ON THE ENDOMORPHISM RING OF AN ABELIAN p-GROUP, AND OF A LARGE SUBGROUP

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For an abelian p-group G, denote the endomorphism ring of G by $E(G)$, the ideal of small endomorphisms by $E_s(G)$ and the quotient ring $E(G)/E_s(G)$ by $S(G)$. It is not difficult to show that for a large subgroup L of G, the map that sends an endomorphism of G to its restriction on L induces a monomorphism $S(G) \to S(L)$. We show that if B_1 is a large subgroup of a group B_2 which is a direct sum of cyclic p-groups and is of cardinality not more than $2^{\aleph_0}$ and R_1 and R_2 are suitable subgroups of $E(B_1)$ and $E(B_2)$, then there are groups G_1 and G_2 having B_1 and B_2 as basic subgroups such that G_1 is large in G_2 and $S(G_i) \cong R_i/(E_s(B_i) \cap R_i)$, $(i = 1, 2)$.

As with Corner's result [2, Th. 2.1], much of the value of this theorem is in producing examples. Although it is clear that these theorems can be used to exhibit counterexamples to several obvious conjectures, we will confine ourselves to giving a counterexample to a theorem of Paul Hill which asserts that the endomorphism ring of a p-group G (with $p \neq 2$) is generated by its units if this property holds for any large subgroup of G. To show that our example has the necessary properties we give, in § 3, some conditions equivalent to a ring of endomorphisms being generated by its units.

All groups considered will be abelian. For a p-group G, we will denote the set of automorphisms of G by $A(G)$ and the subring of $E(G)$ that $A(G)$ generates by $\{A(G)\}$. The ideal of endomorphisms that annihilate the socle of G will be denoted by $N(G)$. As we will be writing morphisms on the right, the restriction of an endomorphism α of a group G to a fully invariant subgroup L of G will be written as $(L \mid \alpha)$. For information on large subgroups and small homomorphisms the reader is referred to Pierce's paper [6]. The topology on a group or its endomorphism ring will always be the p-adic topology. The closure of a set X will be denoted by X^-.

2. **An extension of Corner's Theorem.** Given a torsion complete p-group C with unbounded basic subgroup B of cardinality not more than $2^{\aleph_0}$, Corner's theorem [2, Th. 2.1] gives sufficient conditions on a subring R of $E(B)$ for there to exist a pure subgroup G of C containing B such that $E(G) = E_s(G) \oplus R$. It is necessary for our example in section 4 to weaken slightly the conditions on the ring R. In doing this we obtain a group G such that $E(G) = E_s(G) + R$ without neces-

sarily having $E_s(G) \cap R = (0)$, so that $S(G) \cong R/(E_s(B) \cap R)$. But as will be seen in the example, there is no real loss in this. After we have given the necessary extension of Corner's theorem, we will show that if B is a large subgroup of a basic subgroup B' of a torsion complete group C' (containing C) and R is a suitable ring of extensions of the elements of R to B', then there is a pure subgroup G' of C' containing B', having G as a large subgroup and such that $E(G') = E_s(G') + R'$.

DEFINITION. If α and β are endomorphisms of a group G and $(p^n G)[p](\alpha - \beta) = 0$ for some integer n, then we will say that α is eventually equal to β on $G[p]$ and write $\alpha \doteq \beta$.

We begin with one extension of Corner's theorem that is seen to hold by simply looking at his proof, but is not quite the theorem we need.

LEMMA 1. *Let C be a torsion complete p-group with an unbounded basic subgroup B of cardinality not more than $2^{\aleph_0}$ and let Φ be a closed separable subgroup of $E(B)$ such that $1 \in \Phi$ and*

(C_1). *If $\varphi_1, \varphi_2 \in \Phi$, there is an element $\varphi_3 \in \Phi$ and a small endomorphism $\sigma \in E_s(B)$ such that $\varphi_1 \varphi_2 = \varphi_3 + \sigma$.*

(C_2). *If $\varphi \in \Phi$ and $\varphi \doteq 0$, then $\varphi \in p\Phi$.*
Then there is a pure subgroup G of C containing B such that $E(G) = E_s(G) \oplus \Phi$.

