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A SUFFICIENT CONDITION FOR L^p-MULTIPLIERS

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Suppose $1 \leqq p \leqq \infty$. For a bounded measurable function ϕ on the n-dimensional euclidean space R^n define a transformation T_ϕ by $(T_\phi f)^\wedge = \phi \hat{f}$, where $f \in L^2 \cap L^p(R^n)$ and $\hat{f}$ is the Fourier transform of f:

$$\hat{f}(\xi) = \hat{f} \frac{1}{\sqrt{2\pi}^n} \int_{R^n} f(x) e^{-i\xi x}\, dx \, .$$

If T_ϕ is a bounded transform of $L^p(R^n)$ to $L^p(R^n)$, ϕ is said to be L^p-multiplier and the norm of ϕ is defined as the operator norm of T_ϕ.

THEOREM 1. **Let $2n/(n+1) < p < 2n/(n-1)$ and ϕ be a radial function on R^n, so that, it does not depend on the arguments and may be denoted by $\phi(r)$, $0 \leqq r < \infty$. If $\phi(r)$ is absolutely continuous and**

$$M = \| \phi \|_\infty + \left(\sup_{R>0} R \int_R^{2R} \left| \frac{d}{dr} \phi(r) \right|^2 dr \right)^{1/2} < \infty \, ,$$

then ϕ is an L^p-multiplier and its norm is dominated by a constant multiple of M.

To prove this theorem we introduce the following notations and Theorem 2. For a complex number $\delta = \sigma + i\tau$, $\sigma > -1$, and a reasonable function f on R^n the Riesz-Bochner mean of order δ is defined by

$$s_R^\delta(f, x) = \frac{1}{\sqrt{2\pi}^n} \int_{|\xi| < R} \left(1 - \frac{|\xi|^2}{R^2}\right)^\delta \hat{f}(\xi) e^{i\xi x} d\xi \, .$$

Put

$$t_R^\delta(f, x) = s_R^\delta(f, x) - s_R^{\delta-1}(f, x)$$

and define the Littlewood-Paley function by

$$g_\delta^*(f, x) = \left(\int_0^\infty \frac{|t_R^\delta(f, x)|^2}{R} dR \right)^{1/2} ,$$

which is introduced by E. M. Stein in [3]. Then we have the following.

THEOREM 2. *If $2n/(n + 2\sigma - 1) < p < 2n/(n - 2\sigma + 1)$ and $1/2 < \sigma < (n+1)/2$, then*

$$A \| g_\sigma^*(f) \|_p \leqq \| f \|_p < A' \| g_\sigma^*(f) \|_p \, ,$$

where A and A′ are constants not depending on f.

The first part of inequalities is proved by E. M. Stein [3] for $p=2$ and by G. Sunouchi [4] for $2n/(n+2\sigma-1)<p<2$. The other parts will be shown by the conjugacy method as in S. Igari [2], so that we shall give a sketch of a proof.

Proof of Theorem 2. For $\delta=\sigma+i\tau$, $\sigma>-1$, and $t>0$ let $K_t^\delta(x)$ be the Fourier transform of $[\max\{(1-|\xi|^2t^{-2}),\ 0\}]^\delta$. Since $K_t^\delta(x)$ is radial, we denote it simply by $K_t^\delta(r)$, $r=|x|$. Then $K_t^\delta(r)=2^\delta\Gamma(\delta+1)$ $V_{(n/2)+\delta}(rt)t^n$, where $V_\beta(s)=J_\beta(s)s^{-\beta}$ and J_β denots the Bessel function of the first kind. Considering the Fourier transform of $t_R^\delta(f,x)$ we get

$$t_R^\delta(f,x)=\frac{1}{\sqrt{2\pi}^{\,n}}\int_{\boldsymbol{R}^n}f(y)T_R^\delta(x-y)dy=f*T_R^\delta(x)\ ,$$

where $T_R^\delta(x)=R^{-2}\Delta K_R^{\delta-1}(x)$ and $\Delta=\partial^2/(\partial x_1^2)+\cdots+\partial^2/(\partial x_n^2)$.

Let H be the Hilbert space of functions on $(0,\infty)$ whose inner product is defined by $\langle f,g\rangle=\int_0^\infty f_R\bar{g}_R R^{-1}dR$. For a function $g_R(x)$ in $L^1(\boldsymbol{R}^n;H)$, that is, H-valued $L^1(\boldsymbol{R}^n)$-function, define an operator v^δ by

$$v^\delta(g,x)=\frac{1}{\sqrt{2\pi}^{\,n}}\int_{\boldsymbol{R}^n}<T_R^\delta(y),\bar{g}_R(x-y)>dy\ .$$

By the associativity of convolution relation

$$(1)\qquad \int_{\boldsymbol{R}^n}v^\delta(g,x)\bar{f}(x)dx=\int_{\boldsymbol{R}^n}<g(x),t^{\bar{\delta}}(f,x)>dx$$

for every f in $L^2(\boldsymbol{R}^n)$ and g in $L^2(\boldsymbol{R}^n;H)$, which implies that $v^{\bar{\delta}}$ is the adjoint of t^δ.

