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SOME RESULTS ON NEAREST POINTS AND SUPPORT PROPERTIES OF CONVEX SETS IN c_0

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The space c_0 is shown to contain a closed and bounded symmetric convex body such that no point of its complement has a nearest point in it. Related results involving the existence of functionals which support each member of a family of convex sets are also discussed.

1. Introduction and preliminary results. It has been shown in [3] that if X is a separable conjugate Banach space (i.e., if $X = E^*$ where E is a normed linear space and X contains a countable dense set) and if S is a closed, bounded set in X then, for every nonnegative real number d, there exist x in X and s_0 in S such that

$$d = ||x - s_0|| = \inf\{||x - s||: s \in S\} .$$

Further, it was shown that under the additional assumptions that the unit ball in X and the weak* closed convex hull of S are both strictly convex, the set of points in X admitting nearest points in S is weak* dense in X. The aim of the present paper is to define more precisely the relationship between these geometrical properties and the assumption that X is a separable conjugate space. The paper is concerned, for the most part, with the behaviour of c_0 in this respect. As is well known, this space is separable but not a conjugate space.

Our results show, first of all, that c_0 belongs to the class N_2 ([4]), i.e., the class of those Banach spaces which contain a closed, bounded convex set such that no point in its complement has a nearest point in the set; thus correcting an oversight of Klee. In the third section extensions of this result are presented in two directions. Finally, it is shown that, in a certain sense, the geometry of c_0 on the one hand and that of separable conjugate spaces on the other, are diametrically opposed; here we are indebted to V. L. Klee for remarks (in a private communication) which led us in this direction.

We have tried to obtain results (one way or the other) about m — a conjugate, nonseparable space—but have so far failed.

Before coming to the main theorem we give a preliminary proposition which relates various geometric properties.

PROPOSITION 1. *Let X be a real normed linear space and C a closed, bounded, convex set in X. Let N denote the set of points in $X\setminus C$ which have a nearest point in C. Then the following are equivalent:*

(i) *$N = \phi$;*
(ii) *if B is the closed unit ball and $\lambda > 0$ then $\lambda B + C$ is open;*
(iii) *if $f \in X^*(f \neq 0)$ then either $f(B)$ or $f(C)$ is an open interval.*

Proof. That (i) and (ii) are equivalent follows from Lemma 1 of [3]. Now if $N \neq \phi$ and $a \in N$ then, for some $\lambda > 0$, $(\lambda B + a) \cap C$ consists of a convex, nonempty, subset D of the boundary of $\lambda B + a$ so that a closed hyperplane exists which separates $\lambda B + a$ and C and which contains D. It follows that (iii) implies (i). On the other hand, if (iii) fails, i.e., if there exists a continuous functional which attains either its infimum or its supremum on B and C then a simple argument involving a translation of B (and possibly a reflection in the origin) produces an a in N (cf. also [4] p. 172) so that (i) implies (iii).

2. The space c_0 is of class N_2 ([4]).

THEOREM 1. *There exists a closed, bounded, symmetric, convex body S in c_0 such that if $f \in c_0^*(f \neq 0)$ then either $f(S)$ or $f(B)$ (where B denotes the unit ball in c_0) is an open interval.*

Proof. Let $x = (\xi_1, \xi_2, \xi_3, \cdots)$ denote a typical element of c_0. Consider the following linear functionals and linear operators on c_0:

$$\begin{aligned}
\delta_i(x) &= \xi_i \qquad (i = 1, 2, 3, \cdots) ; \\
f_0 \in c_0^* &= l_1 \text{ defined by } f_0 = (2^{-2}, 2^{-3}, 2^{-4}, \cdots, 2^{-(n+1)}, \cdots) ; \\
Tx &= (\xi_1, \xi_3, \xi_5, \cdots, \xi_{2n-1}, \cdots) ; \\
Ux &= (\xi_2, \xi_3, \xi_4, \cdots, \xi_{n+1}, \cdots) ; \\
Ex = TUx &= (\xi_2, \xi_4, \xi_6, \cdots, \xi_{2n}, \cdots) ; \\
P_i x = UTE^{i-1}x &= (\xi_{k_1^i}, \xi_{k_2^i}, \xi_{k_3^i}, \cdots, \xi_{k_n^i}, \cdots)
\end{aligned}$$

where

$$k_n^i = 2^{i-1}(2n + 1), (i = 1, 2, 3, \cdots) .$$

Let

$$g_i(x) = \delta_i(x) + f_0(P_i(x))$$

and let

$$S = \{x \in c_0 \mid |g_i(x)| \leqq 1, i = 1, 2, 3, \cdots\} .$$

Since each g_i is a continuous linear functional on c_0, S is an intersection of closed half-spaces and, therefore, is closed and convex. Moreover, since $||\delta_i|| = ||P_i|| = 2||f_0|| = 1$, if $||x|| \leqq \frac{2}{3}$ then $x \in S$. Also, if $||x|| > 2$, since there exists i such that $|\xi_i| = ||x||$, we have

$$|g_i(x)| \geqq |\delta_i(x)| - |f_0(P_i(x))| \geqq ||x|| - ||f_0|| \, ||P_i|| \, ||x|| = \tfrac{1}{2}||x|| > 1 .$$

Thus S is bounded and has nonempty interior.