Proof. This is just Corner's theorem with the assumption that Φ be closed under multiplication replaced by (C_1). But the only place in which the multiplicative closure of Φ is used is in the proof of his Lemma 2.10 in which he has a subset X_r of C and a Φ-invariant subgroup H of C containing B and he wants to conclude that $H_r = H + \Sigma\{x_i \Phi \,|\, x_i \in X_r\}$ is a Φ-invariant subgroup of C. That it is a subgroup is a consequence of the fact that Φ is a group. On the other hand, for h in H_r, there is an element h' in H and a finite set of elements $\{\varphi_i\}_0^m$ of Φ such that $h = h' + \sum_0^m x_i \varphi_i$, whence, for any $\varphi \in \Phi$, $(h)\varphi = h'\varphi + \sum_0^m x_i(\varphi_i \varphi)$. Now by condition (C_1), there is, for each i, an element $\varphi_i' \in \Phi$ and a small endomorphism σ_i of B such that $\varphi_i \varphi = \varphi_i' + \sigma_i$. However, since $\sigma_i \in E_s(B)$, we know, from [6, Th. 4.4] that $(C)\sigma_i \subseteq B$, from which we conclude that $(x_i)\sigma \in B \subseteq H$ and

$$(x)\varphi = (h')\varphi + \sum_1^m (x_i)\sigma_i + \sum_1^m (x_i)\varphi_i' \in H + \sum \{x_i \Phi \,|\, x_i \in X_r\} \,.$$

LEMMA 2. *Given any p-group G, $E_s(G)$ is complete, and there is a torsion free, complete subgroup K of $E(G)$ such that $E(G) = E_s(G) \oplus K$ and $1 \in K$.*

Proof. That $E_s(G)$ is pure and complete in $E(G)$ and that $E(G)/E_s(G)$

is torsion free are shown by Pierce [6, Th. 7.5]. It is then easily seen that $E_s(G) \oplus R_p \cdot 1$ is pure and torsion complete in $E(G)$ (where R_p is the p-adic integers) so that it has a complement K' in $E(G)$. The group $K = R_p \cdot 1 \oplus K'$ is then the desired complement of $E_s(G)$.

THEOREM 1. *Let C be a torsion complete p-group with an unbounded basic subgroup B of cardinality not more than $2^{\aleph_0}$ and let R_0 be a countable unital subring of $E(B)$ such that*

(C'). *If $\alpha \in R_0$ is such that $\alpha \doteq 0$, there is an element $\beta \in R_0$ and a small endomorphism σ of B such that $\alpha = p\beta + \sigma$.*

Then there is a pure subgroup G of C containing B such that $E(G) = E_s(G) + (E_s(B) + R_0)^-$.

Proof. Using lemma 2, write $E(B) = E_s(B) \oplus K$ with $1 \in K$, let $\Phi_0 = (R_0 + E_s(B)) \cap K$ and $\Phi = \Phi_0^-$. Certainly Φ is a closed separable subgroup of $E(B)$ containing 1. By the modular law, it is seen that

(1) $\Phi_0 + E_s(B) = R_0 + E_s(B)$.