By the Plancherel formula

$$(2)\qquad \begin{aligned}\|t^\delta(f)\|_{L^2(H)}&=\left(\int_{|\xi|}^\infty\left(1-\frac{|\xi|^2}{R^2}\right)^{2\sigma-2}\frac{|\xi|^4}{R^5}dR\right)^{1/2}\|f\|_{L^2}\\&=B_\sigma\|f\|_{L^2}\ ,\end{aligned}$$

where $B_\sigma=[B(2\sigma-1,2)/2]^{1/2}$, $\delta=\sigma+i\tau$, and $\sigma>1/2$. Therefore $f=(1/B_\sigma^2)v^{\bar{\delta}}t^\delta(f)$ for $f\in L^2(\boldsymbol{R}^n)$. By Schwarz inequality $|<t^\delta(f,x),g(x)>|\leqq\|t^\delta(f,x)\|_H\|g(x)\|_H$. Applying this inequality with (2) to (1) we get

$$(3)\qquad \|v^\delta(g)\|_{L^2}\leqq B_\sigma\|g\|_{L^2(H)}\ .$$

On the other hand

$$\int_{|x|>2|y|} \| T_R^\delta(x+y) - T_R^\delta(x) \|_H dx < A_\sigma e^{\pi|\tau|/2}$$

for $\sigma > \alpha + 1$, $\alpha = (n-1)/2$ (see [4]), where $A_{p,q}$ denotes here and after a constant depending only on p, q and the dimension n. Thus by the well-known argument (see, for example, Dunford-Schwartz [1; p. 1171] we get

$$\| t^\delta(f) \|_{L^q(H)} \leqq A_{q,\rho} e^{\pi|\tau|/2} \| f \|_{L^q} \tag{4}$$

and

$$\| v^\delta(g) \|_{L^q} \leqq A_{q,\rho} e^{\pi|\tau|/2} \| g \|_{L^q(H)} \tag{5}$$

for $1 < q \leqq 2$ and $\delta = \rho + i\tau$, $\rho > \alpha + 1$. Fix such a ρ and a q.

Remark that the Stein's interpolation theorem (see [5; p. 100]) is valid for H-valued L^p-spaces and apply it between the inequalities (2) and (4), and (3) and (5). Then we get

$$\| t^\sigma(f) \|_{L^p(H)} \leqq A_{p,\sigma} \| f \|_{L^p} \tag{6}$$

and

$$\| v^\sigma(g) \|_{L^p} \leqq A_{p,\sigma} \| g \|_{L^p(H)} \tag{7}$$

for $1 < p \leqq 2$ and $\sigma > (n/p) - \alpha$.

Since $f = (1/B_\sigma^2) v^\sigma t^\sigma(f)$, we get Theorem 2 for $2n/(n+2\sigma-1) < p \leqq 2$ from (6) and (7). The case where $2 \leqq p < 2n/(n-2\sigma+1)$ is proved by the equality (1) and the conjugacy method.

Proof of Theorem 1. Let $f \in L^2(\boldsymbol{R}^n)$. By definition

$$t_R^1(T_\phi f, x) = -\frac{1}{\sqrt{2\pi}^n} \int_{|\xi|<R} \frac{|\xi|^2}{R^2} \phi(\xi) \hat{f}(\xi) e^{i\xi x} d\xi \,. \tag{8}$$

Put

$$F(r\omega) = F(\xi) = \frac{-1}{\sqrt{2\pi}^n} \frac{|\xi|^2}{R^2} \hat{f}(\xi) e^{i\xi x} \,,$$

where $r = |\xi|$ and ω is a unit vector. Then

$$t_R^1(T_\phi f, x) = \int_0^R \phi(r) \left(\int_{|\omega|=1} F(r\omega) d\omega \right) r^{n-1} dr \,.$$

The last term is, by integration by parts, equal to

$$\phi(R) \int_0^R r^{n-1} dr \int_{|\omega|=1} F(r\omega) d\omega - \int_0^R \frac{d}{dr} \phi(r) dr \int_0^r s^{n-1} ds \int_{|\omega|=1} F(s\omega) d\omega \,.$$

Thus

$$t_R^1(T_\phi f, x) = \phi(R)t_R^1(f, x) - \int_0^R \frac{d}{dr}\,\phi(r)\,\frac{r^2}{R^2}\,t_r^1(f, x)dr\ .$$

By the Schwarz inequality the last integal is, in absolute value, dominated by

$$\left(\frac{1}{R}\int_0^R \left|\frac{d}{dr}\,\phi(r)\right|^2 r^2dr\right)^{1/2}\left(\frac{1}{R^3}\int_0^R |t_r^1(f, x)|^2 r^2dr\right)^{1/2}.$$

Divide $(0, R)$ into the intervals of the form $(R/2^{j+1}, R/2^j)$ and dominate r^2 by $R^2/2^{2j}$ in each interval. Then the first integral is bounded by

$$\sum_{j=0}^{\infty}\frac{1}{2^{j-1}}\,\frac{R}{2^{j+1}}\int_{R/2^{j+1}}^{R/2^j}\left|\frac{d}{dr}\,\phi(r)\right|^2 dr \leqq 4\sup_{R>0} R\int_R^{2R}\left|\frac{d}{dr}\,\phi(r)\right|^2 dr\ .$$

Therefore

$$\begin{aligned} g_1^*(T_\phi f, x) &\leqq \|\phi\|_\infty\left(\int_0^\infty \frac{|t_R^1(f, x)|^2}{R}dR\right)^{1/2} \\ &\quad + 2\left(\sup_{R>0} R\int_R^{2R}\left|\frac{d}{dr}\,\phi(r)\right|^2 dr\right)^{1/2}\left(\int_0^\infty |t_r^1(f, x)|^2 r^2dr\int_r^\infty \frac{dR}{R^4}\right)^{1/2} \\ &\leqq \frac{2}{\sqrt{3}}\,Mg_1^*(f, x)\ . \end{aligned}$$

Thus, if $2n/(n+1) < p < 2n/(n-1)$, then by Theorem 2 we have

$$\|T_\phi f\|_p \leqq A'\,\|g_1^*(T_\phi(f))\|_p \leqq \frac{2}{\sqrt{3}}\,A'M\,\|g_1^*(f)\|_p \leqq \frac{2}{\sqrt{3}}\,AA'M\,\|f\|_p\ ,$$

which completes the proof.

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