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THE CLASSIFICATION OF CERTAIN CLASSES OF TORSION FREE ABELIAN GROUPS

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Let $\mathscr{A}$ denote the class of torsion free Abelian groups of finite rank. It is shown that for $A \in \mathscr{A}$, there is a quotient divisible subgroup $QD(A)$ such that $A/QD(A)$ is a reduced torsion group. Furthermore, $QD(A)$ and $A/QD(A)$ are unique up to quasi-isomorphism. Let $\mathscr{B}$ denote the subclass of $\mathscr{A}$ of groups A such that for almost all primes p, the p-primary component of $A/QD(A)$ is the direct sum of $r_p(A)$ isomorphic cyclic groups where $r_p(A)$ denotes the p-rank of A. The groups in $\mathscr{B}$ are classified up to quasi-isomorphism, which generalizes the Beaumont-Pierce classification of quotient divisible groups.

The main results of this paper concern the subclass $\mathscr{E}$ of $\mathscr{A}$ of groups A such that $r_p(A) \leqq 1$ for all primes p. The class $\mathscr{E}$ may be profitably treated as a generalization of the class of rank one groups in $\mathscr{A}$.

In §4 $\mathscr{E}$ is characterized as a certain subclass of the class of groups in $\mathscr{A}$ whose isomorphism and quasi-isomorphism classes coincide and the groups in $\mathscr{E}$ are classified up to isomorphism. This generalizes the well-known Baer classification of rank one groups in $\mathscr{A}$ and is related to a question of L. Fuchs concerning the structure of torsion free Abelian groups which have hereditary generating systems. In §5 the endomorphisms of groups in $\mathscr{E}$ are studied. It is shown that every endomorphism of an indecomposable group in $\mathscr{E}$ is an integral multiple of an automorphism. The *special* groups of F. Richman play much the same role in $\mathscr{E}$ that the groups of non-nil type play in the class of rank one groups in $\mathscr{A}$. For example, an indecomposable group A in $\mathscr{E}$ is the additive subgroup of the endomorphism ring of some group in $\mathscr{E}$ if and only if A is a special group.

In the following, Π denotes the set of primes in the ring of integers Z, Z_p the local subring of the field of rationals Q determined by the prime p and $Z(n)$ the cyclic group of order n. The ring of p-adic integers and the field of p-adic numbers are denoted by $Z^{(p)}$ and $Q^{(p)}$ respectively. Let M be a torsion free module over an integral domain R with quotient field $Q(R)$. Then *the rank of* M, denoted by $r_R(M)$, is the $Q(R)$-dimension of $Q(R) \otimes_R M$. If $R = Z$, then we let $r(M) = r_Z(M)$ and call a subgroup N *full* in M if $r(M) = r(N)$. If B is a p-primary abelian group and $B[p] = \{x \in B \mid px = 0\}$, then the rank of B, denoted by $r(B)$, is the $Z(p)$-dimension of $B[p]$. *The p-rank of a torsion free group B*, denoted by $r_p(A)$, is $r(B/pB)$.

Let B and C be Abelian groups. Then $B \otimes C$ and $\mathrm{Hom}(B, C)$ will mean $B \otimes_Z C$ and $\mathrm{Hom}_Z(B, C)$. The endomorphism ring of B is denoted by $\mathrm{End}(B)$. We let B_p denote the p-primary component of B, $d(B)$ be the maximal divisible subgroup of B and for B torsion free, $\Pi(B) = \{p \in \Pi \mid pB = B\} = \{p \in \Pi \mid r_p(B) = 0\}$. If H is a characteristic, then the type determined by H is denoted by $[H]$. Let B be torsion free and $x \in B$. Then the height of x in B is a characteristic which we denote by $H^B(x)$. *The inner type of* B [13], denoted by $\tau_*(B)$, is the greatest lower bound in the lattice of types of *the type set of* B, i.e. $\{[H^B(x)] \mid 0 \neq x \in B\}$. For H a characteristic, we let $B[H] = \{x \in B \mid H^B(x) \geqq H\}$, which is a fully invariant subgroup of B. Epimorphisms [monomorphisms] are denoted by $\twoheadrightarrow [\rightarrowtail]$.

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1. The local-global setting. Throughout A *will denote a torsion free abelian group of finite rank and will be considered as a full subgroup of a fixed finite dimensional rational vector space* V. Any group in $\mathscr{A}$ with the same rank as A can be imbedded in V as a full subgroup and $V \cong Q \otimes A$. Let $V^{(p)} = Q^{(p)} \otimes V$ for $p \in \Pi$. Then we regard the $Q^{(p)}$-module $V^{(p)}$ as an extension of V where $V^{(p)} \cap V^{(p)} = V$ for $p \neq q$. Let B be a subgroup of A. *Then* Z_pB *denotes the* Z_p*-submodule of* V *generated by* B *and* $B^{(p)}$ *denotes the* $Z^{(p)}$*-submodule of* $V^{(p)}$ *generated by* B. Since $V^{(p)}$ is viewed as an extension *of* V *and torsion* free groups are flat, there is a natural identification between $A^{(p)}$ and $Z^{(p)} \otimes A$ given by $\Sigma r_i a_i \rightarrow \Sigma r_i \otimes a_i$ for $r_i \in Z^{(p)}$ and $a_i \in A$. A similar identification occurs between Z_pA and $Z_p \otimes A$. Note that $r(A) = r_{Z^{(p)}}(A^{(p)}) = r_{Q^{(p)}}(V^{(p)}) = r_{Z_p}(Z_pA)$. The following well-known local-global relations will be frequently used:

(i) $A^{(p)} \cap V = Z_pA$

(ii) $\bigcap_{q \in \Pi} Z_pA = A = \bigcap_{q \in \Pi} A^{(p)}$

(iii) $p^n Z_pA \cap A = p^n A = p^n A^{(p)} \cap A$ for $n \geqq 0$

(iv) for B a full subgroup of A, $Z_pA/Z_pB \cong (A/B)_p \cong A^{(p)}/B^{(p)}$.

Two groups B and C are *quasi-isomorphic*, denoted by $B \doteq C$, if there are subgroups $B' \subseteqq B$ and $C' \subseteqq C$ such that $B' \cong C'$ and B/B', C/C' are groups of bounded order. For B and C torsion, $B \cong C$ if and only if $B_p \doteq C_p$ for all p and $B_p \cong C_p$ for almost all p [4]. Thus, if B and C are torsion homomorphic images of A, then $B \doteq C$ if and only if $d(B) \cong d(C)$ and $B_p \cong C_p$ for almost all p. Let B and

C be torsion free. Then $B \doteq\!\!\!\cong C$ is equivalent to the existence of a monomorphism ϕ on B into C such that $C/\phi(B)$ is of bounded order. It is well-known that if $B \subseteqq A$ and $B \cong A$, then A/B is a finite group. This has the important consequence that $A \doteq\!\!\!\cong B$ if and only if each group is imbeddable in the other one. B and C are *quasi-equal*, denoted by $B \doteq C$, if there are positive integers n and m such that $B \supseteqq nC \supseteqq mB$. Two torsion free Z_p modules are quasi-equal if they are quasi-equal as groups. The local-global relations give: $A \doteq B$ *if and only if* $Z_pA = Z_pB$ *for almost all* p *and* $Z_pA \doteq Z_pB$ *for all* p.

LEMMA 1. *Let B and C be full subgroups of A. If $B \doteq C$, then $A/B \doteq\!\!\!\cong A/C$.*

Proof. Since $Z_pB = Z_pC$ for almost all p, $(A/B)_p \cong Z_pA/Z_pB = Z_pA/Z_pC \cong (A/C)_p$ for almost all p. Now $C \supseteqq mB \supseteqq nC$ for some $n, m > 0$. Let $B' = mB$. Since C/B' is bounded, the exact sequence $C/B' \overset{id}{\rightarrowtail} A/B' \twoheadrightarrow A/C$ shows that $d(A/B') \cong d(A/C)$. Since A is torsion free, $A/B \cong mA/mB = m(A/B')$ and so $d(A/B) \cong d(A/B')$. Hence, $A/B \doteq\!\!\!\cong A/C$.