It is evident that since $R_0 + E_s(B)$ is a ring, Φ_0 has the property (C_1) of lemma 1 from which it is easily shown that Φ does also. To show that Φ_0 has the condition (C_2) of Lemma 1, suppose that $\varphi \in \Phi_0$ and $\varphi \doteq 0$. By (1) we can choose $\psi \in R_0$ and $\sigma \in E_s(B)$ such that $\varphi = \psi + \sigma$, so that $\psi \doteq 0$ and by (C'), $\psi = p\psi' + \sigma'$ for some $\psi' \in R_0$ and $\sigma' \in E_s(B)$. Applying (1) again then gives $\varphi = p\varphi' + \sigma''$ with $\varphi' \in \Phi_0$ and $\sigma'' \in E_s(B)$. Inasmuch as $\Phi_0 \cap E_s(B) = (0)$, we infer that $\varphi = p\varphi' \in p\Phi_0$. Now, if $\varphi \in \Phi$ and $\varphi \doteq 0$ there is a sequence $\{\varphi_i\}_1^\infty$ of elements of Φ_0 that converges to φ. The endomorphisms φ_i can be chosen such that $\varphi - \varphi_i \in pK$ for every i, so that $\varphi_i \doteq 0$ and there is an element $\varphi_i' \in R_0$ such that $\varphi_i = p\varphi_i'$. Using the fact that K is torsion free, it is easily seen that $\{\varphi_i'\}_{i=1}^\infty$ is a Cauchy sequence which, since Φ is complete, has a limit $\varphi' \in \Phi$. Therefore $\varphi = p\varphi' \in p\Phi$, Φ has the property (C_2) of Lemma 1, and we can apply this lemma to obtain a pure subgroup G of C containing B such that $E(B) \doteq E_s(G) \oplus \Phi$. Now observe that since $(E_s(B) + R_0)^-$ contains Φ_0 and is closed, we have $(E_s(B) + R_0)^- \supseteq \Phi$ whence $E(G) \subseteq E_s(G) + (E_s(B) + R_0)^-$. On the other hand $E(G) \supseteq E_s(B) + \Phi_0 = E_s(B) + R_0$, so that, since $E(G)$ is closed, $E(G) \supseteq (E_s(B) + R_0)^-$, and we conclude that $E(G) = E_s(G) + (E_s(B) + R_0)^-$.

NOTATION. For an element x in a p-group G we will denote the height of x in G by $h_G(x)$. The nth Ulm invariant of G will be denoted by $f_G(n)$.

LEMMA 3. *Let A be a p-group without elements of infinite height*

with subgroups G, K and L such that (i) G *is pure in A,* (ii) L *is large in G,* (iii) L *is pure and dense in K. Then the subgroup $M = K + G$ of A has K as a large subgroup.*

Proof. According to Pierce [6, Th. 2.6], the large subgroups of a group G are exactly the subgroups of the form

$$G\{n(i)\} = \{g \in G \mid h_G(p^i g) \geqq n(i) , \quad i = 0, 1, 2, \cdots\} ,$$

for all finite U-sequences $\{n(i)\}_0^\infty$, where a strictly increasing sequence $\{n(i)\}_0^\infty$ of nonnegative integers is called a finite U-sequence for a group G of infinite length if $n(i) + 1 < n(i + 1)$ only if $f_G(n(i)) \neq 0$. It is clear that G is pure and dense in M so that the two groups have the same Ulm invariants and every finite U-sequence of G is also a finite U-sequence of M. Thus, if $\{n(i)\}_0^\infty$ is the fininite U-sequence such that

$$(2) \qquad L = \{g \in G \mid h_G(p^i g) \geqq n(i) , \quad i = 0, 1, 2, \cdots\} ,$$

the lemma will be proved if we can show that

$$(3) \qquad K = \{g \in M \mid h_M(p^i g) \geqq n(i) , \quad i = 0, 1, 2, \cdots\} .$$

First observe that for every i, $n(i) \geq i$, so by the purity and denseness of L in K

$$(4) \qquad K[p^r] = (p^{n(i)-i} K)[p^r] + L[p^r] , \qquad r = 1, 2, \cdots,$$

Now suppose that $g \in K$ and i is a nonnegative integer. Then by (4) $g = p^{n(i)-i} g_i + x_i$ with $g_i \in K$ and $x_i \in L$, so that $p^i g = p^{n(i)} g_i + p^i x_i$. But since $x_i \in L$, $p^i x_i = p^{n(i)} y_i$ for some $y_i \in G$, whence $h_M(p^i g) \geqq n(i)$, and we conclude that $g \in M\{n(i)\}$. Conversely, suppose $z \in M\{n(i)\}$ and $z = x + g$ with $x \in K$ and $g \in G$. Then let r be an integer such that z, x, $g \in A[p^r]$ and choose $x' \in K$ and $y \in L[p^r]$ with $x = p^{n(r)-r} x' + y$, so that $p^i(g + y) = p^i z - p^{n(r)-r+i} x'$. For $i < r$, $n(r) - r + i > n(i)$, so that $h_G(p^i(g - y)) = h_M(p^i(g - y)) \geqq n(i)$, while for $i \geqq r$ $p^i(g - y) = 0$, whence $h_M(p^i(g - y)) \geqq n(i)$. Thus $g - y \in G \mid n(i) \mid = L$ and $g \in L$ from which we conclude that $z \in K$, as was to be shown.