It remains to show that if $f \in c_0^*$ then either $f(B)$ or $f(S)$ is an open interval. To see this, define A^* by $A^*\delta_i = g_i (i = 1, 2, 3, \cdots)$ and extend A^* to the whole of $l_1 = c_0^*$ by linearity and continuity. Then

$$(Ax)_i = \delta_i(Ax) = g_i(x)$$

so that the linear transformation A on c_0 can be represented by the infinite matrix:

$$\begin{pmatrix} 1 & 0 & 2^{-2} & 0 & 2^{-3} & 0 & 2^{-4} & 0 & 2^{-5} & 0 & 2^{-6} & 0 & 2^{-7} & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 2^{-2} & 0 & 0 & 0 & 2^{-3} & 0 & 0 & 0 & 2^{-4} & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2^{-2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & & & & & & & & & & & & & & \end{pmatrix}$$

where the ith row has zeros except for the set

$$N_i = \{i\} \cup \{2^{i-1}(2n + 1) \,|\, n = 1, 2, 3 \cdots\}$$

and

$$a_{ii} = 1,\ a_{i,2^{i-1}(2n+1)} = 2^{-(n+1)}\ .$$

It is readily verified that A maps c_0 onto c_0 and is one-one.

Now

$$\begin{aligned} S &= \{x \in c_0 \,|\, |g_i(x)| \leqq 1,\ i = 1, 2, 3, \cdots\} \\ &= \{x \,|\, |A_i^*\delta_i(x)| \leqq 1\ \forall i\} \\ &= \{x \,|\, |\delta_i(Ax)| \leqq 1\ \forall i\} \\ &= \{x \,|\, Ax \in B\} = A^{-1}(B)\ . \end{aligned}$$

Since S is bounded this shows, incidentally, that A has a bounded inverse. Also

$$f(x) = f(AA^{-1}x) = (A^*f)(A^{-1}x)$$

and so

$$f(B) = A^*f(A^{-1}(B)) = A^*f(S)\ .$$

Thus $f(B)$ is a closed interval if and only if $A^*f(S)$ is a closed interval. But those continuous linear functionals which attain their norm on B are (finite) linear combinations of the δ_i's. Hence the functionals which attain their norm on S are (finite) linear combinations of the g_i's. Since these latter functionals, as l_1 sequences, clearly have infinitely many nonzero entries it is obvious that the two sets are disjoint. This concludes the proof.

3. Two extensions. (i) The previous section was concerned with two sets (the unit ball B and the set S) in c_0 such that the sets of nontrivial functionals which support B and S respectively are disjoint. Here we show that it is possible to construct a family $\mathscr{S}$, indexed by the real numbers in the interval $[0, 1)$, of closed, bounded, symmetric, convex bodies in c_0 such that if $f \in c_0^*$ then f attains its supremum on at most one member of $\mathscr{S}$.

Let $\alpha = 0\cdot\alpha_1\alpha_2\alpha_3\cdots$ be a real number in $[0, 1)$ represented as a binary expansion which does not terminate in 1's, i.e., $\alpha_n = 0$ or 1 and there are infinitely many 0's. Let f_α be the element of $c_0^* = l_1$ defined by

$$f_\alpha = (\phi_1, \phi_2, \cdots, \phi_n, \cdots)$$

where

$$\phi_n = \begin{cases} 2^{-(n+1)} & \text{if } \alpha_n = 0 \\ 0 & \text{if } \alpha_n = 1\,; \end{cases}$$

and let

$$g_{i,\alpha} = \delta_i + P_i^* f_\alpha\,.$$

Finally, define

$$S_\alpha = \{x \in c_0 \mid |g_{i,\alpha}(x)| \leqq 1,\ i = 1, 2, 3, \cdots\}\,.$$

