PACIFIC JOURNAL OF MATHEMATICS
Vol. 40, No. 2, 1972

# ON THE OPERATOR $M(Y) = TYS^{-1}$ IN LOCALLY CONVEX ALGEBRAS

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**Let $\mathfrak{A}$ be a locally convex separated unitary algebra over the complex field. If $T$ and $S$ are fixed elements of $\mathfrak{A}$ and $S$ is invertible, it is possible to define on $\mathfrak{A}$ the linear operator**

$$M(Y) = M(T, S)(Y) = TYS^{-1}$$

**for all $Y \in \mathfrak{A}$. The purpose of this paper is to construct a functional calculus with analytic functions for the operator $M(T, S)$, by means of $T$ and $S$, in order to obtain "multiplicative variants" of some results of M. Rosenblum. In the last section these results are applied to normal operators and matrices.**

In what follows $U$ will be a locally convex algebra i.e., an algebra which is a locally convex space and where the multiplication is separately continuous. The topology on $\mathfrak{A}$ is defined by a family of seminorms $\{p_\alpha\}_{\alpha \in J}$ and the space is assumed to be quasi-complete, i.e., every Cauchy net is convergent. It is also supposed that the mappings $Y \to YZ$ (resp. $Z \to YZ$) are uniformly continuous when $Z$ (resp. $Y$) belongs to a bounded set.

Denoting $B(\mathfrak{A})$ the algebra of all continuous linear operators on $\mathfrak{A}$, the topology on $B(\mathfrak{A})$ is defined by the family of seminorms $\{p_{\alpha,B}\}_{\alpha \in J, B \in \mathfrak{B}}$, where

$$p_{\alpha,B}(L) = \sup_{Y \in B} p(L(Y)) ,$$

for each $L \in B(\mathfrak{A})$, $\mathfrak{B}$ being the family of all bounded sets of $\mathfrak{A}$.

When $T, S \in \mathfrak{A}$ and $S$ is invertible, it is shown that the spectrum $\sigma(M)$ of the operator

$$M(Y) = TYS^{-1}$$

is contained in the set $\sigma(T) \cdot \sigma(S^{-1})$ (Proposition 2.4) hence, taking a complex-valued function $f$, analytic in a neighbourhood of the set $\sigma(T) \cdot \sigma(S^{-1})$, we can construct the operator $f(M)$ in each point of $\mathfrak{A}$, as well by means of the functional calculus of $T, S$ and they are equal (Theorem 2.9). Since the logarithm of an element of $\mathfrak{A}$ does not always exist, this case is more general than Rosenblum's results (see [3] and Proposition 3.1).

All the statements of this paper can be applied to linear operators on Banach spaces or on locally convex ones, with supplementary properties.

**1. Preliminaries.** It is useful to recall some of the concepts and results contained in [1] and [4].

On a locally convex algebra it is possible to define the spectrum $\sigma(T)$ and the resolvent $\rho(T)$ of an element $T \in \mathfrak{A}$ (these sets are considered in the complex compactified plane $C_\infty = C \cup \{\infty\}$) [4].

We recall that $C_\infty \ni \lambda \in \rho(T)$ if there is a neighbourhood $V_\lambda$ of $\lambda$ such that:

$1^0$ $(\mu I - T)^{-1} \in \mathfrak{A}$ for any $\mu \in V_\lambda \cap C$ (here $I$ is the identity of $\mathfrak{A}$).

$2^0$ the set $\{(\mu I - T)^{-1};\ \mu \in V_\lambda \cap C\}$ is bounded in $\mathfrak{A}$.

In the following we shall write for $\mu I$ simply $\mu$.

For each $T \in \mathfrak{A}$ let us denote by $\mathfrak{F}(T)$ the set of all analytic functions in a neighbourhood of $\sigma(T) = \complement\rho(T)$. Then, if $\Gamma$ is a contour (throughout we mean by "contour" a finite system of curves, admissible for the integral calculus) "surrounding" $\sigma(T)$, contained in the domain of definition of $f \in \mathfrak{F}(T)$, we put, by definition

$$f(T) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_\Gamma f(\lambda)(\lambda - T)^{-1} d\lambda & \text{if } \sigma(T) \not\ni \infty \\ f(\infty) + \dfrac{1}{2\pi i}\displaystyle\int_\Gamma f(\lambda)(\lambda - T)^{-1} d\lambda & \text{if } \sigma(T) \ni \infty \end{cases}$$

where the integral exists since the space is quasi-complete [1], [4]. For such functions we have the "spectral mapping theorem", namely

$$\sigma(f(T)) = f(\sigma(T))[4]\ .$$

Let us remark that if $L \in B(\mathfrak{A})$, $f \in \mathfrak{F}(L)$ and $\Gamma$ is a contour in the domain of $f$ "surrounding" $\sigma(L)$, then we may define the expression $f(L)(Y)$, by using the natural extension of the formula given above for the elements of $\mathfrak{A}$. Since $\mathfrak{A}$ is quasi-complete, the integrals do also exist.

