

REVERSIBLE HOMEOMORPHISMS OF THE REAL LINE

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Let G be the group of germs of C^k local homeomorphisms of the real line which fix the origin and have nonzero derivative there. In this paper the possibility of factoring an element of G which is conjugate to its inverse into the product of two involutions is investigated. It is shown that it is always possible to do this in the analytic case and not always possible in the continuous case. In the intermediate cases several necessary and sufficient conditions are developed for determining whether or not such a factorization is possible. Included is a construction which allows one to determine an explicit factorization. Indication is given of the application of this material to the same problem in higher dimensions. This work is related to some material in Dynamics.

1. Introduction. If G is an abstract group an element $g \in G$ is called *reversible* in G if there exists an element $h \in G$ such that $hgh^{-1} = g^{-1}$. The product of two involutions is always reversible by an obvious argument. There arises the following question:

Question #. If g is reversible in G can g be factored into the product of two involutions in G ?

D. C. Lewis has decided this issue in the case $G = GL(n, \mathbb{C})$ affirmatively (Lewis [4]).

This paper concerns itself with the investigation of this question in the case where G is the group of germs of continuous or differentiable homeomorphisms of the real line.

Reversible transformations play a role in Dynamics. For further information on this connection see the references in Lewis [4].

2. Definitions and Notation. $C^k = \{F: F \text{ is a local homeomorphism of a neighborhood of } 0 \text{ in } \mathbb{R} \text{ to another such neighborhood which fixes the origin and is of class } C^k \text{ on some neighborhood of } 0, F'(0) \neq 0 \text{ if } k > 0, \text{ for } 0 \leq k \leq \infty \text{ or analytic for } k = \omega\}$. $T^k = \{\text{germs of elements of } C^k\}$

Let $\phi_k: C^k \rightarrow T^k$ be the map which assigns to each element of C^k its germ in T^k . The binary operation of (local) composition of mappings in C^k induces a group multiplication in T^k . T^k will be viewed as a group with this structure henceforward.

If a property is locally true for each $\alpha \in f$, $\alpha \in C^k$, $f \in T^k$ the property will be attributed to f . The identity element of any group under consideration will be called 1.

$P^k = \{\text{set of all real power series of the form } \sum_{r=1}^k a_r x^r, a_1 \neq 0, 1 \leq k \leq \infty\}$

$P^\omega = \{\alpha \in P^\infty: \alpha \text{ has a nonzero radius of convergence}\}$

$P^0 = \{1, -1\}$.

Let $\rho_k: T^k \rightarrow P^k$, $k > 0$ be the mapping which assigns to each element of T^k its Taylor expansion to degree k about the origin. Put $\rho_k(f) = \pm 1$ according as f is the germ of a locally increasing or locally decreasing element of C^0 .

REMARK 1. If $0 \leq k \leq \infty$ or $k = \omega$, ρ_k is onto. This trivial for $k < \infty$ familiar for $k = \omega$ and true for $k = \infty$. For the last case see Borel [2].

The multiplication in T^k induces via the mapping ρ_k a group multiplication in P^k . P^k will be viewed as a group with this multiplication. The elements of P^k are often referred to as jets in the literature. Note that composition of elements in P^k is not multiplication of polynomials but is substitution followed by truncation to degree k . Other homomorphisms which will be found useful are the mappings $T^k \rightarrow T^m$ and $P^k \rightarrow P^m$, $m \leq k$ defined by the inclusion mapping of $C^k \rightarrow C^m$ followed by ϕ_m or $\rho_m \phi_m$.

For $F \in C^k$, let F_+ (resp., F) be the restriction of F to $R_+ = \{x \in R: x \geq 0\}$ (resp., to $R_- = \{x \in R: x \leq 0\}$). F_+ (resp., F_-) is a local homeomorphism fixing 0 of R_+ to R_+ or R_+ to R_- (resp., R_- to R_- or R_- to R_+). The notation f_+ and f_- will denote the corresponding notion for the germ f . f_+^{-1} (resp., f_-^{-1}) will denote $(f_+)^{-1}$ (resp., $(f_-)^{-1}$).

