PACIFIC JOURNAL OF MATHEMATICS
Vol. 37, No. 1, 1971

SOLVABLE AND SUPERSOLVABLE GROUPS IN WHICH EVERY ELEMENT IS CONJUGATE TO ITS INVERSE

J. L. BERGGREN

Let $\mathfrak{S}$ be the class of finite groups in which every element is conjugate to its inverse. In the first section of this paper we investigate solvable groups in $\mathfrak{S}$: in particular we show that if $G \in \mathfrak{S}$ and G is solvable then the Carter subgroup of G is a Sylow 2-subgroup and we show that any finite solvable group may be embedded in a solvable group in $\mathfrak{S}$. In the second section the main theorem reduces the study of supersolvable groups in $\mathfrak{S}$ to the study of groups in $\mathfrak{S}$ whose orders have the form $2^\alpha p^\beta$, p an odd prime.

NOTATION. The notation here will be as in [1] with the addition of the notation $G = XY$ to mean G is a split extension of Y by X. Also, $F(G)$ will denote the Fitting subgroup of G and $\Phi(G)$ the Frattini subgroup of G. We will denote the maximal normal subgroup of G of odd order by $O_{2'}(G)$. Further, $\mathrm{Hol}(G)$ will denote the split extension of G by its automorphism group.

If K and T are subgroups of G we will call K a T-group if $T \leqq N_G(K)$ and we say K is a T-indecomposable T-group if $K = K_1 \times K_2$, where K_1 and K_2 are T-groups, implies $K_1 = \langle 1 \rangle$ or $K_2 = \langle 1 \rangle$.

1. Burnside [2] proved that if P is a Sylow p-subgroup of the finite group G and if X and Y are P-invariant subsets of P which are not conjugate in $N_G(P)$ then they are not conjugate in G. Using Burnside's method one may prove a similar fact about the Carter subgroups. The proof is easy and we omit it.

LEMMA 1.1. *Let C be a Carter subgroup of the solvable group G and let A and B be subsets of C, both normal in C. If $A \neq B$ then A and B are not conjugate in G.*

THEOREM 1.1. *If G is a solvable group in $\mathfrak{S}$ then a Carter subgroup of G is a Sylow 2-subgroup of G.*

Proof. Let C be a Carter subgroup of G. If C has a nonidentity element of odd order then C has a nonidentity central element g of odd order, since C is nilpotent. Then with $A = \{g\}$ and $B = \{g^{-1}\}$ the hypotheses of Lemma 1.1 are satisfied and, since $A \neq B$, g and g^{-1} are not conjugate in G, contradicting our supposition that $G \in \mathfrak{S}$.

Hence C is a 2-group. As C is self-normalizing in G, C must be a Sylow 2-subgroup of G.

NOTE. This proof implies, also, that $Z(C)$ is an elementary abelian 2-group. However, the theorem of Burnside we mentioned can be used to show that if T is a Sylow 2-subgroup of any group $G \in \mathfrak{S}$ (whether solvable or not) then $Z(T)$ is elementary abelian. Thus, if $G \in \mathfrak{S}$ and T is a Sylow 2-subgroup of G the ascending central series of T has elementary abelian factors.

COROLLARY 1.1. *If T is a Sylow 2-subgroup of a solvable group $G \in \mathfrak{S}$ then $N_G(T) = T$.*

Proof. By Theorem 1.1 T is a Carter subgroup of G. Carter subgroups are self-normalizing.

COROLLARY 1.2. *If G and T are as in Corollary 1.1, and if T is abelian, then G has a normal 2-complement.*

Proof. By Corollary 1.1 and the assumption T is abelian, T is in the center of its normalizer. The result follows from a well-known theorem of Burnside.

We now investigate two families of solvable groups in $\mathfrak{S}$.

THEOREM 1.2. *If $G \in \mathfrak{S}$ and a Sylow 2-subgroup of G is cyclic then $G = TK$ where K is an abelian normal subgroup of odd order and $T = \langle \alpha \rangle$ with $\alpha^2 = 1$ and $g^\alpha = g^{-1}$ for all $g \in K$.*

Proof. As G has a cyclic Sylow 2-subgroup, G is solvable. By Corollary 1.2 $G = TK$, $T = \langle \alpha \rangle$ is a Sylow 2-subgroup of G and K is a normal subgroup of odd order. By the Note after Theorem 1.1, $\alpha^2 = 1$. If α did not induce a fixed-point-free automorphism of K then $C_G(T) \cap K \supsetneqq \langle 1 \rangle$, so $N_G(T) \supsetneqq T$, contradicting Corollary 1.1. Thus $g \to g^\alpha$ is a fixed-point-free automorphism of K. It is known that if K has a fixed-point-free automorphism α of order 2 then $\alpha(k) = k^{-1}$ for all $k \in K$ and hence K is abelian.