**2. A functional calculus of the operator M.** For two sets $A$, $B$ in the complex compactified plane with the property that if one contains the point $\infty$ then the other does not contain zero, we denote by $A \cdot B$ the set $\{\lambda\mu;\ \lambda \in A,\ \mu \in B\}$. Also, if $A \not\ni 0$, we denote by $A^{-1}$ the set $\{1/\lambda;\ \lambda \in A\}$. In the following we need the next geometrical result:

LEMMA 2.1. *Let $K$ and $F$ be two closed sets in $C_\infty$, $0 \notin K \not\ni \infty$. If $\lambda_0 \notin K \cdot F$ and $V_0$ is a closed neighbourhood of $\lambda_0$ disjoint from $K \cdot F$, then there is an open set $G_0 \supset K$ such that $V_0 \cap \bar{G}_0 \cdot F = \varnothing$. Moreover, if $\Gamma_0$ is a contour in $G_0$ which surrounds $K$ and separates it from zero, then $\Gamma_\lambda = \lambda\Gamma_0^{-1}$ is a contour such that the set $F$ is "outside" it,* *for each* $\lambda \in V_0$, $0 \neq \lambda \neq \infty$.

*Proof.* If we put

$$G_n = \left\{\mu \in C;\ \mathrm{dist}\,(\mu, K) < \frac{1}{n}\right\},$$

on account of the compactness of $F, K$ and $V_0$ in $C_\infty$, there is an index $n_0$ such that $V_0 \cap \bar{G}_{n_0} \cdot F = \emptyset$. We take $G_0 = G_{n_0}$. We can actually suppose that $\bar{G}_0 \not\ni 0$. Now, let $\Gamma_0$ be a contour in $G_0$ surrounding $K$ and separating it from zero. Then for $\lambda \in V_0$, $0 \neq \lambda \neq \infty$ we have $\Gamma_\lambda = \lambda \Gamma_0^{-1} \subset V_0 \cdot G_0^{-1}$ and $V_0 \cdot G_0^{-1} \cap F = \emptyset$.

LEMMA 2.2. *Let $\Gamma$ be a system of curves in the complex plane, admissible for the integral calculus, $V_0 \subset C_\infty$ a closed set and $F: V_0 \times \Gamma \to \mathfrak{A}$, $G: \Gamma \to \mathfrak{A}$ two continuous functions. Then for any $\lambda \in V_0 \cap C$ we can define on $\mathfrak{A}$ the linear continuous operator*

$$R_\lambda(Y) = \int_\Gamma F(\lambda, \xi) Y G(\xi) d\xi ,$$

*for each* $Y \in \mathfrak{A}$.

*Proof.* By our assumption on the algebra $\mathfrak{A}$, it is easy to see that the product of two continuous functions is also a continuous function (since if $U$ is a neighbourhood of zero in $\mathfrak{A}$ and $B$ is a bounded set then there are two neighbourhoods of zero $U_1$ and $U_2$ such that $U_1 B \subset U$ and $B U_2 \subset U$). Since the mapping $\xi \to F(\lambda, \xi) Y G(\xi)$ is continuous on $\Gamma$ for each $\lambda \in V_0 \cap C$ and the algebra is quasi-complete, then the integral

$$\int_\Gamma F(\lambda, \xi) Y G(\xi) d\xi$$

exists as an element of $\mathfrak{A}$.

Obviously, it defines a linear operator on $\mathfrak{A}$ denoted by $R_\lambda(Y)$. To see that $R$ is a continuous operator on $\mathfrak{A}$, let us denote by $B_1$ the set $\{F(\lambda, \xi); \lambda \in V_0, \xi \in \Gamma\}$ and by $B_2$ the set $\{G(\xi); \xi \in \Gamma\}$ which are bounded in $\mathfrak{A}$. If $U = \{T \in \mathfrak{A}; p_\alpha(T) < \varepsilon\}$ then, by our hypothesis, there is a neighbourhood $U_0$ of zero in $\mathfrak{A}$ such that $B_1 U_0 B_2 \subset (2\pi/|\Gamma|) U$, where $|\Gamma|$ is the length of $\Gamma$. Thus we have

$$p_\alpha(R_\lambda(Y)) \leqq \frac{1}{2\pi} \int_\Gamma |F(\lambda, \xi) Y G(\xi)|_\alpha |d\xi| < \varepsilon$$

whenever $Y \in U_0$, hence $R_\lambda$ is continuous.

