s+1,m} \not\subset p_i * I_j$. This completes the proof.

REMARKS. The theorem as it stands is false for general Λ. For if $\Lambda = n$ by n matrices over a field, $N = a$ column, $M = \Lambda$. Then at least n copies of N are needed to map onto M but max $b(p, M) = 1$. The difficulty in the proof is that in the non-commutative local case the set of not basic elements are not a submodule. The proof above uses heavily that the not basic elements are a submodule locally. In fact the $p_j * I_j$ are exactly the elements of M which are not basic in $(M/I_j)_{p_i}$. I conjecture that if M is generated by n elements over R and q is the biggest integer $\leqq$ the square root of n that $q max b(p, M)$ would work.

Another difficulty with this result is if N were free on a large number of generators then certainly we should be able to notice this and get a much better bound which this theorem cannot detect. Perhaps one could define a function $b(p, N, M)$ which would use the number of copies of N_p needed to map onto M_p. A theorem of this type might give back Serre's theorem except, for general N, one certainly needs the hypothesis that a sum of N's maps onto M.

We recall that in the category of R modules a generator is any module such that a sum of copies of it maps onto R. Or equivalently if for every module M and submodule N with $N \neq M$. There is a map $f: G \rightarrow M$ with image $f \subset N$. Theorem 3 shows for R a module N is a generator if and only if a sum of d copies of N maps onto R where $d = \dim$ m-Spec(R).

REFERENCES

1. O. Forster, *Uber die anzahl der Erzeugenden einen Ideals in einem Noetherschen Ring*, Math. Zeit., **84** (1964), 80-87.

2. I. Kaplansky, *Topics in Commutative Ring Theory*, mimeographed notes University of Chicago 1969.

3. J. P. Serre, *Modules Projective et Espaces Fibres a Fibre Vectorielle*, Seminaire P. Dubreil, Paris (1958), 23-01-23-18.

4. R. G. Swan, *Algebraic K-theory*, Springer-Verlag, Berlin 1968.
5. ———, *The Number of Generators of a Module*, Math. Zeit., **102** (1967), 318–322.

Received August 5, 1970. This work was supported under grant No. GP-9661.

UNIVERSITY OF CALIFORNIA AND
MASSACHUSETTS INSTITUTE OF TECHNOLOGY