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SOME RESULTS ON COMPLETABILITY IN COMMUTATIVE RINGS

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In this paper, R always denotes a commutative ring with identity. The ideal of nilpotents and the Jacobson radical of the ring R are denoted by $N(R)$ and $J(R)$, respectively. The vector $[a_1, \cdots, a_n]$ is called a primitive row vector provided $1 \in (a_1, \cdots, a_n)$; a primitive row vector $[a_1, \cdots, a_n]$ is called completable provided there exists an $n \times n$ unimodular matrix over R with first row $a_1, \cdots, a_n$. A ring R is called a B-ring if given a primitive row vector $[a_1, \cdots, a_n]$, $n \geqq 3$, and

$$(a_1, \cdots, a_{n-2}) \nsubseteq J(R),$$

there exists $b \in R$ such that $1 \in (a_1, \cdots, a_{n-2}, a_{n-1} + ba_n)$. Similarly, R is defined to be a Strongly B-ring (SB—ring), if $d \in (a_1, \cdots, a_n)$, $n \geqq 3$, and $(a_1, \cdots, a_{n-2}) \nsubseteq J(R)$ implies that there exists $b \in R$ such that $d \in (a_1, \cdots, a_{n-2}, a_{n-1} + ba_n)$.

In this paper it is proved that every primitive vector over a B-ring is completable. It is shown that the following are B-rings: π-regular rings, quasi-semi-local rings, Noetherian rings in which every (proper) prime ideal is maximal, and adequate rings. In addition it is proved that $R[X]$ is a B-ring if and only if R is a completely primary ring. It is then shown that the following are SB-rings: quasi-local rings, any ring which is both an Hermite ring and a B-ring, and Dedekind domains. Finally, it is shown that $R[X]$ is an SB-ring if and only if R is a field.

2. B-rings.

LEMMA 2.1. *Let R be a ring with $A \subseteqq J(R)$, A an ideal of R. Then R is a B-ring if and only if R/A is a B-ring.*

Proof. Necessity: Let R be a B-ring and let

$$(1 + A) \in (a_1 + A, \cdots, a_n + A),\ n \geqq 3$$

and

$$(a_1 + A, \cdots, a_{n-2} + A) \nsubseteq J(R/A) = J(R)/A\ ,$$

where $a_i \in R$, $i = 1, \cdots n$. Then $1 + A = \sum_{i=1}^{n} a_i b_i + A$, $b_i \in R$; hence $[a_1, \cdots, a_n]$ is primitive. Since $(a_1, \cdots, a_{n-2}) \nsubseteq J(R)$, it follows that $[a_1 + A, \cdots, a_{n-2} + A, (a_{n-1} + ba_n) + A]$ is primitive for some $b \in R$. Therefore, R/A is a B-ring.

Sufficiency: Suppose R/A is a B-ring and suppose $[a_1, \cdots, a_n]$ is a primitive vector with $(a_1, \cdots, a_{n-2}) \not\subseteq J(R)$. Hence $[a_1 + A, \cdots, a_n + A]$ is a primitive vector; and, since $(a_1, \cdots, a_{n-2}) \not\subseteq J(R)$, we have $(a_1 + A, \cdots, a_{n-2} + A) \not\subseteq J(R/A)$. Since R/A is a B-ring, there exists $b + A \in R/A$ such that $[a_1 + A, \cdots, a_{n-2} + A, (a_{n-1} + ba_n) + A]$ is primitive. It follows that $(1 - u) \in A \subseteq J(R)$, where

$$u = \sum_{i=1}^{n-2} a_i b_i + (a_{n-1} + ba_n)b_{n-1},\ b_i \in R,\ i = 1, \cdots, n-1 .$$

Therefore, u is a unit of R; i.e., $[a_1, \cdots, a_{n-2}, a_{n-1} + ba_n]$ is primitive.

THEOREM 2.1. *If R is a B-ring then every primitive row vector over R is completable.*

Proof. Let R be a B-ring and let $1 \in (a_1, \cdots, a_n)$. The theorem clearly holds for $n = 1$. If $n = 2$, then $1 = a_1x + a_2y$, $x, y \in R$ and the matrix $\begin{pmatrix} a_1 & a_2 \\ -y & x \end{pmatrix}$ is unimodular; hence the result holds for $n = 2$.