PROPOSITION 2.3. *Let $T, S \in \mathfrak{A}$ be such that $S^{-1} \in \mathfrak{A}$. If $\lambda_0 \notin \sigma(T)\cdot \sigma(S^{-1})$ and $V_0$ is a closed neighbourhood of $\lambda_0$ such that $V_0 \cap \sigma(T) \cdot \sigma(S^{-1}) = \emptyset$, then there is an open set $G_0 \supset \sigma(S^{-1})$, $G_0 \not\ni 0$ such that $V_0 \cdot$*

$G_0^{-1} \subset \rho(T)$.

*Moreover, if* $\Gamma_0$ *is a contour in* $G_0$ *which surrounds* $\sigma(S^{-1})$ *and separates it from zero, then we have*

$$(\lambda - M(T, S))^{-1}(Y) = \frac{1}{2\pi i}\int_{\Gamma_0}(\lambda - \xi T)^{-1}Y(\xi - S^{-1})^{-1}d\xi$$

*for each* $Y \in \mathfrak{A}$ *and* $\lambda \in V_0 \cap \boldsymbol{C}$.

*Proof.* By the spectral mapping theorem, $\sigma(S^{-1}) = \sigma(S)^{-1}$ does not contain zero and it is a compact set in $\boldsymbol{C}$, therefore we may apply Lemma 2.1, by putting $F = \sigma(T)$ and $K = \sigma(S^{-1})$. Let $G_0$ and $\Gamma_0$ be as in this lemma. Then $V_0 \cdot \Gamma_0^{-1} \subset V_0 \cdot G_0^{-1}$ does not intersect the set $F = \sigma(T)$, therefore $\lambda\xi^{-1} \in \rho(T)$ for all $\lambda \in V_0 \cap \boldsymbol{C}$ and $\xi \in \Gamma_0$, and $\sigma(T)$ is "outside" $\Gamma_\lambda = \lambda\Gamma_0^{-1}$ $(0 \neq \lambda \neq \infty)$. By Lemma 2.2, the integral

$$\frac{1}{2\pi i}\int_{\Gamma_0}(\lambda - \xi T)^{-1}Y(\xi - S^{-1})^{-1}d\xi = \frac{1}{2\pi i}\int_{\Gamma_0}\left(\frac{\lambda}{\xi} - T\right)^{-1}Y(\xi - S^{-1})^{-1}\frac{d\xi}{\xi}$$

exists, therefore we have for any $\lambda \in V_0$ $(0 \neq \lambda \neq \infty)$

$$\begin{aligned}
(\lambda - M(T, S))&\left(\frac{1}{2\pi i}\int_{\Gamma_0}\left(\frac{\lambda}{\xi} - T\right)^{-1}Y(\xi - S^{-1})^{-1}\frac{d\xi}{\xi}\right) \\
&= \frac{\lambda}{2\pi i}\int_{\Gamma_0}\left(\frac{\lambda}{\xi} - T\right)^{-1}Y(\xi - S^{-1})^{-1}\frac{d\xi}{\xi} \\
&\quad - \frac{1}{2\pi i}\int_{\Gamma_0}\left(T - \frac{\lambda}{\xi} + \frac{\lambda}{\xi}\right)\left(\frac{\lambda}{\xi} - T\right)^{-1}Y(\xi - S^{-1})^{-1}(S^{-1} - \xi + \xi)\frac{d\xi}{\xi} \\
&= -\frac{1}{2\pi i}\int_{\Gamma_0}Y\frac{d\xi}{\xi} + \frac{\lambda}{2\pi i}\int_{\Gamma_0}\left(\frac{\lambda}{\xi} - T\right)^{-1}\frac{d\xi}{\xi} \\
&\quad + \frac{1}{2\pi i}\int_{\Gamma_0}Y(\xi - S^{-1})^{-1}d\xi = Y\,,
\end{aligned}$$

since

$$\frac{1}{2\pi i}\int_{\Gamma_0}\frac{d\xi}{\xi} = 0\,,$$

$$\frac{1}{2\pi i}\int_{\Gamma_0}Y(\xi - S^{-1})^{-1}d\xi = Y$$

by the well-known functional calculus for an element of $\mathfrak{A}$ with the spectrum compact in $\boldsymbol{C}$ and

$$\frac{\lambda}{2\pi i}\int_{\Gamma_0}\left(\frac{\lambda}{\xi} - T\right)^{-1}\frac{d\xi}{\xi^2} = -\frac{1}{2\pi i}\int_{\lambda\Gamma_0^{-1}}(\eta - T)^{-1}d\eta = 0\,,$$

since $\sigma(T)$ is "outside" $\lambda\Gamma_0^{-1}$.