LEMMA 4. *Let K be a large subgroup of a direct sum B of cyclic p-groups. Then an endomorphism α of K can be extended to an endomorphism α' of B if and only if $h_B(x) \leqq h_B(\alpha(x))$ for every $x \in K$.*

Proof. If α has an extension, it clearly does not diminish B-height on K. By a lemma of Pierce [6, Lemma 2.12], if $B = \bigoplus \sum_{i \in I} (b_i)$, then $K = \bigoplus \sum_{i \in I} ((b_i) \cap K)$, so that there is a function m of I into the natural numbers such that $K = \bigoplus \sum_{i \in I} (p^{m(i)} b_i)$. Thus if α does not diminish B-height on K, given $i \in I$, there is an $x_i \in B$ such that

$\alpha(p^{m(i)}b_i) = p^{m(i)}x_i$ and the endomorphism of B given by $\alpha'(b_i) = x_i$ is the desired extension of α.

The following theorem, when taken with Theorem 1 provides the representation of two rings that we seek.

THEOREM 2. *Suppose G is a p-group with basic subgroup B and R is a subring of $E(B)$ containing $E_s(B)$ such that $E(G) = E_s(G) + R$ and suppose that B is a large subgroup of the basic subgroup B' of the torsion complete group C' containing C, the torsion completion of B. Then there is a pure subgroup G' of C' containing B' having G as a large subgroup and such that $E(G') = E_s(G') + R'$ where $R' = \{\alpha \in E(B') \mid (B \mid \alpha) \in R\}$.*

Proof. We let $G' = G + B$, so that by Lemma 3, G is large in G'. Inasmuch as G'/B' is isomorphic to a quotient group of G/B, it is a divisible group. This, together with the fact that B' is pure in C' yields that G' is pure in C'. Now if $\alpha' \in R'$ and $\alpha = (B \mid \alpha')$, then $\alpha \in R \subseteq E(G)$, so that $\alpha' \in E(G')$. Thus $E(G') \supseteq E_s(G') + R'$ and it remains to show the reverse inclusion. Given $\alpha' \in E(G')$, let $\alpha = (G \mid \alpha')$ and let $\sigma \in E_s(B)$ and $\beta \in R$ be such that $\alpha = \sigma + \beta$. If $L = \ker \sigma$ and $B_2 = L \cap B$, we see from [6, Th. 2.8] that B_2 is large in B and hence by [6, Th. 2.13] that it is large in B'. Inasmuch as $\beta = \alpha'$ on B_2, β does not diminish B'-height in B_2, whence, by Lemma 4, there is an element $\beta' \in E(B')$ such that $\beta' = \beta$ on B_2. Let $\beta_1 = (B \mid \beta')$. Then since $B_2(\beta_1 - \beta) = 0$, $\beta_1 = \beta + \sigma_1$ for some $\sigma_1 \in E_s(B) \subseteq R$, and $\beta_1 \in R$ so that $\beta' \in R'$. Now $\alpha - \beta_1 = \sigma - \sigma_1 \in E_s(G)$, and there is a large subgroup K of G and hence of G' annihilated by $\alpha - \beta_1$. Thus $K(\alpha' - \beta') = K(\alpha - \beta) = 0$, and we conclude that $\alpha' = \beta' + \sigma'$ for $\beta' \in R'$ and $\sigma' \in E_s(G')$ as was to be shown.

The following bit of information on the relationship between a group and a large subgroup is helpful in constructing examples.

LEMMA 5. *Let G be a p-group, $\{k(i)\}_1^\infty$ an increasing sequence of nonnegative integers and H the large subgroup of G given by $H = \sum_1^\infty (p^{k(i)}G)[p^i]$. Then*

$$f_H(j) = \sum \{f_G(k(j+1) + j + i) \mid 0 \le i \le k(j+2) - k(j+1)\} \, .$$

Proof. Routine calculations show that for any increasing sequence of nonnegative integers $\{m(i)\}_1^\infty$

$$(\sum_1^{r+1} (p^{m(i)}G)[p^i])[p] \subseteq (p^{m(r+1)}G)[p^r] + \sum_1^r (p^{m(i)}G)[p^i]$$