THEOREM 1.3. *Let G be a finite solvable group in $\mathfrak{S}$ and suppose a Sylow 2-subgroup T of G has order 4. Then T is elementary abelian, G has a normal 2-complement K, and $K^{(1)}$ is nilpotent.*

Proof. As G is solvable, Corollary 1.1 and 1.2 imply that $G =$

TK where $|T| = 4$ and K is a normal subgroup of odd order. The Note after Theorem 1.1 implies T is elementary, say $T = \langle\alpha\rangle \times \langle\beta\rangle$. Let K_α and K_β denote the set of fixed points of the automorphisms of K induced by α and β respectively. Then $\langle 1\rangle = C_K(T) \supseteqq K_\alpha \cap K_\beta$. Hence, as T is abelian, K_α is β-invariant and β induces a fixed-point free automorphism of K_α. Thus K_α is abelian. Then, by [4], $K^{(1)}$ is nilpotent.

Finally, we show that any finite solvable group can be embedded in a solvable group in $\mathfrak{S}$. We shall need the following lemma.

LEMMA 1.2. *Let $G \in \mathfrak{S}$ and let $\langle x\rangle$ be a cyclic group of order p, where p is an odd prime. Let α be an involution and define $H = \langle Gw\langle x\rangle, \alpha\rangle$, where $x^\alpha = x^{-1}$ and $b^\alpha = b$ for all $b \in G$. Then $H \in \mathfrak{S}$.*

Proof. Let $K = G \times G^x \times \cdots \times G^{x^{p-1}}$ be the base subgroup of $Gw\langle x\rangle$. Then $K \in \mathfrak{S}$ since $G \in \mathfrak{S}$. Suppose $h_1 \in H$ and

$$h_1 = x^r g_0 \cdot g_1^x \cdots g_{p-1}^{x^{p-1}} ,$$

where $r \not\equiv 0(p)$. Writing $[j]$ for x^j we may write

$$h_1 = x^r \cdot g_0 \cdot g_r^{[r]} \cdots g_{(p-1)r}^{[(p-1)r]} .$$

Now, if $g \in G$ then $(g^{[i]})^{x^r} = g^{[i+r]}$ implies that

$$(g^{[i]})^{-1} x^{-r} g^{[i]} x^r = (g^{[i]})^{-1} g^{[i+r]} ,$$

and hence $(g^{[i]})^{-1} x^r g^{[i]} = x^r (g^{[i+r]})^{-1} g^{[i]}$. Thus if $\beta = g_{er}^{[(e-1)r]}$ then $(x^r)^\beta = x^r (g_{er}^{-1})^{[er]} (g_{er})^{[(e-1)r]}$. Writing $h_1^\beta = x^r \cdot f_0 \cdot f_r^{[r]} \cdots f_{(p-1)r}^{[(p-1)r]}$, where $f_i \in G$ for all i, we see that $f_{ir} = g_{ir}$ if $i \neq e,\ e-1$ while $f_{er} = 1$. Thus first changing the rightmost $g_{ir}^{[ir]}$ in h_1 to 1 by conjugation and proceeding to the left we may conjugate h_1 to an element $h = x^r g$, where $g \in G = G^{[0]}$.

Pick $a \in G$ such that $g^a = g^{-1}$ and let $u = aa^x \cdots a^{x^{p-1}}$. Then with $\gamma = \alpha u x^{-r}$ we have $h^\gamma = h^{-1}$. It remains to consider elements of H of the form $h = \alpha \cdot x^r \cdot g_0 \cdot g_1^{[1]} \cdots g_{p-1}^{[p-1]}$, where $[j]$ denotes x^j. If $r \not\equiv 0\ (p)$ then let e be an integer such that $2e \equiv -r(p)$. Then h conjugated by x^e has the form $\alpha y_0 y_1^{[1]} \cdots y_{p-1}^{[p-1]}$ where the $y_i \in G$.