Analogously

$$\frac{1}{2\pi i}\int_{\Gamma_0}\left(\frac{\lambda}{\xi} - T\right)^{-1}((\lambda - M(T, S))(Y))(\xi - S^{-1})^{-1}\frac{d\xi}{\xi} = Y ,$$

therefore

$$(\lambda - M(T, S))^{-1}(Y) = \frac{1}{2\pi i}\int_{\Gamma_0}\left(\frac{\lambda}{\xi} - T\right)^{-1} Y(\xi - S^{-1})^{-1}\frac{d\xi}{\xi} .$$

If $\lambda = 0$ then $0 \notin \sigma(T)$, hence $T^{-1} \in \mathfrak{A}$ and

$$(-M(T, S))^{-1}(Y) = -T^{-1}YS = \frac{1}{2\pi i}\int_{\Gamma_0} - T^{-1}Y(\xi - S^{-1})^{-1}\frac{d\xi}{\xi}$$

and this finishes the proof.

PROPOSITION 2.4. *With the same conditions as in the previous proposition, we have*

$$\sigma(M(T, S)) \subset \sigma(T)\cdot\sigma(S^{-1}) .$$

*Proof.* We can suppose $\sigma(T)\cdot\sigma(S^{-1}) \neq C_\infty$. By the preceding proposition, if $\lambda_0 \notin \sigma(T)\cdot\sigma(S^{-1})$ and $V_0$ is a closed neighbourhood of $\lambda_0$, $V_0 \cap \sigma(T). \sigma(S^{-1}) = \emptyset$, then for any $\lambda \in V_0 \cap C$ the operator $(\lambda - M(T, S))^{-1}$ exists and, by Lemma 2.2, it is a continuous operator. We have only to prove that the set

$$\{(\lambda - M(T, S))^{-1}; \lambda \in V_0 \cap C\}$$

is a bounded one in $B(\mathfrak{A})$.

For, let $\{p_\alpha\}_{\alpha \in J}$ the family of semi-norms on $\mathfrak{A}$ and $\{p_{\alpha\,B}\}_{\alpha \in J, B \in \mathfrak{B}}$ the family of semi-norms on $B(\mathfrak{A})$ (see the introduction). Define $B_1 = \{(\lambda - \xi T)^{-1}, \lambda \in V_0 \cap C, \xi \in \Gamma_0\}$ and $B_2 = \{(\xi - S^{-1})^{-1}; \xi \in \Gamma_0\}$ which are bounded in $\mathfrak{A}$. Indeed, $V_0\cdot\Gamma_0^{-1} \subset \rho(T)$ and it is a compact set in $C$, therefore by reasoning with a finite covering, we obtain the boundedness of the family $B_1$. A similar argument is valid for $B_2$. If $B \in \mathfrak{B}$ is arbitrary, then $B_1BB_2$ is also a bounded set, therefore we have

$$\begin{aligned} p_{\alpha,B}((\lambda - M(T, S))^{-1}) &= \sup_{y \in B} p_\alpha((\lambda - M(T, S))^{-1}(Y)) \\ &\leqq \frac{1}{2\pi}\sup_{y \in B}\int_{\Gamma_0} p_\alpha((\lambda - \xi T)^{-1}Y(\xi - S^{-1})^{-1})\,|d\xi| \\ &\leqq C(\alpha, B) < \infty , \end{aligned}$$

where we kept the notations of the preceding proposition. Consequently $\lambda_0 \in \rho(M(T, S))$.

COROLLARY 1. *If $M = M(T, T)$ is an inner automorphism of the algebra $\mathfrak{A}$ then*

$$\sigma(M) \subset \sigma(T) \cdot \sigma(T^{-1}) \, .$$

PROPOSITION 2.5. *Let $T$, $S$ be in $\mathfrak{A}$ with $S^{-1} \in \mathfrak{A}$, and $M = M(T, S)$. If $\lambda_0 \notin \sigma(M) \cdot \sigma(S)$ and $V_0$ is a closed neighbourhood of $\lambda_0$ such that $V_0 \cap \sigma(M) \cdot \sigma(S) = \phi$, then there is an open set $G_0 \supset \sigma(S)$, $G_0 \not\ni 0$ such that $V_0 \cdot G_0^{-1} \subset \rho(M)$.*

*Moreover, if $\Gamma_0$ is a contour in $G_0$ which surrounds $\sigma(S)$ and separates it from zero, we have*

$$(\lambda - T)\Big(\frac{1}{2\pi i}\int_{\Gamma_0}(\lambda - \xi M)^{-1}(Y)(\xi - S)^{-1}d\xi\Big) = Y$$

*for all $Y \in \mathfrak{A}$ and $\lambda \in V_0 \cap \boldsymbol{C}$.*

*Proof*. We apply Lemma 2.1 with $F = \sigma(M)$ and $K = \sigma(S)$. Therefore, if $\Gamma_0$ is a contour as in the quoted lemma, then the integral

$$\frac{1}{2\pi i}\int_{\Gamma_0}(\lambda - \xi M)^{-1}(Y)(\xi - S)^{-1}d\xi$$

exists as an element of $\mathfrak{A}$ for each $\lambda \in V_0 \cap \boldsymbol{C}$.