Let $n \geqq 3$, and suppose the result is established for $k < n$.

Case 1. If $(a_1, \cdots, a_{n-2}) \subseteq J(R)$ and $1 = \sum_{i=1}^{n} a_i b_i$, $b_i \in R$, then $1 - \sum_{i=1}^{n-2} a_i b_i = a_{n-1}b_{n-1} + a_n b_n$ is a unit $u \in R$. Let

$$V = \begin{pmatrix} a_{n-1} & a_n & a_1 & a_2 \cdots a_{n-2} \\ -b_n & b_{n-1} & 0 & 0 \cdots 0 \\ \multicolumn{2}{c}{\mathrm{O}} & \multicolumn{2}{c}{\mathrm{I}^{n-2}} \end{pmatrix} .$$

Then V has determinant u, and it follows that $[a_1, \cdots, a_n]$ is completable.

Case 2. If $(a_1, \cdots, a_{n-2}) \not\subseteq J(R)$, then $1 \in (a_1, \cdots, a_{n-2}, a_{n-1} + ba_n)$, for some $b \in R$. By the induction hypothesis, $[a_1, \cdots, a_{n-2}, a_{n-1} + ba_n]$ is completable to an $(n-1) \times (n-1)$ unimodular matrix D. Let

$$U = \begin{pmatrix} \mathrm{I}^{n-2} & 0 & 0 \\ 0 \cdots & 1 & 0 \\ 0 \cdots & -b & 1 \end{pmatrix} \text{ and let } B = \begin{pmatrix} & a_n \\ \mathrm{D} & 0 \\ & \vdots \\ 0 \cdots 0 & 1 \end{pmatrix} .$$

Then BU is an $n \times n$ unimodular matrix whose first row is $[a_1, \cdots, a_n]$.

For convenience, we introduce the notation $Z(A)$ to mean the set of maximal ideals containing the ideal A; $Z(a)$ will denote the set of maximal ideals containing the element a.

THEOREM 2.2 *If R is a ring such that for every ideal $A \not\subseteq J(R)$, $Z(A)$ is finite, then R is a B-ring.*

Proof. The essentials of the proof are due to Reiner [4]. Let $1 \in (a_1, \cdots, a_n)$, $n \geqq 3$, and $(a_1, \cdots, a_{n-2}) \nsubseteq J(R)$. By the hypothesis on R, $Z(A)$ is finite where $A = (a_1, \cdots, a_{n-2})$. Let $Z(A) = \{M_1, \cdots, M_r\}$, and note that if $b \in R$ and $a_{n-1} + ba_n \notin M_i$, $i = 1, \cdots, r$, then $[a_1, \cdots, a_{n-2}, a_{n-1} + ba_n]$ is primitive.

For any $M_i \in Z(A)$ such that $a_n \in M_i$, we have $a_{n-1} + ba_n \notin M_i$, for all $b \in R$; otherwise, $a_{n-1} \in M_i$, and $(a_1, \cdots, a_n) \subseteqq M_i$ which contradicts the hypothesis that $[a_1, \cdots, a_n]$ is primitive.

For those $M_i \in Z(A)$ for which $a_n \notin M_i$, we have $(a_n, M_i) = (1)$. Hence there exists an x_i such that $a_n x_i \equiv a_{n-1} \pmod{M_i}$. For these M_i, we can find (by the Chinese Remainder Theorem) an element $b \in R$ such that $b \equiv 1 - x_i \pmod{M_i}$. It follows that $a_{n-1} + ba_n \notin M_i$, $i = 1, \cdots, r$. Hence $[a_1, \cdots a_{n-2}, a_{n-1} + ba_n]$ is primitive.

It follows from this theorem that quasi-semi-local rings and Noetherian rings in which every proper prime ideal is maximal (in particular, Dedekind domains) are B-rings.