We now exploit the fact that, since $x^\alpha = x^{-1}$ and $g^\alpha = g$ for all $g \in G = G^{[0]}$, $g_{p-1}^{[p-1]} = (g_{p-1}^{[1]})^\alpha$, $g_{p-2}^{[p-2]} = (g_{p-2}^{[2]})^\alpha$, etc. Thus

$$\alpha^{\gamma(p-1)} = \alpha (g_{p-1}^{-1})^{[p-1]} (g_{p-1})^{[1]},$$

where $\gamma(p-1) = g_{p-1}^{[1]}$. Performing this computation for

$$\gamma(p-1),\ \gamma(p-2),\ \cdots,\ \gamma((p+1)/2) ,$$

where $\gamma(e) = g_e^{[p-e]}$ and observing that $u = \gamma(p-1) \cdots \gamma((p+1)/2)$

has the identity in $G^{[i]}$ as its i-th component for $i > ((p+1)/2)$ we see that h^u has the form $h_1 = \alpha \cdot f_0 \cdot f_1^{[1]} \cdots f_r^{[r]}$ where $r = (p-1)/2$ and $f_i \in G$ for all i. Then $h_1^{-1} = \alpha \cdot f_0^{-1} \cdot ((f_1^{-1})^{[1]} \cdots (f_r^{-1})^{[r]})^{\alpha}$. Now for all $i = 0, \cdots, r$ pick $a_i \in G$ such that $f_i^{a_i} = f_i^{-1}$ and let $u = a_0 \cdot v \cdot v^{\alpha}$ where $v = a_1^{[1]} \cdots a_r^{[r]}$. Taking $x = u\alpha$ it is easy to see that $h_1^{\tau} = h_1^{-1}$, using the fact that $(vv^{\alpha}, \alpha) = (g_0, vv^{\alpha}) = 1$. This disposes of all cases.

Theorem 1.4. *If G is a finite solvable group then there exists a solvable group $L \in \mathfrak{S}$ and a monomorphism $\tau: G \to L$.*

Proof. If G is abelian let $L = \langle G, \alpha \rangle$ where $\alpha^2 = 1$ and $g^{\alpha} = g^{-1}$ for all $g \in G$. Then in L every element of G is conjugate to its inverse and all other elements lie in the coset $G\alpha$ which consists of involutions, so $L \in \mathfrak{S}$ and L is solvable. Hence the theorem is true for all abelian groups G. Induct on $|G|$ and assume it is true for all solvable groups of order less than the order of G. Now let $H \lhd G$ such that $[G: H] = p$, p a prime. Our induction hypothesis says there is a solvable $K \in \mathfrak{S}$ and a monomorphism of HwC_p into KwC_p, where C_p is cyclic of order p. By Satz 15.9 [3] (Chapter I) there is a monomorphism of G into HwC_p, so G may be imbedded in KwC_p. If $p = 2$ then by Theorem 1.1 of [1] $KwC_p \in \mathfrak{S}$, and it is solvable since K is. If $p > 2$ then by Lemma 1.2 KwC_p has a solvable extension $\langle KwC_p, \alpha \rangle \in \mathfrak{S}$.

Thus, in this case as well, G may be imbedded in a solvable group in $\mathfrak{S}$.

This concludes our investigation of solvable groups in $\mathfrak{S}$.

2. In § 1 we showed that if $G \in \mathfrak{S}$ is a solvable group with an abelian Sylow 2-subgroup T then T has a normal complement in G. Of course, if G is supersolvable then (by the Sylow Tower Theorem) T has a normal complement K, regardless whether T is abelian or $G \in \mathfrak{S}$. If we assume that $G \in \mathfrak{S}$, where G is supersolvable, then with the above notation we assert.

THEOREM 2.1. *The Sylow 2-subgroup T is in $\mathfrak{S}$, and K and $\Phi(T)$ are contained in $F(G)$.*

Proof. That $T \in \mathfrak{S}$ was remarked in [1]. Since G is supersolvable $G^{(1)} \leqq F(G)$. Now $G \in \mathfrak{S}$ implies $G/G^{(1)} \in \mathfrak{S}$ and since $G/G^{(1)}$ is abelian $G/G^{(1)}$ is an elementary abelian 2-group. Thus $\Phi(T) \leqq G^{(1)}$, and since $(2, |K|) = 1$, $K \leqq G^{(1)}$.

REMARK. If $G \in \mathfrak{S}$ is supersolvable Theorem 2.1 implies G is a split extension of a nilpotent group K by a two-group T in $\mathfrak{S}$. If S is a Sylow 2-subgroup of $F(G)$ then $S \lhd G$, so $G/S \in \mathfrak{S}$. But by Theorem 2.1 G/S is isomorphic to a split extension EK of the nilpotent group K by an *elementary abelian* two-group E. Thus given a supersolvable G in $\mathfrak{S}$ there exists a supersolvable $G^* \in \mathfrak{S}$ such that $O_{2'}(G^*) \cong O_{2'}(G)$ but G^* has an elementary abelian Sylow 2-subgroup.