From the relation

$$\lambda(\lambda - \xi M)^{-1}(Y) - \xi T(\lambda - \xi M)^{-1}S^{-1} = Y$$

we obtain

$$T(\lambda - \xi M)^{-1} = \frac{1}{\xi}(\lambda(\lambda - \xi M)^{-1}(Y) - Y)S \, ,$$

therefore we can write, for $\lambda \in V_0 \cap \boldsymbol{C}$, $\lambda \neq 0$,

$$\begin{aligned}
&(\lambda - T)\frac{1}{2\pi i}\int_{\Gamma_0}(\lambda - \xi M)^{-1}(Y)(\xi - S)^{-1}d\xi \\
&= \frac{\lambda}{2\pi i}\int_{\Gamma_0}(\lambda - \xi M)^{-1}(Y)(\xi - S)^{-1}d\xi \\
&\quad - \frac{1}{2\pi i}\int_{\Gamma_0}\frac{1}{\xi}(\lambda(\lambda - \xi M)^{-1}(Y) - Y)(-1 + \xi(\xi - S)^{-1})d\xi \\
&= \frac{1}{2\pi i}\int_{\Gamma_0}\frac{\lambda}{\xi}(\lambda - \xi M)^{-1}(Y)d\xi - \frac{1}{2\pi i}\int_{\Gamma_0}\frac{d\xi}{\xi}\,Y \\
&\quad + \frac{1}{2\pi i}\int_{\Gamma_0}Y(\xi - S)^{-1}d\xi = Y \, ,
\end{aligned}$$

since

$$\frac{1}{2\pi i}\int_{\Gamma_0}\frac{d\xi}{\xi} = 0 \, ,$$

$$\frac{1}{2\pi i}\int_{\Gamma_0}(\xi - S)^{-1}d\xi = I$$

and

$$\frac{1}{2\pi i}\int_{\Gamma_0}\frac{\lambda}{\xi}(\lambda - \xi M)^{-1}(Y)d\xi = -\frac{1}{2\pi i}\int_{\lambda\Gamma_0^{-1}}(\eta - M)^{-1}(Y)d\eta = 0$$

because $\sigma(M)$ is "outside" $\lambda\Gamma_0^{-1}$.

If $\lambda = 0$, a direct argument proves the validity of the given formula.

PROPOSITION 2.6. *With the same conditions as in the previous proposition, if* $\sigma(T) \ni \infty$ *then* $\sigma(M) \ni \infty$.

*Proof.* Let us suppose that $\sigma(M) \not\ni \infty$. Then there is a closed neighbourhood of $\infty$, let us say $V_\infty$, where we have

$$(\lambda - \xi M)^{-1}(Y) = \sum_{n=0}^{\infty}\frac{\xi^n M^n(Y)}{\lambda^{n+1}}$$

for all $\xi \in \Gamma_0$ where the series is uniformly convergent in $\mathfrak{A}$.

Hence we have

$$\begin{aligned}\frac{1}{2\pi i}&\int_{\Gamma_0}(\lambda - \xi M)^{-1}(Y)(\xi - S)^{-1}d\xi \\ &= \frac{1}{2\pi i}\int_{\Gamma_0}\sum_{n=0}^{\infty}\frac{\xi^n M^n(Y)}{\lambda^{n+1}}(\xi - S)^{-1}d\xi \\ &= \sum_{n=0}^{\infty}\frac{T^n Y S^{-n}}{\lambda^{n+1}}\frac{1}{2\pi i}\int_{\Gamma_0}\xi^n(\xi - S)^{-1}d\xi \\ &= \sum_{n=0}^{\infty}\frac{T^n Y}{\lambda^{n+1}},\end{aligned}$$

thus the last series is uniformly convergent in a neighbourhood of $\infty$ and defines, for $Y = I$, the inverse $\lambda - T$.

Moreover, the set $\{(\lambda - T)^{-1}; \lambda \in V_\infty\}$ is bounded in $\mathfrak{A}$ since the set

$$\left\{\frac{1}{2\pi i}\int_{\Gamma_0}(\lambda - \xi M)^{-1}(I)(\xi - S)^{-1}d\xi; \lambda \in V_\infty\right\}$$

is bounded in $\mathfrak{A}$ (see the proof of Proposition 2.4), thus $\infty \in \rho(T)$.