LEMMA 2.2. *Let R be an F-ring (i.e., a ring in which every finitely generated ideal is principal) which satisfies the condition that if $1 \in (a_1, a_2, a_3)$ with $a_1 \notin J(R)$ then $1 \in (a_1, a_2 + ba_3)$ for some $b \in R$. Then R is a B-ring.*

Proof. Let $1 \in (a_1, \cdots, a_n)$, $n \geqq 3$, and let $(a) = (a_1, \cdots, a_{n-2}) \nsubseteq J(R)$. Hence $1 \in (a, a_{n-1}, a_n)$. By the hypothesis on R, $1 \in (a, a_{n-1} + ba_n)$; hence, R is a B-ring.

THEOREM 2.3. *If R is an F-ring which satisfies the condition that for every $a, c \in R$ with $a \notin J(R)$, there is an $r \in R$ such that $Z(r) = Z(a) - Z(c)$, then R is a B-ring.*

Proof. The proof is essentially the same as the proof of Theorem 5 of [2]. Let $1 \in (a, b, c)$, $a \notin J(R)$. By the hypothesis on R there exists $r \in R$ such that $Z(r) = Z(a) - Z(c)$. Hence $(c, r) = (1)$, so there exists $q \in R$ such that $1 \in (r, b + qc)$. We claim $(a, b + qc) = (1)$. Otherwise, there exists a maximal ideal M of R such that $(a, b + qc) \subseteqq M$. Hence $M \in Z(a)$ and $M \in Z(b + qc)$. Since $1 \in (r, b + qc)$ it follows that $M \notin Z(r)$, so $M \in Z(c)$. But we now have $M \in Z(b)$, contrary to $(a, b, c) = (1)$. Therefore $(a, b + qc) = (1)$. Lemma 2.2 completes the proof.

THEOREM 2.4. *Every adequate ring is a B-ring.*

Proof. In the proof of Theorem 5.3 of [3], Kaplansky shows that if R is an adequate ring and if $1 \in (a, b, c)$, $a \neq 0$, then there exists $q \in R$ such that $1 \in (a, b + qc)$. Since an adequate ring is an F-ring, the result follows from Lemma 2.2.

THEOREM 2.5. *Every π-regular ring is a B-ring.*

Proof. If R is a π-regular ring, and if $a \in R/N(R)$, then by Lemma 2.2 of [5], a is an associate of $e + \beta$, e an idempotent and β a nilpotent of the π-regular ring $R/N(R)$. Since $\beta = 0$, $a = ue$, u a unit of $R/N(R)$. Therefore, $a^2 = u^2 e$ and $u^{-1}a^2 = ue = a$. Hence, $R/N(R)$ is a regular ring and therefore an adequate ring ([1, Th. 11]). Theorem 2.4 and Lemma 2.1 complete the proof.

THEOREM 2.6. *Let D be an integral domain, K its quotient field. Let $R = \{(a_1, \cdots, a_k, a, a, \cdots) \colon a_i \in K, a \in D\}$, where k is a nonnegative integer* (k may be different for distinct elements of R). *The operations in R are component-wise addition and multiplication. If R is a B-ring then D is a B-domain.*

We illustrate the proof. Suppose R is a B-ring and let $1 \in (a, b, c)$, $a, b, c \in D$, $1 = aa' + bb' + cc'$. Let $\hat{a} = (1, a, a, \cdots)$, $\hat{b} = (0, b, b, \cdots)$, $\hat{c} = (0, c, \cdots)$, $\hat{a}' = (1, a', a', \cdots)$, $\hat{b}' = (0, b', b', \cdots)$, $\hat{c}' = 0, c', c', \cdots)$. Then $\hat{1} = \hat{a}\hat{a}' + \hat{b}\hat{b}' + \hat{c}\hat{c}'$. If $\hat{a} \in J(R)$, then $\hat{1} - \hat{a} = (0, 1 - a, 1 - a, \cdots)$ is a unit of R. Since this is false, $\hat{a} \notin J(R)$, hence $\hat{1} \in (\hat{a}, \hat{b} + \hat{y}\hat{c})$ for some $\hat{y} \in R$. Therefore $\hat{1} = \hat{a}\hat{d} + (\hat{b} + \hat{y}\hat{c})\hat{e}$, where $\hat{d}, \hat{c}, \hat{e} \in R$. Let $\hat{d} = (d_1, \cdots, d_p, d, d, \cdots)$, $\hat{e} = (e_1, \cdots, e_q, e, e, \cdots)$, $\hat{y} = (y_1, \cdots, y_r, y, y, \cdots)$ and let $\lambda = \max(1, p, q, r)$. In the $(\lambda + 1)$st entry of $\hat{a}\hat{d} + (\hat{b} + \hat{y}\hat{c})\hat{e}$, we have $ad + (b + yc)e$; i.e., $1 \in (a, b + yc)$. Hence, D is a B-domain.