Now let $G = TK \in \mathfrak{S}$ be given, where G is supersolvable and T and K are as above. Let $P_1, \cdots, P_r$ be the Sylow subgroups of K, so $K = P_1 \times \cdots \times P_r$. If π_i is the projection of K onto P_i let $H_i = \ker(\pi_i)$. Then $H_i \rhd G$ and $G/H_i \cong TP_i$, a split extension of P_i by T which is supersolvable and in $\mathfrak{S}$. We have now reduced the study of supersolvable groups in $\mathfrak{S}$ to two questions:

(1) Given a 2-group $T \in \mathfrak{S}$ and a p-group P (p an odd prime) find the split extensions TP of P by T which are supersolvable and in $\mathfrak{S}$.

(2) Given split extensions $TP_1, \cdots, TP_n$ of P_i-groups by T (where the p_i are distinct odd primes) which are supersolvable and in $\mathfrak{S}$, when is $TP_1 \curlywedge TP_2 \curlywedge \cdots \curlywedge TP_n \in \mathfrak{S}$? (For a definition of the symbol $\curlywedge$ see [3], Satz 9.11.)

The answer to (2) is *not* "Always." For example let

$$TP_1 = \langle x, y, a, b\rangle$$

where $\langle x, y\rangle$ is the non-abelian group of order 27 and exponent 3, $\langle a, b\rangle$ is the four-group, and $(x, a) = x$, $(x, b) = 1$, $(y, a) = 1$, $(y, b) = y$. Let $TP_2 = \langle u, v, a, b\rangle$ where $\langle u, v\rangle$ is the nonabelian group of order 125 and exponent 5 with $(u, a) = u$, $(u, b) = 1$, $(v, a) = 1$, $(v, b) = v$. Then TP_1 and TP_2 are supersolvable and in $\mathfrak{S}$, but $TP_1 \curlywedge TP_2 \notin \mathfrak{S}$.

The next theorem answers (1) when T and P are abelian. It may be used to show that for certain P no T exists such that $TP \in \mathfrak{S}$.

THEOREM 2.2. *If $G = TK$ is a group in $\mathfrak{S}$ such that K is abelian of odd order ($K \lhd G$) and T is an abelian two-group then T is elementary and we may pick a basis $x_1, \cdots, x_n$ for K and a basis $\alpha, \beta_1, \cdots, \beta_m$ for T such that $x_i^\alpha = x_i^{-1}$ for all $i = 1, \cdots, n$ and $x_i^{\beta_j} = x_i^{\pm 1}$ for all i, j. Conversely any such group is in $\mathfrak{S}$.*

Proof. Since $G/K \cong T$, $T \in \mathfrak{S}$. Being abelian T must be elementary. Since K is a finite T-group we may write $K = K_1 \times \cdots \times K_n$ where each K_i is a T-indecomposable T-group. Now pick any $\gamma \in T$. Since $|\gamma| \leqq 2$ and K_i is abelian of odd order, $K_i = I_\gamma \times F_\gamma$ where

$$I_\gamma = \{x \in K_i \mid x^\gamma = x^{-1}\} \quad \text{and} \quad F_\gamma = \{x \in K_i \mid x^\gamma = x\} .$$

(For clearly $K_i \geqq I_\gamma \times F_\gamma$. For any $x \in K_i$ let $z = xx^\gamma$ and $w = x(x^{-1})^\gamma$. Observe that $z \in F_\gamma$, $w \in I_\gamma$, and $x^2 = zw$. Since $x^2 \in I \times F_\gamma$ and K_i has odd order, $x \in I_\gamma \times F_\gamma$. Thus $K_i = I_\gamma \times F_\gamma$.) Since T is abelian and K_i is a T-group, I_γ and F_γ are also T-groups. But K_i is T-indecomposable so $I_\gamma = \langle 1\rangle$ or $F_\gamma = \langle 1\rangle$. This means that each $\gamma \in T$ either inverts every element of K_i or fixes every element of K_i. Hence in any decomposition of K_i as a direct product of cyclic groups each direct factor is a T-group. As K_i is T-indecomposable we conclude K_i is cyclic. Let $K_i = \langle x_i\rangle$. Because $G \in \mathfrak{S}$ there exists $\alpha \in T$ such that $(x_1 \cdots x_n)^\alpha = x_1^{-1} \cdots x_n^{-1}$. Hence $x_i^\alpha = x_i^{-1}$ for all i and therefore $x^\alpha = x^{-1}$ for all $x \in K$. Now let $\alpha, \beta_1, \cdots, \beta_m$ be a basis of T, where α is as above. We found that for an arbitrary $\gamma \in T$ and an arbitrary $x \in K_i$, $x^\gamma = x$ or $x^\gamma = x^{-1}$. Hence for each j and i, $x_i^{\beta_j} = x_i^\varepsilon$, where $\varepsilon = \pm 1$.