PROPOSITION 2.7. *Let* $T, S \in \mathfrak{A}$ *be such that* $S^{-1} \in \mathfrak{A}$ *and* $M = M(T, S)$. *Suppose* $M$ *invertible on* $\mathfrak{A}$. *If* $\lambda_0 \notin \sigma(T)\cdot\sigma(M)^{-1}$ *and* $V_0$ *is a closed neighbourhood of* $\lambda_0$ *such that* $V_0 \cap \sigma(T)\cdot\sigma(M)^{-1} = \emptyset$ *then there is an open set* $G_0 \supset \sigma(M)^{-1}$, $G_0 \not\ni 0$ *such that* $V_0 \cdot G_0^{-1} \subset \rho(T)$. *Moreover, if* $\Gamma_0$

*is a contour in $G_0$ which surrounds $\sigma(M)^{-1}$, we have*

$$\left(\frac{1}{2\pi i}\int_{\Gamma_0}(\lambda - \xi T)^{-1}(\xi M - 1)^{-1}(Y)\frac{d\xi}{\xi}\right)(\lambda - S) = Y$$

*for all $Y \in \mathfrak{A}$ and $\lambda \in V_0 \cap \boldsymbol{C}$.*

*Proof.* If $1/\xi \in \rho(M)$ then from the relation

$$\left(M - \frac{1}{\xi}\right)\left(M - \frac{1}{\xi}\right)^{-1}(Y) = Y$$

we obtain

$$\left(M - \frac{1}{\xi}\right)^{-1}(Y)S = \xi\left(T\left(M - \frac{1}{\xi}\right)^{-1}(Y) - YS\right).$$

As in the proof of Proposition 2.5, we may apply Lemma 2.1 with $F = \sigma(T)$ and $K = \sigma(M)^{-1}$ and if $\Gamma_0$ surrounds $\sigma(M)^{-1}$, we have for $\lambda$ in a neighbourhood $V_0$ of $\lambda_0$, $0 \neq \lambda \neq \infty$,

$$\begin{aligned}&\left(\frac{1}{2\pi i}\int_{\Gamma_0}(\lambda - \xi T)^{-1}(\xi M - 1)^{-1}(Y)\frac{d\xi}{\xi}\right)(\lambda - S)\\ &\quad = \left(\frac{1}{2\pi i}\int_{\Gamma_0}\left(\frac{\lambda}{\xi} - T\right)^{-1}\frac{d\xi}{\xi}\right)YS + \frac{1}{2\pi i}\int_{\Gamma_0}\left(M - \frac{1}{\xi}\right)^{-1}(Y)\frac{d\xi}{\xi^2} = Y,\end{aligned}$$

since

$$\frac{1}{2\pi i}\int_{\Gamma_0}\left(\frac{\lambda}{\xi} - T\right)^{-1}\frac{d}{\xi^2}\Big) = -\frac{1}{2\pi i}\int_{\lambda\Gamma_0^{-1}}(\eta - T)^{-1}d\eta = 0$$

(because $\sigma(T)$ is "outside" $\lambda\Gamma_0^{-1}$) and

$$\frac{1}{2\pi i}\int_{\Gamma_0}\left(M - \frac{1}{\xi}\right)^{-1}(Y)\frac{d\xi}{\xi^2} = \frac{1}{2\pi i}\int_{\Gamma_0^{-1}}(\eta - M)^{-1}(Y)d\eta = Y$$

as $\Gamma_0^{-1}$ surrounds $\sigma(M)$.

If $0 \notin \sigma(T)\cdot\sigma(M)^{-1}$ then $T^{-1} \in \mathfrak{A}$ and the formula is immediate.

LEMMA 2.8. *Let $F$ and $K$ be two closed sets, $0 \notin K \not\ni \infty$ and $G \supset K.F$ an open set. Then there is an open set $G_0 \not\ni 0$, $G_0 \supset K$ such that $G \supset \bar{G}_0\cdot F$. Moreover, if $\Gamma \subset G$ is a contour surrounding $K\cdot F$ then we can take a contour $\Gamma_0$ in $G_0$ surrounding $K$ and separating it from zero such that $\xi\cdot F$ is "inside" $\Gamma$ for all $\xi \in \Gamma_0$.*

*Proof.* The set $\complement G$ is compact in $\boldsymbol{C}_\infty$, therefore we can apply Lemma 2.1 for any $\lambda_0 \in \complement G$ and, taking a finite covering of $\complement G$, we obtain the set $G_0$. If $\Gamma$ is a contour surrounding $K\cdot F$, we can choose $G_0$ such that $\bar{G}_0$. $F$ is "inside" $\Gamma$, hence there is a contour $\Gamma_0$ in $G_0$

which surrounds $K$ and separates it from zero such that $\Gamma_0 \cdot F$ is "inside" $\Gamma$.