DEFINITION. $k_p(A) = r(d_p(A/I))$ where I is a full, free subgroup of A and $s_p(A) = r(\bigcap_n p^n A)$.

Note that $k_p(A)$ does not depend upon I by Lemma 1. As in [1] we let $\delta_p(A)$ denote the maximal divisible subgroup of $A^{(p)}$, which is the maximal divisible submodule of $A^{(p)}$ regarded as a $Z^{(p)}$ module. Thus, $\delta_p(A)$ is a $Q^{(p)}$ subspace of $V^{(p)}$.

LEMMA 2. (i) $r(A) = r_p(A) + k_p(A)$
(ii) $k_p(A) = r_{Z^{(p)}}(\delta_p(A))$
(iii) $s_p(A) = r(\delta_p(A) \cap V)$

Proof. For (i), let I be a full, free subgroup of A such that $(A/I)_p$ is divisible. Then $(V/I)_p \cong (A/I)_p \oplus (V/A)_p$ and so $r(V) = k_p(A) + r((V/A)_p)$. Since $V/A \cong V/pA$ and $V/pA[p] = A/pA$, $r((V/A)_p) = r(A/pA)$, which gives (i). To show (ii) it will be enough in view of (i) to show $r(A) = r_p(A) + r_{Z^{(p)}}(\delta_p(A))$. Now $Z^{(p)} \otimes A = d(Z^{(p)} \otimes A) \oplus F$ where F is a free $Z^{(p)}$-module [5, 44.2] and so $r(F/pF) = r_{Z^{(p)}}(F)$. Thus, it will be sufficient to show $r_p(A) = r(F/pF)$. The exact sequence $pA \overset{id}{\rightarrowtail} A \twoheadrightarrow A/pA$ implies $Z^{(p)} \otimes pA \overset{e}{\rightarrowtail} Z^{(p)} \otimes A \twoheadrightarrow Z^{(p)} \otimes (A/pA)$ exact. Note that $e(Z^{(p)} \otimes pA) = p(Z^{(p)} \otimes A)$. Thus $F/pF \cong Z^{(p)} \otimes A/p(Z^{(p)} \otimes A) \cong Z^{(p)} \otimes (A/pA) \cong A/pA$, which gives (ii). For (iii), note that $Z_p(\bigcap_n p^n A) = d(Z_pA) = \delta_p(A) \cap V$.

COROLLARY 1. *Let B be a full subgroup of A. Then the following* *conditions are equivalent*: (i) $(A/B)_p$ *is reduced*, (ii) $\delta_p(A) = \delta_p(B)$, (iii) $k_p(A) = k_p(B)$, *and* (iv) $r_p(A) = r_p(B)$.

LEMMA 3. *Let B be a subgroup of A and H a characteristic such that* $\{p \in \Pi \mid H(p) = \infty\} = \Pi(A)$. *Then the following are equivalent*:

(i) $B = A[H]$ *and* $\tau_*(A) \geqq [H]$

(ii) A/B *is torsion with* $(A/B)_p \cong \bigoplus^{r_p(A)} Z(p^{H(p)})$ *for* $p \notin \Pi(A)$ *and* $(A/B)_p = \{0\}$ *for* $p \in \Pi(A)$

(iii) $p^{H(p)}Z_pA = Z_pB$ *for* $H(p) < \infty$ *and* $Z_pA = Z_pB$ *for* $H(p) = \infty$.

Proof. We give a cyclical proof. Assume (i). Then $\tau_*(A) \geqq [H]$ gives A/B torsion. Since $A[H] = \bigcap_{q \notin \Pi(A)} q^{H(q)}A$, $Z_pB = Z_p(p^{H(p)}A)$ for $p \notin \Pi(A)$ and $Z_pB = Z_pA$ for $p \in \Pi(A)$. Thus, $(A/B)_p = \{0\}$ for $p \in \Pi(A)$ and $(A/B)_p \cong A/p^{H(p)}A \cong \bigoplus^{r_p(A)} Z(p^{H(p)})$ for $p \notin \Pi(A)$, which is (ii). Assume (ii). Then $Z_pA = Z_pB$ for $p \in \Pi(A)$. For $p \notin \Pi(A)$, $p^{H(p)}Z_pA \subseteqq Z_pB \subseteqq Z_pA$ and $Z_pA/p^{H(p)}Z_pA$ is a p-group, with the same order as Z_pA/Z_pB. Thus, $p^{H(p)}Z_pA = Z_pB$ for $p \notin \Pi(A)$, which is (iii). Assume (iii). Since $A/A[H]$ is torsion, $\tau^*(A) \geqq [H]$. Since $\Pi(A) = \Pi(B)$, $B = \bigcap_{H(p)<\infty}(p^{H(p)}Z_pA \cap V) = \bigcap_{H(p)<\infty}(p^{H(p)}Z_pA \cap A) = \bigcap_{H(p)<\infty} p^{H(p)}A = A[H]$, which is (i).

COROLLARY 2. *If* $\tau_*(A) \geqq [H]$, *then* $\mathrm{End}\,(A) \cong \mathrm{End}\,(A[H])$.

Proof. For $\phi \in \mathrm{End}\,(A)$, let ϕ' be the restriction of ϕ to $A[H]$. Since $A[H]$ is a full, fully invariant subgroup of A, $\phi \to \phi'$ is a ring monomorphism into $\mathrm{End}\,(A[H])$. For $\lambda' \in \mathrm{End}\,(A[H])$, let λ be its unique extension to A into V. By Lemma 3, $p^{H(p)}Z_pA = Z_pA[H] \supseteqq Z_p(\lambda(A[H])) = p^{H(p)}Z_p\lambda(A)$ for $p \notin \Pi(A)$. Thus, $Z_p\lambda(A) \subseteqq Z_pA$ for all p and so $\lambda \in \mathrm{End}\,(A)$.

2. The quotient divisible core. We recall from [1] that A is a *quotient divisible* [QD] group if A has a full free subgroup I such that A/I is divisible. Note that A is a QD group if and only if for J a full free subgroup of A, $A/J = D \oplus T$ where D is divisible and T is finite. The invariants introduced by Beaumont-Pierce in [1] to classify the QD groups in $\mathscr{A}$ involve the following considerations. Let $\mathscr{L}_p(V)$ denote the lattice of all $Q^{(p)}$-subspaces of $V^{(p)}$ and $\mathscr{L}(V) = \prod_p \mathscr{L}_p(V)$ the direct product of these lattices. If $\delta \in \mathscr{L}(V)$, then the p-component of δ is denoted by δ_p and δ is referred to as a QD *invariant* (associated with V). For ϕ a Q-automorphism of V, let $\phi^{(p)} = \phi \otimes id_{Q^{(p)}}$, which is a $Q^{(p)}$-automorphism of $V^{(p)}$.

DEFINITION. Let $\delta, \delta' \in \mathscr{L}(V)$. Then $\delta \leqq \delta'$ if there is a Q-automorphism ϕ of V such that $\phi^{(p)}(\delta_p) \subseteqq \delta'_p$ for all p. $\delta \sim \delta'$ if $\delta \leqq \delta'$ and $\delta' \leqq \delta$. For A full in V, let $\delta(A) \in \mathscr{L}(V)$ such that $\delta(A)_p = \delta_p(A)$.

Let A and B be full QD subgroups of V. Then the Beaumont-Pierce QD Theorem [1, 5.25] states that:

(i) A is imbeddable in B if and only if $\delta(A) \leqq \delta(B)$,

(ii) $A \doteq B[A \cong B]$ if and only if $\delta(A) = \delta(B)[\delta(A) \sim \delta(B)]$,

(iii) For $\delta \in \mathscr{L}(V)$, there is a full QD subgroup A of V such that $\delta(A) = \delta$.