3. Periodic local homeomorphisms of R . Let

$T_+^k =$ the set of elements of T_-^k which are increasing,

$P_+^k = \rho_k(T_+^k)$

$T_-^k =$ the set of elements of T^k which are decreasing,

$P_-^k = \rho_k(T_-^k)$

REMARK 2.

(i) $T_+^k \cap T_-^k = \emptyset$, $T_+^k \cup T_-^k = T^k$

(ii) T_+^k is a subgroup of T^k which is of index 2 and is therefore normal.

(iii) $T_+^k T_-^k = T_-^k T_+^k = T^k$, $T_-^k T_-^k = T_+^k$

The corresponding statements are true of P_+^k and P_-^k .

Repeatedly used subsequently are following obvious facts:

If $f \in T_+^k$, then $(f^{-1})_+ = f_+^{-1}$ and $(f^{-1})_- = f_-^{-1}$.

If $f \in T_-^k$, then $(f^{-1})_+ = f_-^{-1}$ and $(f^{-1})_- = f_+^{-1}$.

PROPOSITION 1. *If $g \in T_+^k$, $g^m = 1$ for some integer $m \neq 0$, then $g = 1$.*

COROLLARY. *If $g \in T^k$ and $g^m = 1$ for some integer $m \neq 0$, then $g^2 = 1$.*

Proof of Proposition 1. Let $G \in C^k$ such that $\phi_k(G) = g$ and $g^m = 1$. G is clearly monotone near 0. Thus for small $x > 0$, if $x < G(x)$, then $x < G(x) < \dots < G^m(x) = x$. This is impossible. If $x > G(x)$ a similar argument applies. This constitutes a proof of this proposition.

It is to be noted that Proposition 1 follows trivially from a theorem in Bochner [1] in the case $k > 0$.

REMARK 2. $f \in T_-^k$ and $f^2 = 1$, if and only if $f_- = f_+^{-1}$.

4. Factorization of Reversible Transformations of the Real line. In this section the possibility of factoring a reversible element of T^k is investigated.

Df: If $f \in T_-^k$, then \bar{f} is the element of T^0 such that,

$$(\bar{f})_+ = f_+ \text{ and } (\bar{f})_- = f_+^{-1}.$$

LEMMA 1. *If $f \in T_-^k$ and $\rho_k(f)$ is an involution in P^k , then \bar{f} is an involution in T^k .*

Proof. $(f^{-1})_+ = f_-^{-1}$ and $(f^{-1})_- = f_+^{-1}$ since $f \in T_-^k$. $\rho_k(f^2) = 1$ implies $\rho_k(f^{-1}) = \rho_k(f)$. Therefore the right derivatives of f_+ at zero are the same as the left derivatives of f_+^{-1} at zero. Therefore, $\bar{f} \in T^k$ and $\bar{f}^2 = 1$ by Remark 2.

THEOREM 1. *If $f \in T_+^k$ ($0 \leq k \leq \infty$ or $k = \omega$), then f is the product of two involutions in T^m , ($m \leq k$) if and only if there exists $g \in T_-^m$ such that $gfg^{-1} = f^{-1}$ and $\rho_m(g)$ is an involution in P^m .*

Proof of Theorem 1. If $f = hk$, $h, k \in T^m$ and $h^2 = k^2 = 1$, then by Remark 2 and Proposition 1 and its corollary one can conclude that either $h, k \in T^m$ or $f = 1$. If $h, k \in T_-^m$, set $g = h$. If $f = 1$ set g equal to any involution in T_-^m . In either case $\rho_m(g)$ is also an involution.

Assume now that $f^{-1} = gfg^{-1}$ where $g \in T_-^m$ and $\rho_m(g)$ is an involution. If one sets $(\bar{g})_+ = g_+$, $(\bar{g})_- = g_+^{-1}$ it is then easy to verify that $gfg^{-1} = f^{-1}$ implies $\bar{g}f\bar{g}^{-1} = f^{-1}$. This means that $f = \bar{g}(\bar{g}f)$ and

both factors are involutions.

