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UNKNOTTING CONES IN THE TOPOLOGICAL CATEGORY

R. RICHARD SUMMERHILL

Let Q be a topological q-manifold, let X be a compact metric space, and let bQ and aX denote the cones over Q and X, respectively. A proper embedding $f: aX \to bQ$ (i.e., $f(a) = b$ and $f^{-1}[Q] = X$) is unknotted if there is homeomorphism $h: bQ \to bQ$ such that $hf = \bar{f}$, where $\bar{f}$ is the conical extension of f. In this paper it is proved that a proper embedding is unknotted if and only if $bQ - f[aX]$ and $bQ - \bar{f}[ax]$ are of the same homotopy type and the embedding f satisfies a local flatness condition.

In this paper we present a topological analog to Lickorish's theorem concerning the PL unknotting of cones [7]. The PL result states that if one embeds the cone over a complex into a ball (with a codimension restriction) such that the base and only the base of the cone sits in the boundary of the ball, then one can deform the ball (without moving the boundary) so as to straighten out the cone. The codimension requirement is that the dimension of the cone be at least three less than the dimension of the ball.

We consider here a similar problem in the topological category where the complex is replaced by a compact metric space and the ball is replaced by the cone over a topological manifold. Homotopy conditions are used instead of codimension, and, of course, some local flatness condition is needed. This condition generalizes that property for manifolds and is defined by using the inherent fibre structure of the cone.

Our main theorem is then: An embedding of the cone over a compact metric space into the cone over a compact topological manifold is unknotted if and only if (1) certain homotopy properties are satisfied and (2) the embedding is "locally flat."

The proof of this theorem follows precisely the same outline as the proof of the unknotting theorem by Price and Glaser (Theorem 1 of [4]), but uses topological engulfing in place of PL engulfing.

1. Definitions. Throughout this paper the term manifold will be used in the topological sense. That is, a q-manifold Q is a separable metric space in which each point has a closed neighborhood homeomorphic to a q-cell. Let BdQ denote the boundary of Q and Int$Q$ the set Q-BdQ. The manifold Q is *closed* if it is compact and without boundary. We let I denote the closed unit interval $[0, 1]$

and I' the half-open unit interval $[0, 1)$. The symbol 1 will represent the identity map.

Let Q be a compact q-manifold and let X be a compact metric space. The *cone* over X, denoted aX, is the quotient space of $X \times I$ obtained by pinching $X \times 1$ to a point. We denote the point $X \times 1$ by a and identify X with $X \times 0$. X is called the *base* of the cone aX.

An embedding $f: aX \to bQ$ of the cone over X into the cone over Q is *proper* if $f(a) = b$ and $f^{-1}[Q] = X$. If f is a proper embedding, let $\bar{f}: aX \to bQ$ be the map defined by sending a to b, x to $f(x)$ if $x \in X$, and by extending linearly over the line segments ax in aX. Then $\bar{f}$ is a homeomorphism of aX onto $bf[X] \subset bQ$ and is called the *conical extension* of f.

A proper embedding $f: aX \to bQ$ is *unknotted* or *flat* if there is a homeomorphism $h: bQ \to bQ$ which is fixed on Q and such that $hf = \bar{f}$. We say that f is *locally flat at the point* $p \in aX$ if there is an open set U in bQ containing $\bar{f}(p)$ and an embedding $h: U \to bQ$ such that

(1) $h[U]$ is a neighborhood of $f(p)$,

(2) $h^{-1}[f[ax] \cap h[U]] = U \cap \bar{f}[ax]$ for each $x \in X$,

(3) if $U \cap Q \neq \varnothing$, then $h | U \cap Q = 1$, and

(4) if U contains b, then $h(b) = b$.

The embedding f is *locally flat* if it is locally flat at each point of $aX - a$ (the reason for not requiring local flatness at the point a will be apparent in the proof of the main theorem).

REMARK. An embedding f is locally flat at a point $p \neq a$ if and only if there exists an embedding $h: I^q \times I \to bQ$ such that

(1) $h[I^q \times I]$ is a neighborhood of $f(p)$ in bQ and if $p \in X$, then $h[I^q \times 0]$ is a neighborhood of $f(p)$ in Q, and

(2) for each $b \in X$, $h^{-1}[f[ax] \cap h[I^q \times I]] = z \times I$ for some $z \in I^q$.