DEFINITION. Let I be a full free subgroup of A and ϕ be the natural map $A \twoheadrightarrow A/I$. Then $QD(A, I) = \phi^{-1}(d(A/I))$.

LEMMA 4. *Let I and J be full free subgroups of A and let B be a full subgroup of A. Then*:

(i) *B contains I and $\delta(A) = \delta(B)$ if and only if $QD(A, I) \subseteqq B$*

(ii) *if B is QD and $\delta(A) = \delta(B)$, then $QD(A, J) = B$ for some J*

(iii) *$QD(A, I) \doteq QD(A, J)$ and $A/QD(A, I) \cong A/QD(A, J)$.*

Proof. First note that $QD(A, I)$ is a full QD subgroup of A such that $A/QD(A, I)$ is reduced torsion. Part (i) is now immediate from Corollary 1. For (ii), let J be a full free subgroup of B such that B/J is divisible. The first part gives $QD(A, J) \subseteqq B$ and so $B/J \cong QD(A, J)/J \oplus B/QD(A, J)$, which shows that $B/QD(A, J)$ is divisible. Since $\delta(B) = \delta(QD(A, J))$, $B/QD(A, J)$ is reduced by Corollary 1. Hence, $B = QD(A, J)$, which is (ii). For the first part of (iii), you may invoke the Beaumont-Pierce QD Theorem or more directly, note that $QD(A, I) + J$ is a QD subgroup of A which is quasi-equal to $QD(A, I)$. Since $QD(A, I) + J \supseteqq QD(A, I + J) \supseteqq QD(A, I)$ by (i), $QD(A, I) \doteq QD(A, I + J)$. Thus, $QD(A, I) \doteq QD(A, J)$ by symmetry. The second part of (iii) is now immediate from Lemma 1.

For the remainder of this section *I will denote a full free subgroup of A.* Note that A is a locally free group, i.e. Z_pA is a free Z_p module for all p, if and only if $QD(A, I) = I$ and A is a QD group if and only if $QD(A, I) \doteq A$. The quasi-isomorphism class determined by $QD(A, I)$, which by Lemma 4 is independent of the choice of I, will be referred to as *the QD core of A*. The Beaumont-Pierce QD Theorem shows that two groups A and B have the same QD core if and only if $\delta(A) \sim \delta(B)$. The quasi-isomorphism class determined by $A/QD(A, I)$, which by Lemma 4 is independent of choice of I, is closely related to the Richman type of A. See [12] or [13].

Let $A/I = d(A/I) \oplus T$. Then $T \cong A/QD(A, I)$ and $r(T_p) \leqq r_p(A)$ for all p by Lemma 2. Thus, for $r_p(A) > 0$, $(A/QD(A, I))_p = \bigoplus_{i=1}^{r_p(A)} Z(p^{\alpha_i(p)})$ where $0 \leqq \alpha_i(p) \leqq \alpha_j(p) < \infty$ for $j > i$ and $(A/QD(A, I))_p = \{0\}$ for $r_p(A) = 0$.

DEFINITION. For $p \in \Pi$, let $H_*(A, I)(p) = \alpha_1(p)$ if $r_p(A) > 0$ and ∞ otherwise, $H^*(A, I)(p) = \alpha_s(p)$ where $s = r_p(A)$ if $r_p(A) > 0$ and ∞ otherwise, and $\tau^*(A) = [H^*(A, I)]$.

Lemma 4 shows that the types $\tau^*(A)$ and $[H_*(A, I)]$ are independent of the choice of I. The identification of $[H_*(A, I)]$ in (i) of the following was also noted by Warfield [13, p. 194].

LEMMA 5. (i) $\tau_*(A) = [H_*(A, I)]$
(ii) $\tau_*(A) = \tau_*(B)$ *and* $\tau^*(A) = \tau^*(B)$ *whenever* $A \doteq B$
(iii) *A is a QD group if and only if $\tau^*(A)$ is non-nil.*

Proof. For (i), let $H \in \tau_*(A)$ and I be a full free subgroup of $A[H]$. To see that $H_*(A, I) \geqq H$, refer to Lemmas 3 and 4 to note that $QD(A, I) \subseteqq A[H]$ and consider the orders of the finite p-groups $(A/QD(A, I))_p$, $(A/A[H])_p$ and the natural map $A/QD(A, I) \twoheadrightarrow A/A[H]$. On the other hand, let B be a group such that $QD(A, I) \subseteqq B \subseteqq A$ and for $p \notin \Pi(A)$, $(A/B)_p \cong \bigoplus^{r_p(A)} Z(p^\alpha)$ where $\alpha = H_*(A, I)(p)$. Then $\tau_*(A) \geqq [H_*(A, I)]$ by Lemma 3, which gives (i). Finally, (ii) is an easy computation while (iii) is immediate from the definitions.

We have shown that every full subgroup A of V is an extension of a QD group B by a reduced torsion group C and that B and C are unique up to quasi-isomorphism. On the other hand, *let B be a full QD group in V* and *C be a reduced torsion group* such that $r_p(B) \geqq r(C_p)$ for all p. Then there is a full subgroup A in V which is an extension of B by C. This may be seen by observing that $r_p(B) = r((V/B)_p)$ for all p and letting A be the inverse image in V of an appropriate subgroup of the divisible group V/B. Now suppose both A and A' are extensions of B by C. Then it is easily seen that A and A' have the same QD core, $\tau^*(A) = \tau^*(A')$ and $\tau_*(A) = \tau_*(A')$. Thus, $A \cong A'$ whenever $r(B) = 1$. In the next section we study a class of groups A which are determined up to quasi-isomorphism by B and C. In contrast we give the following example of two non-quasi-isomorphic groups A and A' with the same QD core and $A/QD(A, I) \doteq A'/QD(A', J)$.

EXAMPLE 1. Let A and A' be locally free, completely decomposable groups of rank 2 whose type sets, denoted by $T(-)$, satisfy $T(A) \neq T(A')$, sup $T(A) =$ sup $T(A')$, and $\tau_*(A) = \inf T(A) = \inf T(A') = \tau_*(A')$. Such pairs of groups exist in abundance. Let I and J be full free subgroups of A and A' respectively. Then $QD(A, I) = I$ and $QD(A', J) = J$ (since the groups are locally free). Thus, $QD(A, I) \doteq QD(A', J)$. A simple computation shows $\tau^*(A) = \sup T(A) = \tau^*(A')$.

Hence, $A/QD(A, I) \doteqdot A'/QD(A', J)$ (since A and A' have rank 2) and A is not $\doteqdot$ to A' (since $T(A) \neq T(A')$).

3. The Class $\mathscr{B}$.

DEFINITION. $\mathscr{B} = \{A \in \mathscr{A} \mid \tau^*(A) = \tau_*(A)\}$

Note that $\tau^*(A) = \tau_*(A)$ describes the condition that for I a full free subgroup of A, $(A/QD(A, I))_p$ is a direct sum of $r_p(A)$ isomorphic cyclics for almost all p. The finite rank QD groups and the finite rank, homogeneous, completely decomposable groups are examples of groups in $\mathscr{B}$. In fact, the locally free groups in $\mathscr{B}$ are necessarily homogeneous, completely decomposable groups [13, Corollary 5]. Since $\tau^*(-)$ and $\tau_*(-)$ are quasi-isomorphism invariants [Lemma 5], $A \in \mathscr{B}$ whenever $A \doteqdot B$ and $B \in \mathscr{B}$. Finally, we mention that $\mathscr{B}$ is closed with respect to direct summands and finite direct sums of the form $\bigoplus_{i=1}^{n} A_i$ where $A_i \in \mathscr{B}$ and $\tau_*(A_i) = \tau_*(A_j)$ for all i, j.