Conversely, if $G = TK$ is as in the conclusion of the theorem then $g \in G$ either has the form $x_1^{e_1} \cdots x_n^{e_n}$ (which is conjugated to its inverse by α) or the form $\gamma x_1^{e_1} \cdots x_n^{e_n}$, with $\gamma \in T$. In this case it is easy to see that $g^\beta = g^{-1}$, where $\beta = \gamma\alpha$.

As an example of how this theorem might be applied we shall show that if $P = \langle x, y \mid x^{p^{n-1}} = y^p = 1,\ x^y = x^{1+p^{n-2}}\rangle$, where p is an odd prime and $n \geqq 3$, then there is no two-group T and supersolvable extension TP such that $TP \in \mathfrak{S}$. For suppose there were such a T, with $TP \in \mathfrak{S}$. We may assume, by previous remarks, that T is elementary abelian. Then $TP/\Phi(P) \in \mathfrak{S}$ and by the foregoing theorem there exists $\alpha \in T$ such that $x^\alpha = x^{-1}x^{pk}$ and $y^\alpha = y^{-1}x^{pe}$. Then

$$(x^y)^\alpha = (x^{1+p^{n-2}})^\alpha = x^{-1-p^{n-2}}x^{pk}$$

while $(x^\alpha)^{y^\alpha} = (x^{-1}x^{pk})^{y^{-1}} = (x^{-1})^{y^{-1}}x^{pk} = x^{-1+p^{n-2}}x^{pk}$. Since $(x^y)^\alpha = (x^\alpha)^{y^\alpha}$ we conclude that $x^{-p^{n-2}} = x^{p^{n-2}}$. Therefore $x^{2p^{n-2}} = 1$, contradicting the supposition that p is odd. Hence no such G exists.

3. We now give an example of a solvable group satisfying the hypotheses of Theorem 1.3 which does not have a nilpotent normal 2-complement. Thus the second assertion of Theorem 2.1 does not generalize to solvable groups with a normal 2-complement. Let

$$H = \langle x, y, z \mid x^7 = y^3 = z^2 = 1,\ x^y = x^2,\ x^z = x^{-1},\ y^z = y\rangle\ ,$$

so $H = \mathrm{Hol}\,(C_7)$, where C_7 is a cyclic group of order 7. Let

$$C_2 = \langle u \mid u^2 = 1\rangle$$

and define $K = HwC_2$. In K let $a = x$, $b = x^u$, $c = y(y^2)^u$, $d = zz^u$, $e = u$, and consider the subgroup $G = \langle a, b, c, d, e\rangle$. Then G has defining relations $a^7 = b^7 = c^3 = d^2 = e^2 = 1$, $(a, b) = (c, d) = (d, e) = 1$, $a^d = a^{-1}$, $b^d = b^{-1}$, $a^c = a^2$, $b^c = b^4$, $c^e = c^{-1}$, and $a^e = b$.

Consider the subgroup $\langle a, b, d, e\rangle$. Elements of the form ea^ib^j, a^ib^j, da^ib^j, and eda^ib^j are conjugated to their inverses by, respectively, a^jdea^{-j}, d, 1 and e. We may now consider elements $c^\varepsilon e^i d^j a^k b^m$, $\varepsilon = \pm 1$. Such an element is always conjugate to an element of the form $ce^i d^j a^k b^m$. Now $ceda^kb^m$ and cea^kb^m are conjugated to their inverses by ce and ced respectively. Finally ca^kb^m and cda^kb^m are conjugated to their inverses by $a^kb^{5m}ea^{-k}b^{-5m}$ and $a^{2k}b^{4m}ea^{-2k}b^{-4m}$ respectively.

This completes the proof that $G \in \mathfrak{S}$. Notice G satisfies the hypotheses of Theorem 1.3 but the normal 2-complement $K = \langle a, b, c\rangle$ is not nilpotent. In fact $F(K) = K^{(1)}$.

References

1. J. L. Berggren, *Finite groups in which every element is conjugate to its inverse*, Pacific J. Math., **28** (1969), 289-293.

2. W. Burnside, *Theory of Groups of Finite Order*, Dover, New York, 1955.

3. B. Huppert, *Endliche Gruppen I*, Springer-Verlag, New York, 1967.

4. J. N. Ward, *Involutory automorphisms of groups of odd order*, J. Australian Math. Soc., **6** (1966), 480-494.

Received February 10, 1970.

SIMON FRASER UNIVERSITY