THEOREM 2.7. *$R[X]$ is a B-ring if and only if R is a completely primary ring.*

Proof. Sufficiency: Let R be a completely primary ring. Since $R/N(R)$ is a field and since $(R/N(R))[X] \cong R[X]/N(R)[X]$, it follows from Theorem 2.2 that $R[X]/N(R)[X]$ is a B-ring. Since $N(R)[X] = N(R[X])$, the result follows from Lemma 1.2.1.

Necessity: Assume that R is not completely primary and that $R[X]$ is a B-ring. Let r be a nonunit, nonnilpotent element of R. Then $1 \in (r, 1 + X, X^2)$ and $r \notin J(R[X])$. By the assumption that $R[X]$ is a B-ring, we have $1 \in (r, 1 + X + X^2 f(X))$ for some $f(X) \in R[X]$. Let $\bar{a}$ denote the image of $a \in R$ under the natural homomorphism of $R[X]$ onto $(R/rR)[X]$. Then $\bar{1} \in (\bar{0}, \bar{1} + X + X^2\bar{f}(X))$ and $\bar{1} + X + X^2\bar{f}(X)$ is a unit of $(R/rR)[X]$. This is a contradiction since the coefficient of X is not nilpotent.

Since $R[X]$ cannot be completely primary, (clearly, X is neither a unit nor a nilpotent) it follows that for every ring R, $R[X, Y] = R[X][Y]$ is not a B-ring.

3. Strongly B-rings. We now turn our attention to the study of SB-rings. Our main objective here is to compare the theory of this particular subclass of B-rings with that of B-rings given in the last section.

LEMMA 3.1. *R is an SB-ring if and only if for every $s, c_1, c_2, c_3 \in R$ with $s \in (c_1, c_2, c_3)$ and $c_1 \notin J(R)$, it follows that $s \in (c_1, c_2 + bc_3)$ for some $b \in R$.*

Proof. The necessity clearly follows from the definition of an SB-ring.

Sufficiency: Let $r \in (a_1, \cdots, a_n)$, $n \geqq 3$, with $(a_1, \cdots, a_{n-2}) \nsubseteq J(R)$. Without loss of generality, we may assume that $a_{n-2} \notin J(R)$. Suppose $r = \sum_{i=1}^{n} a_i x_i$ and let $s = a_{n-2}x_{n-2} + a_{n-1}x_{n-1} + a_n x_n$. Then $r \in (a_1, \cdots, a_{n-3}, s)$ and $s \in (a_{n-2}, a_{n-1}, a_n)$. Since $a_{n-2} \notin J(R)$, $s \in (a_{n-2}, a_{n-1} + ba_n)$ for some $b \in R$. Therefore $r \in (a_1, \cdots, a_{n-3}, s) \subseteqq (a_1, \cdots, a_{n-1} + ba_n)$, and the proof is complete.

In view of Lemma 3.1, we need only consider triples instead of arbitrary n-tuples in our study of SB-rings.

LEMMA 3.2. *The homomorphic image of an SB-ring is an SB-ring.*

Proof. Let $\bar{R}$ be the image of R under the homomorphism ϕ, and let $\bar{d} \in (\bar{a}_1, \bar{a}_2, \bar{a}_3)$ with $\bar{a}_1 \notin J(\bar{R})$, $\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{d} \in \bar{R}$. Suppose $\bar{d} = \sum_{i=1}^{3} \bar{a}_i \bar{x}_i$, $\bar{x}_i \in \bar{R}$ and let $a_i\phi = \bar{a}_i$, $x_i\phi = \bar{x}_i$, $i = 1, 2, 3$. Let $d = \sum_{i=1}^{3} a_i x_i$. Since $(J(R))\phi \subseteqq J(\bar{R})$, we have $a_1 \notin J(R)$; hence, $d \in (a_1, a_2 + ba_3)$ for some $b \in R$. Since $d\phi = \bar{d}$, we have $\bar{d} \in (\bar{a}_1, \bar{a}_2 + \bar{b}\bar{a}_3)$, where $b\phi = \bar{b}$. Hence $\bar{R}$ is an SB-ring.