LEMMA 6. *Let $H \in \tau_*(A)$. Then $A \in \mathscr{B}$ if and only if $QD(A, J) = A[H]$ for some full free subgroup J in A.*

Proof. Let $A \in \mathscr{B}$ and let I be a full free in $A[H]$. Then $QD(A, I) \subseteqq A[H]$ by Lemmas 3 and 4. Since $H \sim H_*(A, I)$ by Lemma 5, Lemma 3 (ii) gives $(A/A[H])_p \cong (A/QD(A, I))_p$ for almost all p. This says that $A[H]/QD(A, I)$ is finite. Thus, $A[H]$ is a QD group such that $A/A[H]$ is reduced torsion. It follows from Lemma 4 that $A[H] = QD(A, J)$ for some full free J in A. The converse is immediate from Lemma 3.

DEFINITION. A type τ and a QD invariant $\delta \in \mathscr{L}(V)$ are *compatible* if the p^{th}-component of τ is ∞ if and only if $\delta_p = V^{(p)}$. Note that $\tau_*(A)$ and $\delta(A)$ are compatible for $A \in \mathscr{A}$.

THEOREM 1. *Let A and B be groups in $\mathscr{B}$ which are full in V.*

(i) *There is an imbedding of A in B if and only if $\delta(A) \leqq \delta(B)$ and $\tau_*(A) \leqq \tau_*(B)$.*

(ii) *$A \doteq B$ if and only if $\delta(A) = \delta(B)$ and $\tau_*(A) = \tau_*(B)$.*

(iii) *$A \doteqdot B$ if and only if $\delta(A) \sim \delta(B)$ and $\tau_*(A) = \tau_*(B)$.*

(iv) *If $\delta \in \mathscr{L}(V)$ and τ is a type compatible with δ, then there is a group C in $\mathscr{B}$ which is full in V such that $\delta(C) = \delta$ and $\tau_*(C) = \tau$.*

Proof of (i). Let ϕ be an imbedding of A in B and assume that ϕ is extended to a Q-automorphism of V. Then it is clear that

$\phi^{(p)}(\delta_p(A)) \subseteqq \delta_p(B)$ for all p, i.e. $\delta(A) \leqq \delta(B)$. Let I be a full free subgroup of A. Then $\phi(I)$ is a full free subgroup of B and $A/I \cong \phi(A)/\phi(I) \subseteqq B/\phi(I)$. A modest computation shows $H_*(\phi(A), \phi(I)) \leqq H_*(B, \phi(I))$, i.e. $\tau_*(A) \leqq \tau_*(B)$.

Conversely, let $H_1 \in \tau_*(A)$ and $H_2 \in \tau_*(B)$ such that $H_1 \leqq H_2$. Use Lemma 6 to obtain full free subgroups I and J of A and B respectively such that $A[H_1] = QD(A, I)$ and $B[H_2] = QD(B, J)$. Since $\delta(A) \leqq \delta(B)$, the Beaumont-Pierce QD Theorem gives an imbedding $\phi\colon A[H_1] \rightarrowtail B[H_2]$. Assume that ϕ is uniquely extended to A into V and so for $p \in \Pi(B)$, $Z_p\phi(A) \subseteqq V = Z_pB$. For $p \notin \Pi(B)$, $p^{H_1(p)}Z_p\phi(A) = Z_p\phi(A[H_1]) \subseteqq Z_pB[H_2] = p^{H_2(p)}Z_pB$ by Lemma 3. Since division is unique in V, $Z_p\phi(A) \subseteqq p^tZ_pB \subseteqq Z_pB$ where $t = H_2(p) - H_1(p)$. Hence, $\phi(A) \subseteqq B$.

Proof of (*ii*). The "only if" part is immediate. For the converse let $H_1 = H_2$ and I, J be as in the proof of (*i*). Since $\delta(A) = \delta(B)$, $A[H_1] = QD(A, I) \doteq QD(B, J) = B[H_2]$ and so the imbedding ϕ in the previous part may be chosen to be a left multiplication by some positive integer n. The argument in the previous part now gives $nA \subseteqq B$. By symmetry, $mB \subseteqq A$ for some $m > 0$ and so $A \doteq B$.

Proof of (*iii*). Since $A \doteqdot B$ if and only if each group is imbeddable in the other, (iii) is immediate from (i).

Proof of (*iv*). Let B be a full QD subgroup of V with $\delta(B) = \delta$. Let $H \in \tau$ and C be a subgroup of V containing B such that $(C/B)_p \cong \bigoplus^{r_p(B)} Z(p^{H(p)})$ for $p \notin \Pi(B)$. Note that such a C exists since V/B is divisible with $r((V/B)_p) = r_p(B)$. Since τ and δ are compatible, C/B is reduced torsion and Lemma 4(ii) gives $B = QD(C, I)$ where I is a full free subgroup of B such that B/I is divisible. Thus, $C \in \mathscr{B}$ with $\delta(C) = \delta$ and $\tau_*(C) = \tau$.

REMARK 1. Let A be a QD group and H the characteristic such that $H(p) = \infty$ if $\delta_p(A) = V^{(p)}$ and $H(p) = 0$ otherwise. Then by Lemma 5 (iii) $[H] = \tau_*(A)$. Thus, $\tau_*(A)$ may be recaptured from $\delta(A)$ whenever A is a QD group. This shows that Theorem 1 is a generalization of [1, 5.25]. We mention that the Warfield Duality [13] may be used to show that for D a torsion free, rank 1 group with $\tau_*(D) = \tau_*(A)$ and I a full free subgroup of A, then the following are equivalent: (i) $A \in \mathscr{B}$, (ii) $QD(A, I) \doteqdot \mathrm{Hom}(D, A)$, (iii) $A \doteqdot D \otimes QD(A, I)$. This, of course, yields another proof of Theorem 1.

LEMMA 7. *If I is a full free subgroup of A and $A \in \mathscr{B}$, then* $Q \otimes \mathrm{End}(A) \cong Q \otimes \mathrm{End}(QD(A, I))$.

Proof. Let $H \in \tau_*(A)$ and J be a full free subgroup of A such that $A[H] = QD(A, J)$ [Lemma 6]. Thus, $A[H] \doteq QD(A, I)$ and so $Q \otimes \text{End}(A) \cong Q \otimes \text{End}(A[H]) = Q \otimes \text{End}(QD(A, I))$ by Corollary 2.

For the following corollary, recall that A is *quasi-decomposable* if it is quasi-equal to the direct sum of two nonzero torsion free groups. A is *strongly indecomposable* if it is not quasi-decomposable.

COROLLARY 3. *Let I be a full free subgroup of A and $A \in \mathscr{B}$. Then the following are equivalent:*

(i) *A is quasi-decomposable*

(ii) *$QD(A, I)$ is quasi-decomposable*

(iii) *there are nonzero subspaces U and W such that $V = U \oplus W$ and $\delta_p(A) = \delta_p(A) \cap U^{(p)} \oplus \delta_p(A) \cap W^{(p)}$ for all p.*

Proof. Since a group B in $\mathscr{A}$ is quasi-decomposable if and only if $Q \otimes \text{End}(B)$ is decomposable as a module over itself [11], the equivalence of (i) and (ii) is immediate from Lemma 7. (iii) is a necessary and sufficient condition for a QD group in $\mathscr{A}$ to be quasi-decomposable [1, 5.26]. Since $\delta(A) = \delta(QD(A, I))$, (ii) and (iii) are equivalent. Note that the quotient-divisibility of A must be added to the hypotheses of [1, 5.26].

4. The Class $\mathscr{E}$.

DEFINITION. $\mathscr{E} = \{A \in \mathscr{A} \mid r_p(A) \leq 1 \text{ for all } p \in \Pi\}$.