As before S_α is a closed, bounded, convex set with interior and, as before, each S_α is supported by finite linear combinations of the functions $\{g_{i,\alpha} \mid i = 1, 2, 3, \cdots\}$. If $\alpha \neq \alpha'$ then, for some n, $\alpha_n \neq \alpha'_n$ and we can suppose that $\alpha_n = 1$, $\alpha'_n = 0$. Let f support S_α. Then f is a finite linear combination of the $g_{i,\alpha}$. Let i_0 be the maximal index which occurs in this combination. Then the sequence representing f has a zero in the $k_0 = 2^{i_0-1}(2n + 1)$ place. If f' supports $S_{\alpha'}$, in order that f' have a zero in that same place either f' has no contribution from $g_{i_0,\alpha'}$ or f' does have a contribution from $g_{k_0,\alpha'}$. In the first case f' has zeros on the whole set

$$N_{i_0} = \{2^{i_0-1}(2k + 1) \mid k = 1, 2, 3, \cdots\}$$

and therefore differs from f, while in the second case, since $k_0 > i_0$, f has zeros on the whole set N_{k_0} and again $f \neq f'$. We thus have the desired conclusion.

(ii). The set S in the proof of Theorem 1 is similar to the unit ball in the sense that it is the image of the unit ball under an invertible linear transformation and hence each of its faces has finite codimension. We show here that it is possible to construct a set S' with the required properties and which is, moreover, strictly convex, i.e., its faces are 0-dimensional. With the same notation as in §2, let

$$p(x) = \|x\| + \left(\sum_{n=1}^{\infty} 2^{-2n} g_n(x)^2\right)^{1/2}$$

and let $S' = \{x \in c_0 \mid p(x) \leqq 1\}$. Since, as can be readily seen, p is a norm on c_0 which is equivalent to the original norm, S' is a centrally symmetric, convex body which is closed and bounded. Further, since $g_n(x) = 0$ for all n if and only if $x = 0$ (this follows from the fact that S in §2 is bounded), S' is strictly convex; (cf. Köthe [5] p. 365). To complete the proof we show that if x is in the complement of S' then x has no nearest point in S'. Suppose, on the contrary, that there exists x_0 with $p(x_0) > 1$ and s_0 with $p(s_0) = 1$ such that

$$\|x_0 - s_0\| = \inf \{\|x_0 - s\| \mid s \in S'\} .$$

Now, since $x_0, s_0 \in c_0$ there exists N such that

$$|\delta_j(x_0 - s_0)| < \tfrac{1}{2}\|x_0 - s_0\| \text{ and } |\delta_j(s_0)| < \tfrac{1}{2}\|s_0\|$$

for all $j > N$, (clearly the numbers on the right are nonzero). Let $n = 2^{i-1}(2k+1)$. If $s_0 = (\sigma_1, \sigma_2, \cdots, \sigma_n, \cdots)$, consider $s_0' = (\sigma_1, \sigma_2, \cdots, \sigma_n - \varepsilon, \cdots)$. Clearly $g_j(s_0) = g_j(s_0')$ for all j except $j = n$ and $j = i$. For these integers we have $g_n(s_0') = g_n(s_0) - \varepsilon$ and $g_i(s_0') = g_i(s_0) - \varepsilon 2^{-(k+1)}$. Hence

$$2^{-2n}(g_n(s_0')^2 - g_n(s_0)^2) = 2^{-2n}\varepsilon^2 - 2^{-2n}\varepsilon g_n(s_0)$$

and

$$2^{-2i}(g_i(s_0')^2 - g_i(s_0)^2) = 2^{-2(i+k+1)}\varepsilon^2 - 2^{-2i-k-1}\varepsilon g_i(s_0) .$$

We can assume that $g_i(s_0)$ is positive (otherwise replace ε by $-\varepsilon$). Then, since $g_n(s_0) \to 0$ as $n \to \infty$, choose k so large and ε sufficiently small so that

(a) $n = 2^{i-1}(2k+1) > N$
(b) $\varepsilon < \frac{1}{2}\|s_0\|$
(c) $\varepsilon < \frac{1}{2}\|x_0 - s_0\|$
(d) $2^{-2n+k+1}\varepsilon + 2^{-2i-k-1}\varepsilon + 2^{-2n+k+1}|g_n(s_0)| < 2^{-2i}g_i(s_0)$.

Then $p(s_0') < p(s_0) = 1$ but $\|x_0 - s_0'\| = \|x_0 - s_0\| = \inf \{\|x_0 - s\| \mid s \in S\}$. This is clearly absurd since x_0 cannot have a nearest point which is interior to the set and the proof is complete.

4. How to support a family. It was shown in §3(i) that in c_0 it is possible to construct an uncountable family of closed bounded convex bodies such that no linear functional supports more than one of them. In contrast we have the following theorem.