THEOREM 3.1. *Every quasi-local ring is an SB-ring.*

Proof. Let $d \in (a_1, a_2, a_3)$, with $a_1 \notin J(R)$, R a quasi-local ring. Since $a_1 \notin J(R)$, a_1 is a unit of R; hence, $d \in (a_1, a_2 + ba_3) = (1)$ for every $b \in R$.

LEMMA 3.3. *Let $A = (a_1, \cdots, a_n)$, $n \geqq 3$, be an ideal in a Dedekind domain R. If $B = (a_1, \cdots, a_{n-2}) \neq (0)$, then $A = (a_1, \cdots, a_{n-2}, a_{n-1} + ba_n)$ for some $b \in R$.*

Proof. Let $A = \prod_{i=1}^t M_i^{\alpha_i}$ and let $B = \prod_{i=1}^t M_i^{\beta_i}$ be the representations of the ideals A and B as a product of powers of distinct maximal ideals. Since $B \subseteqq A$, we may order the M_i so that $0 \leqq \alpha_i < \beta_i$ for $1 \leqq i \leqq r$, and $\alpha_i = \beta_i$ for $r+1 \leqq i \leqq t$. Let $1 \leqq k \leqq r$. We claim that either a_{n-1} or a_n does not belong to $M_k^{\alpha_k+1}$. For suppose both a_{n-1} and a_n belong to $M_k^{\alpha_k+1}$. Then $A \subseteqq M_k^{\alpha_k+1}$, a contradiction. Since the $M_k^{\alpha_k+1}$ are relative prime, the Chinese Remainder Theorem guarantees the existence of a $b \in R$ satisfying:

$$b \equiv 0 \pmod{M_k^{\alpha_k+1}} \quad \text{if} \quad a_{n-1} \notin M_k^{\alpha_k+1}$$
$$b \equiv 1 \pmod{M_k^{\alpha_k+1}} \quad \text{if} \quad a_{n-1} \in M_k^{\alpha_k+1} ,$$

for $k = 1, 2, \cdots, r$. It follows that $a_{n-1} + ba_n \notin M_k^{\alpha_k+1}$ for $k = 1, 2, \cdots, r$. Let $(a_1, \cdots, a_{n-2}, a_{n-1} + ba_n) = \prod_{i=1}^t M_i^{\mu_i}$. Since $(a_1, \cdots, a_{n-2}, a_{n-1} + ba_n) \subseteqq A = \prod_{i=1}^t M_i^{\alpha_i}$, it follows that $\mu_i \geqq \alpha_i$, $i = 1, 2, \cdots, t$. Since $B = \prod_{i=1}^t M_i^{\beta_i} \subseteqq \prod_{i=1}^t M_i^{\mu_i} \subseteqq \prod_{i=1}^t M_i^{\alpha_i} = A$, and since $\beta_i = \alpha_i$, $r+1 \leqq i \leqq t$, it follows that $\mu_i = \beta_i = \alpha_i$, $r+1 \leqq i \leqq t$. If $\mu_i > \alpha_i$ for some i with $1 \leqq i \leqq r$, then $a_{n-1} + ba_n \in M_i^{\mu_i} \subseteqq M_i^{\alpha_i+1}$, a contradiction. Hence, $\mu_i = \alpha_i$, $i = 1, 2, \cdots, t$. Equivalently, $(a_1, \cdots, a_{n-2}, a_{n-1} + ba_n) = A$.