We note that $\mathscr{E}$ is a subclass of $\mathscr{B}$ which contains the torsion free, rank one groups. Furthermore, the reduced groups A in $\mathscr{E}$ are up to isomorphism precisely the finite rank, pure subgroups of $\prod_{p \in \Pi} Z^{(p)}$, the Z-adic completion of the integers. The class $\mathscr{E}$ is closed with respect to pure subgroups, torsion free homomorphic images and tensor products. Here we make use of the facts that for $A, B \in \mathscr{A}$, $r_p(A \otimes B) = r_p(A) r_p(B)$ and for B pure in A, $r_p(A) = r_p(B) + r_p(A/B)$. Recall that a group A in $\mathscr{A}$ is *cohesive* if $A \in \mathscr{E}$ and $s_p(A) = 0$ for $p \notin \Pi(A)$ [4]. If A is a non-cohesive group in $\mathscr{E}$, i.e. $0 < s_p(A) < r(A)$ for some p, then A is not homogeneous (since there are $0 \neq x, y \in A$ such that $H^A(x)(p) = \infty$ and $H^A(y)(p) < \infty$). Thus, the homogeneous groups in $\mathscr{E}$ are cohesive. On the other hand, Theorem 4 in [4] shows the existence of homogeneous and non-homogeneous cohesive groups of any rank greater than one. Richman's *special* groups [12] are a subclass of the homogeneous groups in $\mathscr{E}$. Reduced groups in $\mathscr{E}$ can be decomposable. For example, if $\{\Pi', \Pi''\}$ is a nontrivial partition of Π, $Z(\Pi') = \bigcap_{p \in \Pi'} Z_p$, $Z(\Pi'') = \bigcap_{p \in \Pi''} Z_p$, and $A = Z(\Pi') \oplus Z(\Pi'')$, then $A \in \mathscr{E}$. On the other hand, the reduced cohesive groups in $\mathscr{E}$ are purely indecomposable, i.e. every pure subgroup is indecomposable [4, C5]. Later we show that $\mathscr{E}$ contains abundant non-cohesive, indecomposable groups which need not be purely indecomposable.

LEMMA 8. *$A \in \mathscr{E}$ if and only if $A \in \mathscr{A}$ and all finite homomorphic images of A are cyclic.*

Proof. If $A \in \mathscr{E}$ and A/B is a group of order n, then A/B is the homomorphic image of the cyclic group A/nA. Conversely, since A has finite rank, A/pA is a finite group and thus, cyclic.

DEFINITION. $\mathscr{C} = \{A \in \mathscr{A} \mid A \dot{\cong} B$ implies $A \cong B\}$. $\mathscr{D} = \{A \in \mathscr{A} \mid B \subseteqq A$ and $B \cong A$ implies $B = nA$ for some $n\}$.

Note that if $A \in \mathscr{A}$ and End (A) is a subring of Q, then $A \in \mathscr{D}$ and that $A \in \mathscr{C}$ if and only if $A \doteq B$ implies $A \cong B$. The rank one groups are in $\mathscr{C}$ and more generally, any completely decomposable group of finite rank whose type set is a chain in the lattice of types is in $\mathscr{C}$ [1, 9.6]. Furthemore, the rank two groups in $\mathscr{C}$ have been explicitly computed [2, 9.6]. The groups in $\mathscr{C}$ would appear to have a simpler structure than arbitrary groups in $\mathscr{A}$. For example, if $A \in \mathscr{C}$, then it is immediate from Jónsson's Theorem [9, 2.6] that A is indecomposable if and only if A is strongly indecomposable.

THEOREM 2. $\mathscr{E} = \mathscr{C} \cap \mathscr{D}$.

Proof. Let $A \in \mathscr{E}$. If C is any subgroup of A such that $C \supseteqq nA$ for some $n > 0$, then C/nA is a subgroup of the cyclic group A/nA. It follows that $C/nA = t(A/nA) = tA/nA$ for some divisor t of n and so $C = tA$. In particular, if $B \subseteqq A$ and $A \cong B$, then A/B is finite and so $B = tA$ for some $t > 0$, i.e. $A \in \mathscr{D}$. If $A \doteq B$, then $A \supseteqq mB \supseteqq nA$ for some $n, m > 0$. Let $C = mB$. Then the above gives $B \cong C = tA \cong A$ for some $t > 0$, i.e. $A \in \mathscr{C}$. Thus, $\mathscr{E} \subseteqq \mathscr{C} \cap \mathscr{D}$. On the other hand, let $A \in \mathscr{C} \cap \mathscr{D}$ and suppose $pA \subseteqq B \subset A$ for some $p \notin \Pi(A)$. Since $A \in \mathscr{C}$, $A \cong B$ which together with $A \in \mathscr{D}$ gives $nA = B$ for some $n > 0$. Since $nZ_pA \subset Z_pA$, p divides n and so $pA = nA = B$, i.e. $A \in \mathscr{E}$.

REMARK 2. Since $\mathscr{E} \subset \mathscr{C}$, Lemma 2 gives an affirmative answer to the conjecture in [2, p. 41]. This has been previously noted by other authors. Let A be a rank two group with a maximal independent set $\{x_1, x_2\}$ and $(A: x_1, x_2) \rightarrow \Sigma$ be the characteristic in [2, 2.2]. Then a direct computation gives $\Sigma(p) + H^A(x_1)(p) + H^A(x_2)(p) = \infty$ if and only if $r_p(A) \leqq 1$. It follows from [2, Theorem 9.6] that the indecomposable rank two groups in $\mathscr{C}$ are precisely the indecomposable rank two groups in $\mathscr{E}$. Finally note that Theorem 2 shows that for $A \in \mathscr{A}$ such that End (A) is a subring of Q (and such groups exist in abundance), $A \in \mathscr{C}$ if and only if $A \in \mathscr{E}$.

COROLLARY 4. $\mathscr{E} = \{A \in \mathscr{A} \mid A \doteq B$ *implies* $nA = mB$ *for some* $n, m > 0\}$.

Proof. A modest computation shows that the above set is equal to $\mathscr{C} \cap \mathscr{D}$.

COROLLARY 5. *The Krull-Schmidt Theorem holds in* $\mathscr{E}$.

Proof. Let $A \in \mathscr{E}$ and $A = \bigoplus_{i=1}^{n} A_i = \bigoplus_{j=1}^{m} B_j$ where A_i and B_j are indecomposable. Since $\mathscr{E}$ is closed with respect to direct summands and $\mathscr{E} \subseteqq \mathscr{C}$, A_i and B_j are strongly indecomposable. Jónsson's Theorem gives $n = m$ and for some permutation σ, $A_i \doteq B_{\sigma(i)}$ and so $A_i \cong B_{\sigma(i)}$ for all i.

COROLLARY 6. *Let A be a group in $\mathscr{E}$ which is full in V. A is decomposable if and only if there are nonzero Q-subspaces U and W of V such that $V = U \oplus W$ and for each p, either $U \subseteqq d(Z_pA)$ or $W \subseteqq d(Z_pA)$.*

Proof. Assume $A = B \oplus C$ where B and C are nonzero. Let U and W be the nonzero subspaces of V generated by B and C respectively. Then $V = U \oplus W$. Since $r_p(-)$ distributes through direct sums, B is p-divisible or C is p-divisible. It follows that for each p, either $U = Z_pB \subseteqq d(Z_pA)$ or $W = Z_pC \subseteqq d(Z_pA)$. Conversely, since $U^{(p)} \subseteqq d(Z_pA)^{(p)} \subseteqq \delta_p(A)$ or $W^{(p)} \subseteqq d(Z_pA)^{(p)} \subseteqq \delta_p(A)$ for each p, the modular law gives $\delta_p(A) = (\delta_p(A) \cap U^{(p)}) \oplus (\delta_p(A) \cap W^{(p)})$ for each p. It follows from Corollary 3 that A is quasi-decomposable. Since $\mathscr{E} \subseteqq \mathscr{C}$, A is decomposable.

THEOREM 3. *Let A and B be groups in $\mathscr{E}$ which are full in V.*

(i) *There is an imbedding of A into B if and only if $\tau_*(A) \leqq \tau_*(B)$ and $\delta(A) \leqq \delta(B)$.*

(ii) *$A \cong B$ if and only if $\tau_*(A) = \tau_*(B)$ and $\delta(A) \sim \delta(B)$.*

(iii) *Let τ be a type and $\delta \in \mathscr{L}(V)$ be a compatible QD invariant where $1 + r_{Z^{(p)}}(\delta_p) \geqq r_Q(V)$ for all p. Then there is a $C \in \mathscr{E}$ which is full in V such that $\tau_*(C) = \tau$ and $\delta(C) = \delta$.*

Proof. This is immediate from Theorems 1 and 2 and Lemma 2.