Let X_0 be a compact subset of X and let $f: aX \to bQ$ be a proper embedding. We say that f is an *allowable* embedding and write $f: a(X, X_0) \to b(Q, BdQ)$ if $f^{-1}[bBdQ] = aX_0$. That is, the cone over X_0 (and nothing else) maps into the cone over BdQ. Note that $f | aX_0$: $aX_0 \to bBdQ$ is a proper embedding.

Now let N be a compact n-manifold, Y a compact metric space, and $g: aY \to bN$ a proper embedding. Then (N, Y, g) satisfies (*) or (**), respectively, if

(*) the pair $(bN - g[aY], N - g[Y])$ is $(n-2)$-connected, or

(**) b has arbitrarily small neighborhoods U in bN such that the pair $(bN - g[aY], U - g[aY])$ is $(n-2)$-connected.

An allowable locally flat embedding $f: a(X, X_0) \to b(Q, BdQ)$ is said to be *simple* if

(1) (Q, X, f) satisfies (*) and (**), and

(2) $(BdQ, X_0, f \mid X_0)$ satisfies (*) and (**).

Note that if Q is a closed manifold, then only (1) is meaningful.

REMARK. The theorems to be proved here do not require the full strength of (*) and (**). Either (*) or (**) may be weakened by replacing $(n-2)$ with 2.

2. Embeddings into cones over closed manifolds.

In this section let X be a compact metric space, Q a closed q-manifold, and $f: aX \to bQ$ a proper embedding. To show that f is unknotted we shall prove that the pairs $(bQ - b, f[aX] - b)$ and $(bQ - b, \bar{f}[aX] - b)$ are homeomorphic. This is accomplished by obtaining a collar of Q in bQ in which the fibers of the collar and the fibers of the cone are aligned and then by pushing the collar toward the cone point.

The existence of the desired type of collar follows by carefully examining the proof of Brown's local collaring theorem (Theorem 1 of [2]). One need only note that the procedure of piecing local collars together can be accomplished without destroying the fiber preserving property.

LEMMA 1. *If $f: aX \to bQ$ is a locally flat embedding, then there is an embedding $h: Q \times I' \to bQ$ such that*

(1) *$h(x, 0) = x$ for each $x \in Q$ and*

(2) *$h[f(x) \times I'] = h[Q \times I'] \cap f[ax]$ for each $x \in X$.*

The open subset $U = h[Q \times I']$ of bQ, where h is an embedding as in Lemma 1, is called a *strong collar* of $(Q, f[X])$. If t is a real number, $0 < t < 1$, the subset $h[Q \times [0, t)]$ is called a *subcollar* of U and $h[Q \times [0, t]]$ is called a *closed subcollar* of U.

LEMMA 2. *Let $f: aX \to bQ$ be a locally flat proper embedding, let U be a strong collar of $(Q, f[X])$, and let V be a subcollar of U. If C is a compact subset of bQ not containing b, then there is a homeomorphism $h: bQ \to bQ$ such that*

(1) *$h \mid V \cup b = 1$,*

(2) *$h[f[ax]] = f[ax]$ for each $x \in X$, and*

(3) *$h[U] \supset C \cap f[aX]$.*

Proof. Cover the set $f[X] - V$ with finitely many open sets granted by the definition of locally flat and then push U up toward the cone point by sliding it through these open sets.

LEMMA 3. *If* $f\colon aX \to bQ$, $g \geqq 4$, *is a simple embedding, then* $bQ - b$ *is a strong collar of the pair* $(Q, f[X])$.

Proof. Let U be a strong collar of $(Q, f[X])$, let $g\colon Q \times I' \to bQ$ be the defining embedding for U, and for each $i = 1, 2, \cdots$, let $U_i = g[M \times [0, i/(i+1))]$. Let $V_1, V_2, \cdots$ be a monotone decreasing sequence of open subsets of bQ which squeeze down on the point b and such that the pair $(bQ - f[aX], V_i - f[aX])$ is $(q-2)$-connected for each $i = 1, 2, \cdots$. Let $D_i = bQ - V_i$.

We construct a sequence of homeomorphisms $h_1, h_2, \cdots$ such that

(a) $h_i | Q = 1$

(b) $h_i[U_i] \supset D_i$ for each $i = 1, 2, \cdots$,

(c) $h_i | U_{i-1} = h_{i-1} | U_{i-1}$ for each $i = 2, 3, \cdots$, and

(d) $h_i[f[ax]] = f[ax]$ for each $i = 1, 2, \cdots$.

By induction assume that $h_i, \cdots, h_{i-1}$ have been chosen and apply Lemma 2, with C replaced by D_i and V replaced by $h_{i-1}[U_{i-1}]$, to obtain a homeomorphism h' satisfying the conclusions (1) and (2) of Lemma 2 and such that $h'h_{i-1}[U_i] \supset D_i \cap f[aX]$. Then employ the topological engulfing methods of Connell [3] and Newman [8]. The homotopy conditions on $(bQ - f[aX], Q - f[X])$ and $(bQ - f[aX], V_i - f[aX])$ are sufficient to apply the proof of Theorem 1 of [3] to obtain a homeomorphism $h''\colon bQ - f[aX] \to bQ - f[aX]$ such that $h''h'h_{i-1}[U_i] \supset D_i$. It is easily seen that h'' can be defined so as not to move points outside a compact set and therefore can be extended by the identity on $f[aX]$. Then the homeomorphism $h_i = h''h'h_{i-1}$ completes the induction argument.

Finally, set $h = \lim h_i | U$. Then h is a homeomorphism from U onto $bQ - b$ which preserves the alignment between the fibers of U and $f[aX]$ and hence $bQ - b$ is a strong collar of $(Q, f[X])$.

The following proposition is essentially a corollary of the previous lemma.

PROPOSITION 1. *If* $f\colon aX \to bQ$, $q \geqq 4$, *is a simple embedding, then there is a homeomorphism* $h\colon bQ \to bQ$ *which is fixed on* Q *and such that* $h[f[ax]] = \bar{f}[ax]$ *for each* $x \in X$.

LEMMA 4. *Let* Y *be a compact metric space and let* X *be a compact subset of* Y. *Suppose* $f\colon X \times I \to Y \times I$ *is an embedding such that* $f | X \times \{0, 1\} = 1$ *and* $f[x \times I] = x \times I$ *for each* $x \in X$. *Then there is a homeomorphism* $h\colon Y \times I \to Y \times I$ *such that*

(1) $h | Y \times \{0, 1\} = 1$,

(2) $h[y \times I] = y \times I$ *for each* $y \in Y$, *and*

(3) $hf = 1$.

Proof. Let $p_2: Y \times I \to I$ denote the projection on the second factor and let t_0 be some fixed real number, $0 < t_0 < 1$. Then $p_2 f: X \times t_0 \to \mathrm{I}$ is a continuous map and $p_2 f[X \times t_0] \subset (0, 1)$. By the Tietze extension theorem there is a map $p: Y \times t_0 \to (0, 1)$ which extends $p_2 f \mid X \times t_0$. Define a map $g: Y \times t_0 \to Y \times I$ by $g(y, t_0) = (y, p(Y, t_0))$. Then g embeds $Y \times t_0$ into $Y \times I$, extends $f|X \times t_0$, and has the property that $g(y, t_0) \in y \times (0, 1)$ for each $y \in Y$.

Now define a map $\bar{h}: Y \times I \to Y \times I$ by

$$\bar{h}(y, t) = \begin{cases} \left(y, \dfrac{1 - p(y, t_0)}{1 - t_0} (t - t_0) + p(y, t_0) \right) \text{ if } t_0 \leqq t \leqq 1 \\ \left(y, \dfrac{p(y, t_0)}{t_0} t \right) \text{ if } 0 \leqq t \leqq t_0 \, . \end{cases}$$

Then $\bar{h}$ is a homeomorphism and satisfies (1) and (2). If we set $h' = h^{-1}$, then h' satisfies (1) and (2) and $h'g(y, t_0) = (y, t_0)$ for each $y \in Y$. In particular, $h'f(x, t_0) = h'g(x, t_0) = (x, t_0)$ for each $x \in X$.

The desired homeomorphism is now constructed as the limit of a sequence of homeomorphisms obtained by applying the above construction as t_0 varies over the dyadic rationals in I. Let $g: Y \times I \to Y \times I$ be the homeomorphism h' constructed above for $t_0 = 1/2$. Let $g_2': Y \times [0, 1/2] \to Y \times [0, 1/2]$ and $g_2'': Y \times [1/2, 1] \to Y \times [1/2, 1]$ be homeomorphisms constructed as above for $t_0 = 1/4$, $t_0 = 3/4$ and the embedding $g_1 f$, respectively. Combining g_2' and g_2'' at $Y \times 1/2$, we obtain a homeomorphism $g_2: Y \times I \to Y \times I$ such that

(a) $g_2 \mid Y \times k/2 = 1$ for each $k = 0, 1,$ and 2,

(b) $g_2 g_1 f(x, k/4) = (x, k/4)$ for each $x \in X$ and $k = 0, 1, 2, 3,$ and 4, and

(c) if $(y, t) \in Y \times [(k - 1)/2, k/2]$, then

$$g_2(y, t) \in y \times [(k - 1)/2, k/2] \text{ for each } k = 1 \text{ and } 2 \, .$$

Continuing this procedure for all the diadic rationals in I, we obtain a sequence $g_1, g_2, g_3, \cdots$ such that

(d) $g_n \mid Y \times k/2^{n-1} = 1$ for each $k = 0, 1, \cdots, 2^{n-1}$,

(e) $g_n \cdots g_1 f(x, k/2^n) = (x, k/2^n)$ for each $x \in X$ and $k = 0, 1, \cdots, 2^n$, and

(f) if $(y, t) \in Y \times [(k - 1)/2^{n-1}, k/2^{n-1}]$, then $g_n(y, t) \in y \times [(k - 1)/2^{n-1}, k/2^{n-1}]$ for each $k = 1, 2, \cdots, 2^{n-1}$.

For each $n = 1, 2, \cdots$, let $h_n = g_n \cdots g_1$. Then $h = \lim h_n$ is a homeomorphism and satisfies the desired conclusions.

In terms of cones, the previous lemma becomes

PROPOSITION 2. *Let Y be a compact metric space and let X be a* *compact subset of* Y. *Suppose* $f\colon aX \to aY$ *is an embedding such that* $f \mid X \cup a = 1$ *and* $f[ax] = ax$ *for each* $x \in X$. *Then there is a homeomorphism* $h\colon aY \to aY$ *such that*

(1) $h \mid Y \cup a = 1$,

(2) $h[ay] = ay$ *for each* $y \in Y$, *and*

(3) $hf = 1$.

We are now in a position to prove the unknotting theorem for closed manifolds. Proposition 1 indicates that the embedded cone can be pushed onto the straight cone in such a way that the corresponding fibers are aligned and then Proposition 2 shows that these fibers can be matched in a pointwise fashion.

THEOREM 1. *An embedding* $f\colon aX \to bQ$, $q \geqq 4$, *is unknotted if and only if it is simple.*

Proof. The "only if" part is trivial. Suppose then that f is simple and let $h_1\colon bQ \to bQ$ be the homeomorphism granted by Proposition 1; that is $h_1 \mid Q \cup b = 1$ and $h_1[f[ax]] = \bar{f}[ax]$ for each $x \in X$. Then $h_1 f \bar{f}^{-1}\colon bf[X] \to bf[X] \subset bQ$ is an embedding satisfying the hypotheses of Proposition 2 (recall that $\bar{f}$ is the conical extension of f) and therefore there is a homeomorphism $h_2\colon bQ \to bQ$ such that $h_2 \mid Q \cup b = 1$, $h_2[by] = by$ for each $y \in Q$, and $h_2 h_1 f \bar{f}^{-1} = 1$. Then $h = h_2 h_1$ is the homeomorphism which unknots f.

If Y is a topological space, an ambient isotopy of Y is a level preserving homeomorphism $H\colon Y \times I \to Y \times I$ such that $H \mid Y \times 0 = 1$. The statement that H is fixed on a subset A of Y means that $H \mid A \times I = 1$.

COROLLARY 1. *If* $f\colon aX \to bQ$, $q \geqq 4$, *is a simple embedding, then there is an ambient isotopy* H *of* bQ *which is fixed on* $Q \cup b$ *and such that* $H_1 f = \bar{f}$. *Moreover, if* X_0 *is a compact subset of* X *and* $f \mid aX_0 = \bar{f} \mid aX_0$, *then* H *may be chosen so as to be fixed on* $f[aX_0]$.

Proof. Let $h\colon bQ \to bQ$ be a homeomorphism which unknots f. Define $H\colon bQ \times I \to bQ \times I$ by letting it equal the identity on $(bQ \times 0) \cup [(Q \cup b) \times I]$ and h on $bQ \times 1$. Then extend to all of $bQ \times I$ by coning over the point $(b, 1/2)$.

COROLLARY 2. *Let* $f\colon aM \to bQ$, $q \geqq 4$, *be a proper embedding where* M *is a compact manifold. Suppose the pair* $(bQ - f[aM], Q - f[M])$ *is* $(q-2)$*-connected and suppose* b *has arbitrarily small neighborhoods* U *in* bQ *such that the pair* $(bQ - f[aM], U - f[aM])$ *is* $(q -$

2)-connected. If $f[aM - a]$ is a locally flat submanifold of $bQ - b$, then f is unknotted.

Proof. It need only be shown that f is locally flat in the sense defined above. But this follows trivially from the fact that any locally flat embedding (in the manifold sense) of the k-cell D^k into E^n extends to a homeomorphism of E^n.

3. Embeddings into cones over manifolds with boundary. In this section we extend the unknotting theorem to include manifolds with possibly nonempty boundary. The proof is essentially the same as that of Theorem 1 and we therefore only indicate in what order the various engulfing stages are to be done.

THEOREM 2. *An allowable embedding $f: a(X, X_0) \rightarrow b(Q, BdQ)$, $q \geqq 5$, is unknotted if and only if it is simple. Moreover, if f has the property that $f|aX_0 = \bar{f}|aX_0$, then the unknotting homeomorphism may be chosen so that it is the identity on $bBdQ$.*

Proof. Let D be a compact subset of $bQ - b$. As in Lemmas 1 and 2 there is a strong collar $U = h[Q \times I']$ of $(Q, f[X])$ containing $D \cap f[ax]$. In addition, U can be selected so that $h^{-1}[bBdQ] = BdQ \times I'$.

Now consider $U \cap bBdQ$. Using properties (*) and (**) with respect to $(BdQ, X_0, f|X_0)$, and the engulfing theorem of Connell [3] and Newman [8], $U \cap bBdQ$ can be pushed (in $bBdQ$) toward b to cover $D \cap bBdQ$. Since this engulfing homeomorphism is actually realized by an ambient isotopy of $bBdQ$, it can be extended to bQ without moving $f[aX]$. Thus we may assume that $U \cap bBdQ$ contains $D \cap bBdQ$.

Applying the engulfing theorem again, this time using (*) and (**) with repect to (Q, X, f), the collar U can be pushed toward b until it contains all of D. This is the necessary condition needed to complete the proof of Lemma 3 in this case and hence the first part of the theorem is proved.

Now suppose $f|aX_0 = \bar{f}|aX_0$ and let h be an unknotting homeomorphism. We need to adjust h so that $h|bBdQ$ is the identity. By applying Corollary 1 to $h|bBdQ$, there is an ambient isotopy H which realizes $h|bBdQ$ and is fixed on $f[aX_0]$ and $BdQ \cup b$. Then H can be used in conjunction with the cone over a collar of $BdQ - f[X_0]$ in $Q - f[X]$ to obtain a homeomorphism h': $bQ \rightarrow bQ$ which is fixed on $\bar{f}[aX]$ and $Q \cup b$ and such that $h'h|bBdQ$ is the identity. Then $h'h$ is the desired unknotting homeomorphism.

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UNIVERSITY OF MISSOURI, COLUMBIA