As an immediate consequence, we have:

THEOREM 3.2. *A Dedekind domain is an SB-ring.*

LEMMA 3.4. *Let R be a B-ring, let $e = e^2 \in R$, and let $e \in (a_1, \cdots, a_n)$ with $(a_1, \cdots, a_{n-2}) \not\subseteqq J(R)$, $n \geqq 3$. Then $e \in (a_1, \cdots, a_{n-2}, a_{n-1} + ba_n)$ for some $b \in R$.*

Proof. Since the case $e = 1$ is covered by the hypothesis, we may assume $e \neq 1$. Let $e = \sum_{i=1}^n a_i x_i = \sum_{i=1}^n (a_i e)(x_i e)$. Hence, $1 = (a_1 e + 1 - e)(x_1 e + 1 - e) + \sum_{i=2}^n (a_i e)(x_i e)$. Thus,

$$1 \in (a_1 e + 1 - e, a_2 e, \cdots, a_n e) .$$

If $a_1 e + 1 - e \in J(R)$, then $1 - (a_1 e + 1 - e) = e(1 - a_1)$ is a unit of R, a contradiction since $e = e^2, e \neq 1$. Thus, since R is a B-ring, we have $1 \in (a_1 e + 1 - e, a_2 e, \cdots, a_{n-2} e, a_{n-1} e + ba_n e)$ for some $b \in R$. Therefore, $e \in (a_1 e, a_2 e, \cdots, a_{n-2} e, a_{n-1} e + ba_n e) \subseteqq (a_1, a_2, \cdots, a_{n-2}, a_{n-1} + ba_n)$.

COROLLARY. *If R is a regular ring then R is an SB-ring.*

Proof. The result is immediate from Theorem 2.5 and Lemma 3.4; since, for every $r \in R$, r is an associate of some idempotent $e \in R$ ([1, Lemma 10]).

THEOREM 3.3. *If a B-ring R is also an Hermite ring, then R is an SB-ring.*

Proof. Let $d \in (a_1, a_2, a_3) = (a)$, $a_1 \notin J(R)$. By Corollary 5 of [1], there exist b_1, b_2, b_3 such that $a_1 = b_1a$, $a_2 = b_2a$, $a_3 = b_3a$, and $(b_1, b_2, b_3) = (1)$. Since R is a B-ring and since $b_1 \notin J(R)$, there exists a $q \in R$ such that $(b_1, b_2 + qb_3) = (1)$. Therefore, $(a) = (b_1a, b_2a + qb_3a) = (a_1, a_2 + qa_3)$. Hence, $d \in (a_1, a_2 + qa_3)$.

COROLLARY. *Every adequate domain is an SB-ring.*

Proof. An adequate domain is both an F-domain and a B-ring. Since every F-domain is an Hermite ring, the result follows from Theorem 3.3.

COROLLARY. *If R is an F-ring with infinitely many maximal ideals and, if for every ideal $A \nsubseteq J(R)$, $Z(A)$ is finite, then R is an SB-ring.*

Proof. R is necessarily a B-ring by Theorem 2.2. By the proof of Corollary 2 of [2], R is also an Hermite ring. Theorem 3.3 completes the proof.

THEOREM 3.4. *$R[X]$ is an SB-ring if and only if R is a field.*

Proof. The sufficiency follows from Theorem 3.2. To prove the necessity, let $r \in R$, $r \neq 0$. Then $r \in (X^2, X, r)$ and $X^2 \in J(R[X])$. If $R[X]$ is an SB-ring then $r \in (X^2, X + rb(X))$ for some $b(X) \in R[X]$. Let $r = X^2f(X) + (X + rb(X))g(X)$, where $f(X)$ and $g(X) \in R[X]$, and let f_i, g_i, b_i represent the coefficient of X^i in the polynomials $f(X)$, $g(X)$, $b(X)$, respectively. Equating coefficients in the above equation gives $r = rb_0g_0$ and $0 = g_0 + r(b_0g_1 + g_0b_1)$. Hence r divides g_0 and therefore $r = r^2k$ for some $k \in R$. Hence $rk = (rk)^2$; therefore, rk is an idempotent of R. Since $R[X]$ is a B-ring, R must be a completely primary ring by Theorem 2.7. It follows that the idempotent rk is either 0 or 1. Since $rk = 0$ and $r = r^2k$ imply $r = 0$, we conclude that $rk = 1$; i.e., r is a unit of R. Hence, R is a field and the proof is complete.

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