REMARK 3. Let $r(A) = 1$ and δ be the QD invariant associated with a 1 dimensional V such that $\delta_p = V^{(p)}$ if $\tau_*(A)$ has ∞ at p and $\delta_p = 0$ otherwise. Then $\delta(A) = \delta$. Thus, $\delta(A)$ may be recaptured from $\tau_*(A)$ whenever $r(A) = 1$. This shows that Theorem 3 generalizes the well-known theorem of Baer [5, 44.2]. In addition, we note that Lemma 8 together with [5, 86.6] identifies $\mathscr{E}$ as the class of torsion free Abelian groups which have hereditary generating systems [5, p. 332]. Therefore, we have a solution to the torsion free part of Problem 84 in [5] at least to the extent that our classification of the groups in $\mathscr{E}$ and characterizations of the class $\mathscr{E}$ can be said to determine the *structure* of the groups in $\mathscr{E}$.

For the remainder of this section we consider the adequacy of our theory for $\mathscr{E}$. For example, Kaplansky in [10] proposed three test problems which any adequate classification of a class of Abelian groups should be able to answer. Let us consider these test problems suitably adjusted to groups A, B and C in $\mathscr{A}$: (I) If A is isomorphic to a subgroup of B and B is isomorphic to a subgroup of A, then is $A \cong B$?, (II) If $A \oplus A \cong B \oplus B$, then is $A \cong B$?, (III) If $A \oplus C \cong B \oplus C$, then is $A \cong B$?. *All three problems have affirmative answers for* $A \in \mathscr{C}$ *and therefore, for* $A \in \mathscr{E}$. For note that $\mathscr{C} = \{A \in \mathscr{A} \mid (I)$ has an affirmative answer$\}$ and Jónsson's Theorem together with the symmetry and transitivity of the relation of quasi-isomorphism shows (II) and (III) have affirmative answers for $A \in \mathscr{C}$. On the other hand, our theory for $\mathscr{E}$ is only as adequate as our ability to effectively compute with the invariants $\delta(-)$ and $\tau_*(-)$. The inner type $\tau_*(-)$ is certainly a manageable invariant. The following example shows that the QD invariants are at least adequate for constructing large families of indecomposables in $\mathscr{E}$ with certain preassigned divisibility properties.

EXAMPLE 2. Assume $r_Q(V) = n > 2$. Let $\{\alpha_p\}_{p\in\pi}$ be a sequence of integers where $0 \leqq \alpha_p \leqq n$ for all p, $\alpha_p < n$ for at least $n+1$ primes p and $\alpha_q < n-1$ for at least one prime q. Further, let τ be a type such that $\tau(p) = \infty$ if and only if $\alpha_p = n$. Then there is an uncountable family $\{A_i\}_{i\in\lambda}$ such that

(i) A_i is an indecomposable group in $\mathscr{E}$ which is full in V

(ii) $s_p(A_i) = \alpha_p$ for all p and $\tau_*(A_i) = \tau$

(iii) $A_i \not\cong A_j$ for $i \neq j$

(iv) if $\alpha_q = 0$ for some q, then $\operatorname{Hom}(A_i, A_j) = \{0\}$ for $i \neq j$.

We give the construction in three steps:

I. Let $\{U_p\}_{p\in\pi}$ be a family of Q-subspaces of V such that $r_Q(U_p) = \alpha_p$ for each p. Then there is an uncountable subfamily $\{\delta^i\}_{i\in\lambda}$ of $\mathscr{L}(V)$

such that for each $i \in \lambda$ and $p \in \pi$, $(\delta^i)_p \cap V = U_p$, $r_{Z^{(p)}}((\delta^i)_p) \geqq n-1$, and $\delta^i \not\sim \delta^j$ for $i \neq j$.

Proof. If $U_p = V$ (i.e. $\alpha_p = n$), then let $\delta_p = V^{(p)}$. For $\alpha_p < n$, let $V = U_p \oplus U$ and select a pure subgroup B of $Z^{(p)}$ with $r(B) = n - \alpha_p$. Imbed B as a full subgroup of U. Then $r(d(B^{(p)})) = n - \alpha_p - 1$ (since $B \in \mathscr{E}$) and $d(B^{(p)}) \cap U = \{0\}$ (since $s_p(B) = 0$). Let $\delta_p = U_p^{(p)} \oplus d(B^{(p)}) \subsetneqq V^{(p)}$. Then $r_{Z^{(p)}}(\delta_p) = n-1$. Since $U_p \subsetneqq \delta_p$, the modular law gives $\delta_p \cap V = U_p \oplus (\delta_p \cap U) = U_p$.

Let q be a prime such that $n - \alpha_q > 1$. Then it is well-known that there is an uncountable family $\{B_i\}_{i \in \rho}$ of pure subgroups of $Z^{(q)}$ where $r(B_i) = n - \alpha_q$ and $B_i \not\cong B_j$ for $i \neq j$. As in the previous paragraph, let $V = U_q \oplus U$, imbed B_i in U, and let $\delta_q^i = U_q^{(q)} \oplus d(B_i^{(q)})$. We may regard $d(B_i^{(q)}) \in \mathscr{L}_q(U)$ and it follows that for $i \neq j$, $\delta_q^i \neq \delta_q^j$ (since if $\delta_q^i = \delta_q^j$, then by modularity $\delta(B_i) \sim \delta(B_j)$ and Theorem 3 would give $B_i \cong B_j$).

For $i \in \rho$, define δ^i by $(\delta_i)_p = \delta_p$ for $p \neq q$ and $(\delta^i)_q = \delta_q^i$. Then $\mathscr{F} = \{\delta^i\}_{i \in \rho}$ is an uncountable family of distinct QD invariants in $\mathscr{L}(V)$ such that $(\delta^i)_p \cap V = U_p$ and $r_{Z^{(p)}}((\delta^i)_p) \geqq n-1$. Since the group of automorphisms of V is countable, the equivalence class determined by a $\delta^i \in \mathscr{F}$ w/r to the equivalence relation $\sim$ on $\mathscr{L}(V)$ is countable. It follows that there is an uncountable subset $\lambda \subsetneqq \rho$ such that $\delta^i \not\sim \delta^j$ for $i \neq j$ and $i, j \in \lambda$.

II. Let $\alpha_1, \cdots, \alpha_{n+1}$ be integers such that $0 \leqq \alpha_i < n$. Then there are $n+1$ subspaces $\{V_i\}_{i=1}^{n+1}$ of V such that:
(a) $r_Q(V_i) = \alpha_i$, and (b) if $V = U \oplus W$ and for each i, either $V_i \supseteqq U$ or $V_i \supseteqq W$, then $U = \{0\}$ or $W = \{0\}$.

Proof. It is enough to assume $\alpha_i = n-1$ in (a) and to construct a family of $n+1$ hyperspaces which satisfy condition (b). Let X be a basis for V. Choose $V_1, \cdots, V_n$ as the hyperspaces generated by the n subsets of X with $n-1$ elements. Let V_{n+1} be any hyperspace which does not interset X. Then $\{V_i\}_{i=1}^{n+1}$ satisfies condition (b).