THEOREM 2.9. *Let $T, S \in \mathfrak{A}$ be such that $S^{-1} \in \mathfrak{A}$. If $\sigma(T)\cdot\sigma(S^{-1}) \neq C_\infty$ and $f$ is an analytic function, defined in an open set containing $\sigma(T)\cdot\sigma(S^{-1})$, then there is a contour $\Gamma_0$ surrounding $\sigma(S^{-1})$ and separating it from zero such that $f(\xi T)$ is defined for each $\xi \in \Gamma_0$ and*

$$f(M(T, S))(Y) = \frac{1}{2\pi i}\int_{\Gamma_0} f(\xi T)Y(\xi - S^{-1})^{-1}d\xi ,$$

*for all $Y \in \mathfrak{A}$ (where the left side is defined as in the Introduction).*

*Proof.* We apply Lemma 2.5 with $F = \sigma(T)$ and $K = \sigma(S^{-1})$. Let $\Gamma$ and $\Gamma_0$ be as in this lemma. Suppose that $\sigma(M) \ni \infty$. Then, by Proposition 2.4, we must have $\sigma(T) \ni \infty$. Hence, by Proposition 2.3, we can write

$$\begin{aligned} f(M(T, S)(Y)) &= f(\infty)Y + 1/2\pi i\int_{\Gamma} f(\lambda)(\lambda - M(T, S))^{-1}(Y)d\lambda \\ &= f(\infty)Y + 1/2\pi i\int_{\Gamma} f(\lambda)\Big(\frac{1}{2\pi i}\int_{\Gamma_0}(\lambda - \xi T)^{-1}Y(\xi - S^{-1})^{-1}d\xi\Big)d\lambda .\end{aligned}$$

By interchanging the order of integration, we obtain

$$\begin{aligned} &\frac{1}{2\pi i}\int_{\Gamma} f(\lambda)\Big(\frac{1}{2\pi i}\int_{\Gamma_0}(\lambda - \xi T)^{-1}Y(\xi - S^{-1})^{-1}d\xi\Big)d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_0}\Big(\frac{1}{2\pi i}\int_{\Gamma} f(\lambda)(\lambda - \xi T)^{-1}d\lambda\Big)Y(\xi - S^{-1})^{-1}d\xi \\ &= \frac{1}{2\pi i}\int_{\Gamma_0}(f\xi T) - f(\infty))Y(\xi - S^{-1})^{-1}d\xi \\ &= \frac{1}{2\pi i}\int_{\Gamma_0} f(\xi T)Y(\xi - S^{-1})^{-1}d\xi - f(\infty)Y \end{aligned}$$

since, by Lemma 2.5, $\sigma(\xi T) = \xi\sigma(T)$ is "inside" $\Gamma$ for all $\xi \in \Gamma_0$.

In this manner we obtain

$$f(M(T, S))(Y) = \frac{1}{2\pi i}\int_{\Gamma_0} f(\xi T)Y(\xi - S^{-1})^{-1}d\xi .$$

If $\sigma(T) \not\ni \infty$ we have $\sigma(M) \not\ni \infty$ and a similar calculus leads to the same formula. No other case is possible because of Proposition 2.6, and this finishes our proof.

**3. Some applications of the functional calculus.** First of all, we shall show that, in a certain sense, the commutator of two elements

[3] can be found again as a function of the operator $M$.

PROPOSITION 3.1. *Let $T$, $S$ be in $\mathfrak{A}$, with compact spectra in $C$ and with $S^{-1} \in \mathfrak{A}$. If none of the sets $\sigma(T)$, $\sigma(S)$ and $\sigma(T)\cdot\sigma(S^{-1})$ separates the complex plane, we have the relation*

$$(\log M(T, S))(Y) = (\log T)Y - Y(\log S) ,$$

*for all $Y \in \mathfrak{A}$.*

*Proof.* By our assumption, $\log T$, $\log S$ and $\log M(T, S)$ exist and, from Theorem 2.9, it follows that

$$\begin{aligned}(\log M(T, S))(Y) &= \frac{1}{2\pi i}\int_{\Gamma_0} \log (\xi T)Y(\xi - S^{-1})^{-1}d\xi \\ &= \frac{1}{2\pi i}\int_{\Gamma_0} (\log \xi)Y(\xi - S^{-1})^{-1}d\xi + \frac{1}{2\pi i}\int_{\Gamma_0} (\log T)Y(\xi - S^{-1})^{-1}d\xi \\ &= Y(\log S^{-1}) + (\log T)T = (\log T)Y - Y(\log S) .\end{aligned}$$

Let now $E(\cdot)$ and $F(\cdot)$ two selfadjoint spectral measures on a Hilbert space, defined on Borel sets of the complex plane. Then the mappings

$$\mathfrak{E}(\sigma_1)(Y) = E(\sigma_1)Y$$
$$\mathfrak{F}(\sigma_2)(Y) = YF(\sigma_2)$$

are two commuting spectral measures, $Y$ being an arbitrary linear bounded operator; therefore the mapping

$$(\mathfrak{E} \times \mathfrak{F})(\sigma_1 \times \sigma_2)(Y) = E(\sigma_1)YF(\sigma_2)$$

induces a spectral measure on the space of the operators and it is possible to integrate with respect to it (see [2] for details).