$$= \sum_1^r (p^{m(i)}G)[p^i] \, ,$$

so that by induction on r we have $(\sum_1^r (p^{m^{(i)}}G)[p^i]) \subseteqq (p^{m^{(1)}}G)[p]$, and $(\sum_1^\infty (p^{m^{(i)}}G)[p^i])[p] = (p^{m^{(1)}}G)[p]$. It is also easily seen that for integers i, j and k with $i < k$, $p^i(p^jG)[p^k]) = (p^{i+j}G)[p^{k-i}]$. Therefore given any integer j, $p^jH = \sum_{i=j+1}^\infty (p^{k(i)+j}G)[p^{i-j}]$, and $(p^jH)[p] = (p^{k(j+1)+j}G)[p]$. Inasmuch as the dimension of the vector space

$$(p^{k(j+1)+j}G)[p]/(p^{k(j+2)+j+1}G)[p]$$

is the sum of the dimensions of the vector spaces

$$(p^{k(j+1)+j+i}G)[p]/(p^{k(j+2)+j+i+1}G)[p]$$

for $i = 0, 1, 2, \cdots, (k(j+2) - k(j+1))$, we have the desired result.

3. Endomorphism rings generated by their units.

LEMMA 6.   *The following conditions are equivalent for an endomorphism α of a p-group G: (a) There is a bounded projection π in $E(G)$ such that $(1 - \pi)\alpha \in N(G)$,   (b) $\alpha \in E_s(G) + N(G)$,   (c) $\alpha \doteq 0$.*

Proof. Inasmuch as a bounded endomorphism is small and for any projection π, $\alpha = \pi\alpha + (1 - \pi)\alpha$, we see that (a) implies (b). Since (b) obviously implies (c), it remains to show that (c) implies (a). Suppose $(p^nG)[p]\alpha = 0$. It is not difficult to show that for any integer n (cf. the proof of Lemma 1 in [5]) there are subgroups G_0 and G_1 of G such that $G = G_0 \oplus G_1$, $p^nG_0 = 0$ and $G_1[p] = (p^nG)[p]$. Thus, if we let π be the projection on G_0 along G_1, then π is bounded and $G_0(1 - \pi) = 0$ while $(1 - \pi)\alpha = \alpha$ on $G_1[p] = (p^nG)[p]$, so that $G_1[p](1 - \pi)\alpha = 0$ and $(1 - \pi)\alpha \in N(G)$.

THEOREM 3.   *Given a p-group G with $p \neq 2$, all of the following rings are generated by units if any one of them is: (a) $E(G)$,   (b) $E(G)/E_s(G)$,   (c) $E(G)/(E_s(G) + N(G))$.*

Proof. Since the property of being generated by units is inherited by the quotient rings of a ring, we need only show that if $E(G)/(E_s(G) + N(G))$ is generated by units, then $E(G)$ is also. We first prove that in any case $E_s(G) + N(G) \subseteqq \{A(G)\}$. If $\sigma \doteq 0$, then let π be a finite projection such that $(1 - \pi)\sigma \in N(G)$ and write $\sigma = (1 - \pi)\sigma + \pi\sigma(1 - \pi) + \pi\sigma\pi$. For any endomorphism $\varphi \in N(G)$, $\sum_0^\infty \varphi^i$ is also an endomorphism of G, so that $1 - \varphi$ has an inverse and $\varphi \in \{A(G)\}$. Thus $(1 - \pi)\sigma \in \{A(G)\}$. On the other hand, $\pi\sigma\pi$ can be viewed as an endomorphism of the bounded group $G\pi$ and as such is the sum of automorphisms of this group ([1, Th. 2.3]). But clearly this expression of $\pi\sigma\pi$ as a sum of automorphisms can be extended to $\pi\sigma\pi$ as an endomorphism G, so that $\pi\sigma\pi \in \{A(G)\}$. Finally, since $1 - \pi\sigma(1 - \pi)$ has $1 + \pi\sigma(1 - \pi)$