III. We now construct the required family of groups. Since $\alpha_p < n$ for at least $n+1$ primes, we may assume that the $n+1$ subspaces of V constructed in (II) are in the family of subspaces $\{U_p\}_{p \in \pi}$ in (I). By hypothesis τ is compatible with each δ^i in (I). Now use Theorem 3 to obtain a group A_i in $\mathscr{E}$ with $\delta(A_j) = \delta^i$ and $\tau_*(A_i) = \tau$. Since $U_p = (\delta^i)_p \cap V = d(Z_p A)$, A_i is indecomposable [Corollary 6 and (II)] and $s_p(A_i) = \alpha_p$ [Lemma 2]. Since $\delta^i \not\sim \delta^j$ for $i \neq j$, $A_i \not\cong A_j$ for $i \neq j$ [Theorem 3]. Hence, $\{A_i\}_{i \in \lambda}$ is the required family. For part (iv), let $\phi \in \operatorname{Hom}(A_i, A_j)$ for $i \neq j$ and assume that ϕ is not the zero map. Suppose $r_q(\ker \phi) = 1$, then $r_q(\phi(A_i)) = 0$. Since $s_q(A_j) = 0$, the only q-divisible subgroup of A_j is $\{0\}$ and so $\phi(A_i) = \{0\}$, a contradiction. Thus, $r_q(\ker \phi) = 0$ and so $\ker \phi = \{0\}$. Since ϕ is an imbedding of A_i in A_j, Theorem 3 gives $\delta(A_i) \leqq \delta(A_j)$. Since $s_p(A_i) = s_p(A_j)$ for all p and $A_i, A_j \in \mathscr{E}$, it follows that $\delta(A_i) \sim \delta(A_j)$. Since $\tau_*(A_i) = \tau_*(A_j)$, Theorem 3 gives $A_i \cong A_j$, a contradiction of (iii). Thus, ϕ is the zero map.

In the preceding example the $A_i's$ will be purely indecomposable if $\alpha_p = 0$ for some p [Corollary 6]. On the other hand, if $\alpha_p > 0$ for all p, then the $A_i's$ need not be purely indecomposable. In the following example we construct *an indecomposable group A in $\mathscr{E}$ which is not purely indecomposable*, i.e. A contains a pure, decomposable subgroup.

EXAMPLE 3. Assume $r_Q(V) = n > 2$. Let q and r be distinct primes and α_q, α_r be positive integers such that $\alpha_q + \alpha_r < n$. Further, let $\mathscr{F} = \{U_p\}_{p \in \pi}$ be a family of subspaces of V such that:

(i) $r_Q(U_q) = \alpha_q$, $r_Q(U_r) = \alpha_r$ and $U_r \cap U_q = \{0\}$,

(ii) for all p, $U_p \supseteqq U_q$ or $U_p \supseteqq U_r$.

Use the construction in (I) to obtain a group A in $\mathscr{E}$ which is full in V and $\delta_p(A) \cap V = U_p$ for each p. Since $d(Z_pA) = \delta_p(A) \cap V$, A is indecomposable by condition (i) and Corollary 6. Now let B be the pure hull in A of the subgroups $U_q \cap A$ and $U_r \cap A$. Then B is a full subgroup of $U_q \oplus U_r$ and for all $p \in \pi$, either $U_q \subseteqq d(Z_pB)$ or $U_r \subseteqq d(Z_pB)$. Since $B \in \mathscr{E}$, Corollary 6 shows that B is decomposable. Hence, A is indecomposable but not purely indecomposable.

We briefly mention another method of constructing uncountable families of indecomposable groups in $\mathscr{E}$ which have the same rank and are pairwise non-isomorphic. Let $\delta \in \mathscr{L}(V)$ such that $1 + r_{Z(p)}(\delta_p) \geqq n$ for all p where $n = r_Q(V)$, $r_{Z(p)}(\delta_p) < n$ for an infinite number of primes p, and δ does not satisfy part (iii) of Corollary 3. Note that such a δ exists by (I) and (II) in Example 2. Then there are an uncountable number of distinct types $\{\tau_i\}_{i \in \lambda}$ which are compatible with δ. Now apply Theorem 3 to obtain for each i, a rank n group A_i in $\mathscr{E}$ with $\tau_*(A_i) = \tau_i$ and $\delta(A_i) = \delta$. The resulting $A_i's$ are pairwise non-isomorphic [Theorem 3] and indecomposable [Corollary 3].

5. The endomorphisms of groups in $\mathscr{E}$.

LEMMA 9. *Let A be a reduced group in $\mathscr{E}$ and $A = \bigoplus_{i=1}^n A_i$ where A_i is indecomposable. Then* End $(A) \cong \bigoplus_{i=1}^n$ End (A_i).

Proof. It is sufficient to show Hom $(A_i, A_j) = \{0\}$ for $i \neq j$. Let

$\phi \in \mathrm{Hom}(A_i, A_j)$. If A_i is p-divisible, then so is $\phi(A_i)$. Thus, $Z_p\phi(A_i) \subseteqq d(Z_pA_j)$ whenever $r_p(A_i) = 0$. On the other hand, $Z_p\phi(A_i) \subseteqq Z_pA_j = d(Z_pA_j)$ whenever $r_p(A_j) = 0$. Since $A \in \mathscr{C}$, for each p, $r_p(A_i) = 0$ or $r_p(A_j) = 0$. It follows that $\phi(A_i) = \bigcap_{p \in \Pi} Z_p\phi(A_i) \subseteqq \bigcap_{p \in \Pi} d(Z_pA_j) = \{0\}$ (since A_j is reduced). Hence, ϕ is the zero map.

THEOREM 4. *If A is an indecomposable group in $\mathscr{C}$, then every endomorphism of A is an integral multiple of an automorphism of A.*

REMARK. The author is indebted to the referee for suggesting the following proof, which is considerably more natural than the original one.

Proof. If $\phi \in \mathrm{End}(A)$ is 1-1, then Theorem 2 gives $\phi(A) = nA$ for some $0 \neq n \in Z$ and so $(1/n)\phi$ is an automorphism of A. Therefore, it is sufficient to show that nonzero endomorphisms of A are monomorphisms. Suppose not. Then choose a nonzero singular endomorphism ϕ such that $r(\phi(A))$ is minimal. Let $B = \phi(A)$. Then B is clearly indecomposable and by induction on $r(A)$, $\phi|_B$ is a multiple of an automorphism of B. Hence, $\phi(B) = nB$ for some $n > 0$ and so $\ker \phi \cap B = \{0\}$. Let $n\alpha \in nA$. Then $\phi(n\alpha) \in nB = \phi(B)$ and so $n\alpha \in \ker \phi \oplus B$. Thus, $A \doteq B \oplus \ker \phi$ but A is indecomposable and in $\mathscr{C}$, a contradication.

We remark that a theorem of P. Griffith's [7] says that a pure subgroup B of the Z-adic completion of Z is purely indecomposable if and only if the nonzero endomorphisms of pure subgroups of B are monomorphisms. For finite rank B, this result is immediate from Theorem 4. Another immediate consequence of Theorem 4 is that the characteristic and fully invariant subgroups of an indecomposable group in $\mathscr{C}$ coincide.

Before stating some other consequences of Theorem 4, we recall some elementary facts about the Z-adic completions of reduced groups in $\mathscr{C}$. If R is a ring with identity, then we denote the additive subgroup of R by R^+ and *require that subrings contain the identity of R*. Let A be a reduced group in $\mathscr{C}$. Then the Z-adic completion of A, denoted by $\hat{A}$, is $\prod Z^{(p)}$ where p runs over $\Pi \backslash \Pi(A)$ [8]. In addition, $\hat{A}$ is a commutative ring with identity under component-wise multiplication. For $\alpha \in \hat{A}$, let l_α denote the left-multiplication by α. It is well-known that an endomorphism of $\hat{A}^+$ is a l_α for some $\alpha \in \hat{A}$. The automorphisms of $\hat{A}^+$ correspond to the left multiplications by units. Regard A as a pure dense subgroup of $\hat{A}$. Since $\hat{A}^+$ is pure injective [6], every endomorphism (automorphism) of A uniquely extends to an endomorphism (automorphism) of $\hat{A}^+$. In particular, the identity map on A extends to l_1. It follows that End (A) is isomorphic to a subring of $\hat{A}$. Finally, if R is a pure subring of $\hat{A}$, then $\Pi(R^+) = \Pi(A)$ and End $(R^+) \cong R$ (since $1 \in R$).