PROPOSITION 3.2. *Suppose that $\mathfrak{A}$ is the algebra of all linear operators on a Hilbert space and $T, S \in \mathfrak{A}(S^{-1} \in \mathfrak{A})$ two normal operators. If $E$, $F$ are the spectral measures of $T$ and $S$ respectively, then for any function $f$, analytic in a neighbourhood of $\sigma(T)\cdot\sigma(S^{-1})$, we have*

$$f(M(T, S))(Y) = \iint f(\lambda\mu^{-1})dE_\lambda Y dE_\mu ,$$

*for all $Y \in \mathfrak{A}$.*

*Proof.* Using the same notations as in Theorem 2.9, we have

$$f(M(T, S))(Y) = \frac{1}{2\pi i}\int_{\Gamma_0} f(\xi T)Y(\xi - S^{-1})^{-1}d\xi$$

$$= \frac{1}{2\pi i}\int_{\Gamma_0} f(\xi T)\, Y\Big(\int(\xi - \mu^{-1})^{-1} dF_\mu\Big) d\xi$$

$$= \int\Big(\frac{1}{2\pi i}\int_{\Gamma_0} f(\xi T)(\xi - \mu^{-1})^{-1} d\xi\Big) Y dF_\mu$$

$$= \int\Big(\frac{1}{2\pi i}\int_{\Gamma_0} (Sf(\xi\lambda) dE_\lambda)(\xi - \mu^{-1})^{-1} d\xi\Big) Y dF_\mu$$

$$= \iint\Big(\frac{1}{2\pi i}\int_{\Gamma_0} f(\xi\lambda)(\xi - \mu^{-1})^{-1} d\xi\Big) dE_\lambda Y dF_\mu$$

$$= \iint f(\lambda\mu^{-1}) dE_\lambda Y dF_\mu \ .$$

PROPOSITION 3.3. *Assume that the Hilbert space in the previous proposition is finite dimensional and that*

$$T = \sum_{j=1}^{p} \lambda_j E_j, \qquad S = \sum_{k=1}^{2} \mu_k F_k$$

*(where $\{E_j\}$ and $\{F_k\}$ are now finite orthogonal resolutions of the identity). If $f$ is an analytic function in an open set containing the set $\{\lambda_1, \cdots, \lambda_p\}\cdot\{\mu_1^{-1}, \cdots, \mu_2^{-1}\}$ then we have*

$$f(M(T, S))(Y) = \sum_{j=1}^{T} \sum_{k=1}^{T} f(\lambda_j \mu_k^{-1}) E_j Y F_k \ ,$$

*for all $Y \in \mathfrak{A}$.*

The proof follows easily from the preceding proposition.

PROPOSITION 3.4. *Suppose that $\mathfrak{A}$, $T$ and $S$ are as in Proposition 3.3. Then a necessary and sufficient condition that the equation $TY = ZS$ have a solution $Y \in \mathfrak{A}$ is that $\lambda_r = 0$ implies $E_r Z = 0$.*

*Proof.* Let $Y$ be a solution of the equation $TY = ZS$, hence

$$TYS^{-1} = \sum_{j=1}^{p} \sum_{k=1}^{2} \lambda_j \mu_k^{-1} E_j Y F_k = Z \ .$$

From this relation we obtain easily

$$E_r TYS^{-1} F_s = \lambda_r \mu_s^{-1} E_r Y F_s = E_r Z F_s \ ,$$

thus $\lambda_r E_s Y F_s = \mu_s E_r Z F_s$.

If $\lambda_r = 0$, since $\mu_s \neq 0$ for all $s$, we have $E_r Z F_s = 0$, hence $\sum_s E_r Z F_s = E_r Z = 0$.

Conversely, if $\lambda_r = 0$ implies $E_r Z = 0$, let us consider the matrix

$$Y = \sum_{\lambda_j \neq 0} \sum_{k=1}^{L} \frac{\mu_k}{\lambda_j} E_j Z F_k \ .$$

We have

$$TYS^{-1} = \sum_{l=1}^{q} \lambda_l E_l \sum_{\lambda_j \neq 0} \sum_{k=1}^{q} \frac{\mu_l}{\lambda_j} E_j Z F_k \sum_{t=1}^{q} \mu_t^{-1} F_t$$

$$= \sum_{l=1}^{p} E_l Z \sum_{t=1}^{q} F_t = Z ,$$

consequently $Y$ is a solution of the equation.

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Received May 11, 1970.

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