Some of the preceding material can be utilized to demonstrate the impossibility of factoring each reversible element of T^0 into the product of two involutions. This is embodied in the following proposition:

PROPOSITION 3. *There exists a reversible element of T^0 which cannot be factored into the product of two involutions.*

Proof of Proposition 3. Suppose there exists an element $f \in T_+^0$ such that f is reversible and $f_- = 1$ and $f_+ \neq 1$. It is clear that $gfg^{-1} = f^{-1}$ implies that $g \in T_+^0$. If such an element exists it cannot be factored as the product of two involutions by Theorem 1. An element of this type is constructed below:

Let $f = \phi_0(F)$ where $F_- = x$, $F \in C^0$

$$\text{and } F_+ = \begin{cases} F_0 & x \in (1/2, 1] \\ F_1 & x \in (1/4, 1/2] \\ \vdots & \\ F_k & x \in (2^{-k-1}, 2^{-k}] \\ \vdots & \vdots \end{cases}$$

and

$$F_{k+1} = \hat{F}_k \text{ and } \hat{F}_k(x) = (1/2 F_k(2X))^{-1}$$

and

$$F_0(x) = \begin{cases} x + \frac{1}{M} \exp [(x-1)^{-1} - (x-1/2)^{-1}] & x \in (1/2, 1) \\ 1 & x = 1 \end{cases}$$

with M large enough to ensure that F_0 is one to one on its domain of definition. It can be easily verified that F is continuous on $(-\infty, 1]$. Moreover $2F(1/2x) = F^{-1}(x)$. Therefore $f = \phi_0(F)$ is reversible and not the product of two involutions. This completes the proof of the proposition.

REMARK 3. Composition in the group P^n is given as follows:
If

$$\alpha = \sum_{k=1}^n a_k x^k \quad \text{and} \quad \beta = \sum_{k=1}^n b_k x^k,$$

then

$$\beta\alpha = \sum_{k=1}^n A_k x^k,$$

where

$$A_k = \sum \frac{r!}{r_1! r_2! \cdots r_k!} b_r a_1^{r_1} a_2^{r_2} \cdots a_k^{r_k},$$

$$r_1 + 2r_2 + \cdots + kr_k = k$$

and

$$r = r_1 + r_2 + \cdots + r_k.$$

LEMMA 2. Suppose $\alpha \in \mathbf{P}^k$, $m \leq k$

(a)
$$\text{If } \alpha = x + a_m x^m + o(x^m),$$

then

$$\alpha^{-1} = x - a_m x^m + o(x^m).$$

(b)
$$\text{If } \alpha = -x + a_m x^m + o(x^m),$$

then

$$\alpha^{-1} = -x + (-1)^m a_m x^m + o(x^m).$$

LEMMA 3. Suppose $\alpha \in \mathbf{P}^k$, α is reversible, $m \leq k$ and $\alpha = x + a_m x^m + o(x^m)$, $a_m \neq 0$.

If $\beta\alpha\beta^{-1} = \alpha^{-1}$ and $\beta = \sum_{s=1}^k b_s x^s$, then $b_1 = -1$. Moreover, m must be even.

LEMMA 4. If $\alpha \in \mathbf{P}^2$ and $\alpha = -x + a_2 x^2$, then α is an involution.

The verification of Lemmas 2–4 is straightforward and will therefore be omitted.

Df: If $f \in \mathbf{T}^k$, $k > 0$, $m \leq k$, let

$$f^{(m)}(0) = \left. \frac{d^m F(x)}{dx^m} \right|_{x=0} \quad \text{for any } F \in \mathbf{C}^k$$

such that $\phi_k(F) = f$.

THEOREM 2. If $f \in \mathbf{T}_+^k$, $k \geq 2$, f reversible and $\rho_k(f) \neq 1$ then f can be factored as the product of two involutions in \mathbf{T}^2 .

Proof of Theorem. Let $f^{-1} = gfg^{-1}$; Lemma 3 and Lemma 4 show

that $\rho_2(g)$ is an involution; now Theorem 1 applies to the image of f in T^2 .

COROLLARY. *If $f \in T_+^\omega$ and f reversible in T^ω , then f can be factored as the product of two involutions in T^2 .*

This follows from Theorem 2 and the observation that if $f \in T_+^\omega$ and $f^{(m)}(0) = 0$ for $m > 0$, then $f = 1$.