COROLLARY 7. *If A is a reduced group in $\mathscr{E}$, then* End (A) *is a ring direct sum of Principal Ideal Domains. In particular, if A is indecomposable, then* End (A) *is a Principal Ideal Domain which is isomorphic to a pure subring of $\hat{A}$ and $Q \otimes$* End (A) *is an algebraic number field.*

Proof. In view of Lemma 9 it is enough to assume that A is a reduced, indecomposable group in $\mathscr{E}$. Regard A as embedded in $\hat{A}$ as a pure dense subgroup and so by the previous remarks End (A) may be regarded as a subring R of $\hat{A}$. In this setting Theorem 4 says that the elements of R are integral multiples of units in $\hat{A}$. Thus, R is a domain and the purity of A in $\hat{A}$ shows that $\hat{A}/R$ is torsion free as a group, i.e. R is a pure subring of A. Let I be a nonzero ideal in R and $W = \{n \in Z \mid n > 0$ and $l_n \in I\}$. $W \neq \Phi$ by Theorem 4. Let $m = \min W$ and $\lambda \in I$. Then it follows from a routine use of the division algorithm that $\lambda = l_m \phi$ for some $\phi \in R$. Thus, R is a Principal Ideal Domain. That $Q \otimes$ End (A) is an algebraic number field is immediate from Theorem 4 and the finiteness of the rank of Hom (A, A).

COROLLARY 8. *If A is a reduced, indecomposable group in $\mathscr{E}$, then $r(A) = r(\mathrm{Hom}(A, A)) r_R(A)$ where $R =$* End (A).

Proof. A is a torsion free module over the Principal Ideal Domain R. Let $x_1, \cdots x_n$ be a maximal R-independent set in A where $n = r_R(A)$. Then $I = \bigoplus_{i=1}^{n} Rx_i$ is a full free R-submodule of A and so by Theorem 4, I as a group is full in A. This gives the equation.

Before stating the final two corollaries to Theorem 4, we recall two definitions from [12]. A is called *strongly homogeneous* if given two pure rank one subgroups B and C, there is an automorphism ϕ of A such that $\phi(B) = C$. A is called a *special* group if it is a strongly homogeneous QD group in $\mathscr{E}$. All rank one groups are trivially strongly homogeneous and the rank one special groups are precisely those with non-nil type. We will need the following property of special groups [12]: If A is a special group, then $A \cong$ Hom (A, A).

COROLLARY 9. *If A is a reduced group in $\mathscr{E}$, then* Hom (A, A) *is a direct sum of special groups. In particular, if A is indecomposable, then* Hom (A, A) *is a special group.*

Proof. By Lemma 9 it is enough to assume that A is indecomposable. Regard End (A) as a pure subring of $\hat{A}$. Let C be the pure hull in Hom (A, A) of the identity and B be any rank one pure subgroup of Hom (A, A). B contains a unit b by Theorem 4 and so l_b is an automorphism of Hom (A, A). Since $l_b(C) \cap B$ is a nonzero pure subgroup of both $l_b(C)$ and B, $l_b(C) = B$. Furthermore, Hom (A, A) is a QD group (since it is homogeneous of the type of C, which is nonnil). Thus, Hom (A, A) is a special group.

The previous corollaries are combined in the following in order to survey the finite rank pure subrings of the Z-adic completions of groups in $\mathscr{E}$.

COROLLARY 10. *Let $\Pi' \subseteqq \Pi$ and R be a finite rank pure subring of $T(\Pi') = \prod Z^{(p)}$ where p runs over Π'. Then*:

(i) *R is a ring direct sum of Principal Ideal Domains*

(ii) *R^+ is a direct sum of special groups and every group, which is a direct sum of special groups and whose Z-adic completion is $T(\Pi')$, is the additive subgroup of some R.*

(iii) *In particular, if R is a Principal Ideal Domain, then R^+ is special and every special group, whose Z-adic completion is $T(\Pi')$, is the additive subgroup of some Principal Ideal Domain R.*

Proof. These are all immediate from Corollaries 7 and 9 together with the facts that End $(R^+) \cong R$, Hom $(R^+, R^+) \cong R^+$ and for A special, Hom $(A, A) \cong A$.

The following shows that the relationship between a strongly homogeneous group in $\mathscr{E}$ and its endomorphism ring is the same as that between a rank one group and its endomorphism ring.

THEOREM 5. *Let A be an indecomposable group in $\mathscr{E}$ and I a full free subgroup of A. Then the following are equivalent:*

(i) *A is strongly homogeneous*

(ii) *$QD(A, I)$ is a special group which is isomorphic to* Hom(A, A)

(iii) *A is a torsion free, rank one* End (A)*-module.*

Proof. Assume (i). Let J be a full free subgroup of A such that $QD(A, J)$ is a characteristic subgroup of A [Lemma 6]. Since $QD(A, I) \cong QD(A, J)$, it is enough to show $QD(A, J)$ is a special group isomorphic to Hom (A, A). Let B and C be pure rank one subgroups of $QD(A, J)$. Then there is an automorphism ϕ' of A such that $\phi'(PH(B)) = PH(C)$ where $PH(-)$ denotes the pure hull in A. Let ϕ be the restriction of ϕ' to $QD(A, J)$. Then ϕ is an automorphism of $QD(A, J)$. It follows from the purity of B and C in $QD(A, J)$ that $PH(B) \cap QD(A, J) = B$ and $PH(C) \cap QD(A, J) = C$. Thus, $\phi(B) = C$ and so $QD(A, J)$ is special. This together with Corollary 2 gives $QD(A, J) \cong \text{Hom}(A, A)$, which is (ii). Assume (ii). Then $r(A) = r(\text{Hom}(A, A))$ and (iii) follows from Corollary 8. Assume (iii). Let B and C be pure rank one subgroups of A and $0 \neq x \in B$, $0 \neq y \in C$. Then there are nonzero $\phi', \lambda' \in \text{End}(A)$ such that $\phi'(x) = \lambda'(y)$. By Theorem 4 $\lambda' = n\lambda$ and $\phi' = m\phi$ for some automorphisms ϕ and λ and some nonzero n, m. Note that $B = PH((mx))$ and $C = PH((ny))$. Let $\theta = \lambda^{-1}\phi$. Since θ is an automorphism of A, $\theta(B)$ is a pure rank 1 subgroup of A which contains ny. Thus, $C \subseteqq \theta(B)$ and it follows from purity that $C = \theta(B)$, which is (i).

COROLLARY 11. *If A is an indecomposable group in $\mathscr{E}$ of prime rank, then A is either strongly homogenous or* End (A) *is a subring of Q.*

Proof. Immediate from Corollary 8 and Theorem 5.

COROLLARY 12. *Let A be a reduced group in $\mathscr{E}$. Then* $r(A) = r(\text{Hom}(A, A))$ *if and only if A is a direct sum of strongly homogeneous groups.*

Proof. Immediate from Lemma 9 and Theorem 5.

We follow Reid [11] in calling A *irreducible* if A has no nontrivial pure, fully invariant subgroups. Since strongly homogeneous groups in $\mathscr{A}$ have no nontrivial pure, characteristic subgroups, strongly homogeneous groups are always irreducible.

COROLLARY 13. *Let $A \in \mathscr{E}$. Then A is irreducible if and only if A is strongly homogeneous.*

Proof. We need only check the "only if" and may assume that A is reduced. Since A is an irreducible group in $\mathscr{E}$, A is indecomposable. Therefore, A is a torsion free module over its endomorphism ring R. Let $0 \neq x \in A$. Then Rx is a nonzero fully invariant subgroup of A and is full in $PH(Rx)$. It follows that $PH(Rx)$ is a nonzero pure, fully invariant subgroup of A. Thus, $PH(Rx) = A$ which implies $r(A) = r(R)$ and so A is strongly homogeneous [Corollary 12].

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