as an inverse, $\pi\sigma(1 - \pi) \in \{A(G)\}$ and we conclude that $\sigma \in \{A(G)\}$ If $\varphi \in E(G)$ is such that $\varphi + E_s(G) + N(G)$ is a unit, there is an integer n and an endomorphism $\psi \in E(G)$ such that $\psi\varphi = 1 = \varphi\psi$ on $(p^n G)[p]$. By a theorem of Pierce [6, Th. 13.1] we then have that $((p^n G)|\varphi)$ is an automorphism of $p^n G$ so that by a theorem of Fuchs [3, Lemma 1], there is an automorphism φ' of G such that $\varphi = \varphi'$ on $p^n G$. Thus, $\varphi = \varphi' + \sigma$ for some $\sigma \in E_s(G)$. Now if we assume that $E(G)/(E_s(G) + N(G))$ is generated by its units and α is in $E(G)$, then there is a set of endomorphisms $\{\alpha_i\}_1^m$ of G such that $\alpha_i + E_s(G) + N(G)$ is a unit for every i and $\alpha \doteq \sum_1^m \alpha_i$. By what we just showed, there is, for each i, an automorphism α_i' of G such that $\alpha_i \doteq \alpha_i'$, whence $\alpha \doteq \sum_1^m \alpha_i'$ and $\alpha \in \{A(G)\}$.

4. **Counterexample.** We give in this section, a counterexample to the theorem of P. Hill [4, cor. 4.6] which says that if L is a large subgroup of the p-group G ($p \neq 2$), then $E(G) = \{A(G)\}$ if and only if $E(L) = \{A(L)\}$. We fix the following notation: $B' = \bigoplus \sum_0^\infty (b_i')$, $(E(b_i') = i + 1)$ is a standard basic p-group ($p \neq 2$) with torsion completion C' and $B = \sum_1^\infty (p^{n(i)} B)[p^i]$ is a large subgroup of B' where $n(1) = n(2) = 0$, and $n(i) = \sum \{j \mid 1 \leq j \leq i\}$ for $i \geq 2$. Let C be the torsion completion of B in C'.

LEMMA 7.   *The group B can be written as $B = \bigoplus \sum_1^\infty B_i$ where $B_i = \bigoplus \sum_{j=0}^{i-1} (c_{ij})$, with $E(c_{ij}) = i$ and $c_{ij} = p^{r(j)} b_{s(j)}'$ for $0 \leq j < i$, for strictly increasing sequences $\{r(j)\}_{j=0}^{i-1}$ and $\{s(j)\}_{j=0}^{i-1}$.*

Proof. We infer from [6, Th. 2.12] that

$$(1) \qquad\qquad B = \bigoplus \sum_0^\infty ((b_i') \cap B) .$$

On the other hand, according to Lemma 5, every direct decomposition of B into cyclic subgroups involves exactly i summands of order p^i. Thus, if we let B_i be the sum of those summands in (1) of order p^i, we have that $B_i = \bigoplus \sum_0^{i-1} (p^{r(j)} b_{s(j)}')$ for some strictly increasing sequence $\{s(j)\}_0^{i-1}$. But for $0 \leq j < k < i$, $E(b_{s(j)}') < E(b_{s(k)}')$ while

$$E(p^{r(j)} b_{s(j)}') = E(p^{r(k)} b_{s(k)}') ,$$

so that $r(j) < r(k)$. If we then let $c_{ij} = p^{r(j)} b_{s(j)}'$ we obtain the desired result.

Now we give endomorphisms $\{P_i\}_{-\infty}^\infty$, $\{E_i\}_0^\infty$, $\{F_i\}_1^\infty$, of B by defining them on the basis of each B_m. In all that follows, when we are restricting attention to a particular B_m, we will denote by $i \oplus j$ the addition of the integers i and j modulo m.

$$(c_{mj})P_i = c_{m,j\oplus i}$$

$$(c_{mj})E_i = \begin{cases} 0, j \neq i \\ c_{mj}, j = i \end{cases}$$

$$(c_{mj})F_i = \begin{cases} 0, j \neq m - i \\ c_{mj}, j = m - i \end{cases}.$$

Let R_0 be the subring of $E(B)$ generated by $\{P_i\}_{-\infty}^{\infty}$, $\{E_i\}_0^{\infty}$, $\{F_i\}_1^{\infty}$. