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ON POLYNOMIALS APPROXIMATING THE SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS

A. BACOPOULOS AND A. G. KARTSATOS

Suppose that $L(x)$ is a differential operator and $R(t)$ a continuous function, and consider the differential equation (*) $L(x) = R(t)$. Then a problem in approximation theory is whether we can approximate a solution $x(t)$ of (*) uniformly with a sequence of polynomials P_n for which we have $||R(t) - L(P_n))|| \leqq \eta_n$, where $||\cdot||$ is a certain norm and η_n a specific sequence of nonnegative constants. This is done here for a first order nonlinear differential operator L and for two different norms, the uniform norm and the L_p norm $(1 \leqq p < +\infty)$.

Consider the differential equation:

$$(1) \qquad L(x) \equiv x' + Q(t, x) = R(t)$$

where the functions Q, R are defined and continuous on $[0, 1] \times (-\infty, +\infty)$ and $[0, 1]$ respectively. Suppose also that there exists a unique solution $x(t)$ *of* (1) satisfying $x(0) = 0$. On $C[0, 1]$ consider the norms:

$$(2) \qquad ||f|| = \sup_{t \in [0,1]} |f(t)|,\ ||f||_p = \left[\int_0^1 |f(t)|^p dt\right]^{1/p} (p \geqq 1)\ ,$$

and let

$$(3) \qquad \mu_n = \inf_{P_n \in \Pi_n} ||L(x) - L(P_n)||,\ \mu_n^* = \inf_{P_n^* \in \Pi_n} ||L(x) - L(P_n^*)||_p\ ,$$

where $x(t)$ is the solution of (1) with $x(0) = 0$, and Π_n is the set of all polynomials of degree less than or equal to n, which satisfy the condition $P_n(0) = 0$, (or $P_n^*(0) = 0$). By (3), if ε_n, (or ε_n^*), is a sequence of positive constants such that $\lim_{n\to\infty} \varepsilon_n = 0$, (or $\lim_{n\to\infty} \varepsilon_n^* = 0$), then there exist sequences of polynomials P_n, (or P_n^*), $\in \Pi_n$ such that

$$(4) \qquad ||L(x) - L(P_n)|| \leqq \mu_n + \varepsilon_n,\ ||L(x) - L(P_n^*)||_p \leqq \mu_n^* + \varepsilon_n^*\ ,$$

for every $n = 1, 2 \cdots$.

Our aim here is to show that, for quite a large class of equations of the type (1), it is possible to have the polynomials satisfying the first or the second of (4) converge uniformly, along with their derivatives, to the solution $x(t)$ and its derivative respectively.

It should be noted that if the infimum in either one of (3) is attained for every n (and this is not always true), then we can choose $\varepsilon_n = 0$, (or $\varepsilon_n^* = 0$), $n = 1, 2, \cdots$, and consider in (4) only the equality sign.

The results of this paper are related to those of Huffstutler and Stein [2], [3], which are taken as special cases for certain classes of equations of the form (1).

In what follows, $D = \max_{1 \leqq i \leqq m-1} \{1, \sup_{t \in [0,1]} \{|x(t)|^i\}\}$.

2. Main result. THEOREM. *Let the function Q be such that*

$$|Q(t, u) - Q(t, v)| \leqq A \sum_{k=1}^{m} |u^k - v^k|$$

for every $(t, u, v) \in [0, 1] \times (-\infty, +\infty) \times (-\infty, +\infty)$ where

$$A \text{ is} \begin{cases} \text{any positive constant for } m = 1 \\ \text{such that } AD < [m(m-1)]^{-1} \text{ for } m > 1 , \end{cases}$$

and suppose further that a sequence of polynomials P_n, (or P_n^*), *satisfies the given initial condition and the first of* (4), (*the second of* 4), *for every n. Then the sequence P_n, (or P_n^*), converges uniformly to the solution $x(t)$ on [0.1]. In addition, the sequence P_n', (or $P_n^{*\prime}$), converges uniformly* (w.r.t. the L_p norm) *to the derivative $x'(t)$.*

Proof. Case I (Uniform norm). We show first that $\lim_{n\to\infty} L(P_n(t)) = L(x(t))$ uniformly on [0, 1]. In fact, there exists a sequence of polynomials S_n, of degree less than or equal to n, such that $S_n(0) = 0$, and $\lim_{n\to\infty} S_n^{(i)}(t) = x^{(i)}(t)$, $i = 0,1$, uniformly on [0,1]. We can take, for example, the Bernstein polynomials

$$S_n \equiv B_n(x; t) = \sum_{k=0}^{n} x\left(\frac{k}{n}\right) \binom{n}{k} t^k (1-t)^{n-k} . \tag{5}$$

Thus, by use of (3), we obtain

$$\begin{aligned} \|L(x) - L(P_n)\| &\leqq \mu_n + \varepsilon_n \leqq \|L(x) - L(S_n)\| + \varepsilon_n \\ &\leqq \|x' - S_n'\| + \|Q(t, x) - Q(t, S_n)\| + \varepsilon_n \\ &\leqq \|x' - S_n'\| + A \sum_{n=1}^{m} \|x^k - S_n^k\| + \varepsilon_n , \end{aligned} \tag{6}$$

and the sequences in the last member of (6) tend to zero, which shows the uniform convergence of $L(P_n(t))$.

We show next that the sequence P_n is uniformly bounded on [0, 1]. Let $u_n(t) = x(t) - P_n(t)$ and $F_n(t) = L(x(t)) - L(P_n(t))$, $t \in [0, 1]$; then from (1) we obtain

$$\begin{aligned} F_n(t) &= L(x) - L(x - u_n) = x' + Q(t, x) - [x' - u_n' + Q(t, x - u_n)] \\ &= u_n' + [Q(t, x] - Q(t, x - u_n)] \end{aligned}$$

which gives

$$
\begin{aligned}
|u_n(t)| &\leqq \int_0^1 |F_n(s)|\,ds + \int_0^t |Q(s, x(s)) - Q(s, x(s) - u_n(s))|\,ds \\
&\leqq \int_0^1 |F_n(s)|\,ds + A\int_0^t \Big[\sum_{k=1}^m |x^k(s) - (x(s) - u_n(s))^k|\Big]\,ds \\
&\leqq A_n + A\int_0^t \Big[\sum_{k=1}^m |x^k(s) - (x^k(s) - kx^{k-1}(s)u_n(s) + \cdots \\
&\qquad + (-1)^k u_n^k(s))|\Big]\,ds \\
&\leqq A_n + A\int_0^t \Big[\sum_{k=1}^m (k|x(s)|^{k-1}|u_n(s)| + \cdots + |u_n(s)|^k)\Big]\,ds \qquad (7)\\
&\leqq A_n + AD\int_0^t \Big[\sum_{k=1}^m (k|u_n(s)| + (k(k-1)/2!)|u_n(s)|^2 + \cdots \\
&\qquad + |u_n(s)|^k)\Big]\,ds \\
&\leqq A_n + AD\int_0^t \Big[\sum_{k=1}^m (1 + |u_n(s)|)^k\Big]\,ds \\
&\leqq A_n + mAD\int_0^t (1 + |u_n(s)|)^m\,ds\ ,
\end{aligned}
$$

where A_n is a constant determined by (6).

If $m = 1$, then the uniform boundedness follows easily from (7) by a direct application of Gronwall's inequality ([1], p. 8). Let $m > 1$, $q_n(t) = \int_0^t (1 + |u_n(s)|)^m ds$, and choose $\varepsilon > 0$ such that $AD < [(1+\varepsilon)^{m-1} m(m-1)]^{-1}$ and $A_n < \varepsilon$ for every $n \geqq$ (some) N. Then from (7) we have

$$
\begin{aligned}
q_n'(t) &\leqq [1 + A_n + mADq_n(t)]^m \\
&\leqq [1 + \varepsilon + mADq_n(t)]^m \quad (n \geqq N)
\end{aligned}
\tag{8}
$$

which, dividing by the last member and integrating from 0 to $t \geqq 0$, yields

$$
(1 + \varepsilon) + mADq_n(t) \leqq [(1 + \varepsilon)^{1-m} - m(m-1)AD]^{-1/(m-1)}\ , \tag{9}
$$

which shows the uniform boundedness of

$$
[q_n'(t)]^{1/m} - 1 = |u_n(t)| = |x(t) - P_n(t)|
$$

and, consequently, the uniform boundedness of the sequence P_n.

Now, we use the uniform boundedness of P_n in order to show their convergence to the solution $x(t)$. From (1) we obtain

$$
\begin{aligned}
|x(t) - P_n(t)| &\leqq \int_0^t |F_n(s)|\,ds + \int_0^t |Q(s, x(s) - Q(s, P_n(s))|\,ds \\
&\leqq \int_0^t |F_n(s)|\,ds + A\sum_{k=1}^m \int_0^t |x - P_n||x^{k-1} + x^{k-2}P_n \qquad (10)\\
&\qquad + \cdots + P_n^{k-1}|\,ds \\
&\leqq \int_0^t |F_n(s)|\,ds + mAK\int_0^t |x(s) - P_n(s)|\,ds
\end{aligned}
$$

where $K = \max_{1 \leqq k \leqq m} \sup_{t \in [0,1]} \{|x^{k-1}(t) + x^{k-2}(t)P_n(t) + \cdots + P_n^{k-1}(t)|\}$ independent of n, due to the uniform boundedness of the P_n's. Thus, an application of Gronwall's inequality in (10) gives

$$|x(t) - P_n(t)| \leqq \left(\int_0^t |F_n(s)|ds\right). \quad e^{mAK} \leqq \left(\int_0^1 F_n(t)\ dt\right) e^{mAK}.$$

Since the right side $\to 0$ as $n \to \infty$, this proves the uniform convergence of the sequence P_n.

The proof of the uniform convergence of the derivatives of the P_n's follows from

$$\begin{aligned} |x'(t) - P_n'(t)| &\leqq |L(x) - L(P_n)| + |Q(t, x) - Q(t, P_n)| \\ &\leqq |F_n(t)| + A \sum_{k=1}^{m} |x^k(t) - P_n^k(t)| \\ &\leqq |F_n(t)| + A \sum_{k=1}^{m} [|x(t) - P_n(t)||x^{k-1}(t) + x^{k-2}(t)P_n(t) \\ &\qquad + \cdots + P_n^{k-1}(t)|] \\ &\leqq |F_n(t)| + mAK|x(t) - P_n(t)|, \end{aligned} \tag{11}$$

and the final expression $\to 0$ as $n \to \infty$.

Case II (L_p norm). Suppose that P_n^* is a sequence of polynomials which satisfies the second of (4). Then (6) holds with P_n replaced by P_n^*, and $||\cdot||$ by $||\cdot||_p$, since the uniform convergence of $S_n^{(i)}$, $i = 0, 1$, implies their convergence w.r.t. the L_p norm. Thus, $L(P_n^*)$ converges w.r.t. the L_p norm to $L(x)$. In order to show that the P_n^*'s are uniformly bounded, choose $\varepsilon^* > 0$, N such that $AD < [(1 + \varepsilon^*)^{m-1}m(m - 1)]^{-1}$ and $||L(x) - L(P_n^*)||_p < \varepsilon^*$ for every $n \geqq N$. Then we obtain (as in (7))

$$\begin{aligned} |u_n^*(t)| &\leqq \int_0^t |F_n^*(s)|ds + mAD \int_0^t (|u_n^*(s)| + 1)^m\, ds \\ &\leqq \left[\int_0^t |F_n^*(s)|^p ds\right]^{1/p} + mAD \int_0^t \left(|u_n^*(s)| + 1\right)^m ds \\ &\leqq \varepsilon^* + mAD \int_0^t (|u_n^*(s)| + 1)^m\, ds, \end{aligned} \tag{12}$$

and the proof follows as in the case of the uniform norm. The uniform convergence of the P_n^*'s follows from an inequality similar to (10), and the L_p norm convergence of the derivatives of the P_n^*'s from

$$\begin{aligned} ||u_n^{*\prime}(t)||_p &\leqq ||F_n^*||_p + ||Q(t, x) - Q(t, x - u_n^*)||_p \\ &\leqq ||F_n^n||_p + ADE \sum_{k=1}^{m} ||u_n^*||_p \end{aligned}$$

(E is a suitable constant), and the final expression $\to 0$ as $n \to \infty$.

It should be mentioned here that if $\lim_{n\to\infty} n^2 \max_{t\in[0,1]} |P_n^*(t) - x(t)| = 0$, then the sequence $P_n^{*\prime}$ converges uniformly to $x'(t)$, and this can be shown as in [2].

3. Example. Consider the differential equation

$$(*) \quad x' + (3/37)t^2x^2/(1 + x^2) - (1/16)[t^2/(1 + t^2)]x^4/(1 + x^4) = (t - 1/2)^{1/3}.$$

Here we have $m = 4$, and

$$\begin{aligned} |Q(t, u) - Q(t, v)| &\leqq (3/37)|u^2 - v^2| + (1/16)|u^4 - v^4| \\ &\leqq (3/37)(|u^2 - v^2| + |u^4 - v^4|) \end{aligned}$$

i.e., $A = 3/37$. If $x(t)$ is a solution of (*) with $x(0) = 0$, then we have

$$\begin{aligned} |x(t)| &\leqq (3/37)\int_0^1 t^2 dt + (1/16)\int_0^1 [t^2/(1 + t)^2]\, dt + \int_0^1 (t - 1/2)^{1/3}\, dt \\ &< 3/37 + 1/16 < 1 \end{aligned}$$

and, consequently, we have $D = 1$. Moreover, if we suppose the existence of a second solution $y(t)$ of (*) with $y(0) = 0$, then we get

$$\begin{aligned} |Q(t, x) - Q(t, y)| &\leqq A(|x^2 - y^2| + |x^4 - y^4|) \\ &\leqq A|x - y|\,(|x + y| + |x|^3 + x^2|y| + |x|y^2 + |y|^3) \\ &\leqq 6A|x - y|. \end{aligned} \tag{13}$$

Now, integration of $x' - y' = -[Q(t, x) - Q(t, y)]$ and use of (13) and Gronwall's inequality, shows the uniqueness of the solution $x(t)$, $t \in [0, 1]$ of (*) with $x(0) = 0$. Furthermore, $AD = 3/37 < [m(m - 1)]^{-1} = 1/12$, and the theorem applies to the equation (*). This example is not contained in any of the results in [4], since the function $Q - R$ is not analytic for $-1 \leqq t \leqq 1$ and all x.

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MICHIGAN STATE UNIVERSITY
AND
UNIVERSITY OF ATHENS