The question of factorization of reversible elements of T_-^ω is settled positively in the following material.

LEMMA 5. *If $g, h \in T_+^\omega$ and $g^2 = h^2$, then $g = h$.*

Proof of Lemma 5. If $g-h$ has zeros in every neighborhood of 0, then $g = h$ since g and h are analytic. Assume therefore that there exists a neighborhood $(0, \varepsilon]$, $\varepsilon > 0$, such that $g(x) > h(x)$ and $g(x), h(x)$ are monotone in $(0, \varepsilon]$. Choose $x_0 > 0$ sufficiently small such that $x_0, g(x_0), h(x_0), g^2(x_0) \in (0, \varepsilon]$. Then $g(x_0) > h(x_0)$ and $g^2(x_0) > gh(x_0) > h^2(x_0)$. This contradicts $g^2 = h^2$.

THEOREM 3. *If $f \in T_-^\omega$ and f is reversible in T^ω , then f is an involution in T^ω (hence $1 \cdot f$ is the product of two involutions in T^ω).*

Proof of Theorem 3. Suppose $f^{-1} = gfg^{-1}$. The proof of this theorem is divided into two cases according as $g \in T_+^\omega$ or $g \in T_-^\omega$.

Case I. $g \in T_+^\omega$.

Moreover, $gfg^{-1} = f^{-1}$ iff $g_-f_+g_+^{-1} = f_-^{-1}$ and $g_+f_-g_-^{-1} = f_+^{-1}$ iff $f_- = g_+f_+^{-1}g_-^{-1}$ and $f_+ = g_-^{-1}f_-^{-1}g_+$.

$$\text{iff } \begin{cases} f_- = g_+f_+^{-1}g_-^{-1} \\ g_+^2 = f_+^{-1}g_-^2f_+ = (f_+^{-1}g_-f_+)^2 \end{cases} \quad (\#).$$

Since g_+ is locally increasing and analytic the foregoing implies that

$$g_+ = f_+^{-1}g_-f_+ \quad (*).$$

using Lemma 5.

Therefore

$$f_- = g_+f_+^{-1}g_-^{-1} \quad \text{by } (\#).$$

$$f_- = (f_+^{-1}g_-f_+)f_+^{-1}g_-^{-1} \quad \text{by } (*).$$

Therefore $f_- = f_+^{-1}$ so f is an involution.

Case II. Suppose $g \in T^\omega$.

Therefore, $gf g^{-1} = f^{-1}$ is equivalent to $g_+ f_- g_+^{-1} = f_-^{-1}$ and $g_- f_+ g_-^{-1} = f_+^{-1}$. The foregoing is equivalent to $g_- = f_- g_+ f_-$ and $g_+ = f_+^{-1} g_- f_+^{-1}$. This means $g_- g_+ = (f_- g_+)^2$ and $g_- g_+ = (f_+^{-1} g_-)^2$. Therefore, $(f_- g_+)^2 = (f_+^{-1} g_-)^2$. Lemma 5 implies $f_- g_+ = f_+^{-1} g_-$. This means that $f_- = f_+^{-1}$. The last statement and Remark 2 implies that f is an involution. This concludes the proof of this theorem.

REMARK. It is known that Lemma 5 is false in T^0 . It is not known if Lemma 5 is true in T^k $1 \leq k \leq \infty$. The truth of Lemma 5 for T^k would permit one to state the analogue of Theorem 6 for T^k . If Lemma 5 is false for T^k one could conclude that reversible elements of T^k could not be factored into the product of two involutions.

There is a square root lemma, weaker than Lemma 5, which will be proved which provides an additional criterion for the factorability of a reversible element of T^k into the product of two involutions.

LEMMA 6. If $f, g \in T^k_+$, $f'(0) = a \neq 1$, $k \geq 2$, $f^2 = g^2$, then $f = g$.

Proof of Lemma 6. If $f^2 = g^2 = l$, then $l'(0) = a^2$. By Sternberg [5] Theorem 2 there exists $h \in T^{k-1}_+$ such that $h l h^{-1} = \phi_{k-1}(a^2 x)$. It is now shown that the only differentiable square root of $\phi_{k-1}(a^2 x)$ in T^{k-1}_+ is $\phi_{k-1}(ax)$. Suppose there is another square root of $\phi_{k-1}(a^2 x)$ given by $\phi_{k-1}(K(x))$. Consider some set $(0, \eta)$ on which $K^2(x) = a^2 x$, $(\eta > 0)$. It is clear that $K(x) > ax$ or $K(x) < ax$ cannot hold in $(0, \eta)$. Therefore there is some point $x_0 \in (0, \eta)$ where $ax = K(x)$. Assume further that $a < 1$. If this is not the case one can apply the argument to l^{-1} .

In the argument following it will be convenient to utilize the notation $(F^m(y_0))'$ for $d/dx F^m(x)|_{x=y_0}$.

Let $x_k = K^k(x_0) = a^k x_0$. Since $a < 1$ $\lim_{k \rightarrow \infty} x_k = 0$ monotonically. $K'(x) \neq ax$ at some point in (x_1, x_0) ; otherwise $K(x) = ax$ on $(0, \eta)$ and there is nothing to prove. Assume, therefore, that there exists $y_0 \in (x_1, x_0)$ such that $K'(y_0) = \beta \neq a$. Let $Z_k = K^{2k-1}(y_0)$. Since $x_{2k} < Z_k < x_{2k-1}$ one observes that $\lim_{k \rightarrow \infty} Z_k = 0$. Since $K^2(x) = a^2 x$, one observes that $K'(0) = a$ by the chain rule and the fact that $K'(0) > 0$. This implies $\lim_{k \rightarrow \infty} K'(Z_k) = a$.

But,

$$\begin{aligned} K'(Z_k) &= K' K^{2k-1}(y_0) = \frac{(K^{2k}(y_0))'}{(K^{2k-1}(y_0))'} \\ &= \frac{(K^{2k}(y_0))'}{(K^{2k-2}(y_0))' K'(y_0)'} \end{aligned}$$

Therefore,

$$K'(Z_k) = \frac{\alpha^{2k}}{\alpha^{2k-2}\beta} = \frac{\alpha^2}{\beta} \neq \alpha.$$

Therefore $\lim_{k \rightarrow \infty} K'(Z_k) \neq K'(0)$ which contradicts the continuity of $K'(x)$. This means that $\phi_{k-1}(ax)$ is the only square root of $\phi_{k-1}(a^2x)$ in $T_+^{k-1}(R, 0)$.

Since hfh^{-1} and hgh^{-1} are differentiable square roots of $\phi_{k-1}(a^2x)$, $hfh^{-1} = hgh^{-1}$ which implies $f = g$. This concludes the proof.

The foregoing leads to the following theorem.

THEOREM 4. *If $f \in T_+^k$, $k \geq 2$, f reversible in T_+^k and $f^{-1} = gfg^{-1}$ $g \in T_+^k$ and $g'(0) \neq 1$, then f is an involution.*

Proof of Theorem 4. The proof utilizes the construction in the proof of Theorem 3 and will therefore be omitted.

5. Conclusions. In this paper Question (#) of the introduction is shown to have answer "no" for continuous homeomorphisms and nearly "yes" for analytic homeomorphisms. For C^k homeomorphisms, $1 \leq k \leq \infty$, the reply to the Question is shown to depend on the existence of unique square roots in a particular group and the non-existence of an element $f \in T_+^k$ such that $\rho_k(f) = 1$ and f is reversible in T_+^k . It is not known whether $gfg^{-1} = f^{-1}$ in T^ω , $f \in T_+^\omega$, $f \neq 1$ implies that \bar{g} is more than twice differentiable. One conjectures that this is not the case since it may be shown that $\alpha\beta\alpha^{-1} = \beta^{-1}$ in P^3 , $\beta \neq 1$ and $\beta = x + b_3x^3$, does not ensure that $\alpha^2 = 1$.

One may generalize some of this material to higher dimensions. The techniques of this paper are applicable to some germs of transformations of R^n which fix a sufficiently nice $n - 1$ manifold containing the origin. This and related questions will be treated in a subsequent paper.

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