THEOREM 2. *Let $\{C_1, C_2, \cdots, C_n \cdots\}$ be a countable family of closed and bounded sets in a separable Banach space X which is the conjugate of some Banach space Y. Let S denote the set of points y in Y with the property that for each $i = 1, 2, \cdots$ there is a $c_i \in C_i$ such that*

$$\sup \{\langle y, c\rangle : c \in C_i\} = \langle y, c_i\rangle^1 .$$

Then S is a dense G_δ.

Proof.[2] The set S_i of all $y \in Y$ for which $\sup\{\langle y, c\rangle : c \in C_i\}$ is attained is, by a result of Asplund (cf. Theorem 3 and proof of Proposition 5 in [1]), a dense G_δ. It follows that $S = \bigcap_1^\infty S_i$ too is a dense G_δ.

To show that countability of the family $\{C_1, C_2, \cdots\}$ in Theorem 2 is essential we bring the following

PROPOSITION 2. *To every continuous linear functional u on l_1 with $\|u\| = 1$ there is a closed and bounded convex set C such that u fails to attain its supremum on it.*

Proof. It suffices to show that a C as required exists for each $u = (u_1, u_2, \cdots) \in m$ for which a natural number k exists with $\|u\| = |u_k| = 1$ as the unit ball may clearly serve as C for all other u of norm 1. Clearly, if C satisfies the conclusion for a given u then $-C$ does for $-u$; thus we may assume that $u_k = 1$. Now the sequence $\{u_{k+1}, u_{k+2}, \cdots\}$ contains a subsequence $\{u_{n_1}, u_{n_2}, \cdots\}$ which is either nonincreasing or nondecreasing. The proofs being similar in both cases we assume that

$$u_{n_1} \leqq u_{n_2} \leqq \cdots$$

i.e., the subsequence is nondecreasing.

Let $A = \{x^{(1)}, x^{(2)}, \cdots\} \subset l_1$ be defined by setting

$$x_i^{(m)} = \begin{cases} 1 - \dfrac{1}{m} & \text{for } i = k \\ 1 & \text{for } i = n_m \\ 0 & \text{otherwise} \end{cases}$$

and set $C = \overline{\text{co}}\, A$. Suppose there is a

[1] Here, and in the sequel, we find it advantageous to use the customary $\langle x, f\rangle$ for $f(x)$.

[2] We are indebted to the referee for suggesting this proof.

$$z = (z_1, z_2, \cdots, z_{k-1}, z_k, z_{k+1}, \cdots) \in C$$

at which u attains its supremum. Then, as can be readily seen, $z_1 = z_2 = \cdots = z_{k-1} = 0$, $\sum_{m=1}^{\infty} z_{n_m} = 1$ and $z_k = 1$. Since $z_{n_M} \neq 0$ for some natural number M, if $\delta = |z_{n_M}|/2M$ then $||z - x|| \geqq \delta$ for all $x = \sum_{i=1}^{n} \lambda_i x^{(i)}$ with $\lambda_i \geqq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$. Indeed,

$$||z - x|| \geqq |z_{n_M} - x_{n_M}| \geqq |z_{n_M}| - \lambda_{n_M}$$

so that we may assume that $\lambda_{n_M} \geqq \dfrac{1}{2} |z_{n_M}|$. But then

$$||z - x|| \geqq 1 - \sum_{i=1}^{n}\left(1 - \frac{1}{i}\right)\lambda_i = \sum_{i=1}^{n} \frac{\lambda_i}{i} \geqq \frac{\lambda_{n_M}}{M} \geqq \frac{|z_{n_M}|}{2M} = \delta .$$

It follows that $z \notin C$ so that u cannot attain its supremum on C, as asserted.

5. Concluding remarks and problems.

1. We have already pointed out that the behaviour of m in this respect is unknown.

2. A procedure first given by Day [2] was shown by Rainwater [6] to yield a locally uniformly convex unit ball in c_0. It is possible that a similar procedure applied to our construction of the set S' in §3 (ii) will give a locally uniformly convex set with the same properties.

3. It is unknown whether, given any closed and bounded convex body S_1 in c_0, it is possible to construct a second set S_2 such that no functional supports both S_1 and S_2 (i.e., whether Theorem 1 remains true under any equivalent renorming of c_0).

4. The above construction can be used in a more general $c_0(\Gamma)$.

5. The definition of the set S in Theorem 2 can be modified so as to obtain a stronger conclusion. Indeed, without changing the proof, one can require that each element $y \in S$ strongly expose each of the sets C_i; i.e., whenever $\{c_i^{(n)}: n = 1, 2, \cdots\} \subset C_i$ and $\langle y, c_i^{(n)} \rangle \to \langle y, c_i \rangle$ then $c_i^{(n)} \to c_i (i = 1, 2, \cdots)$.

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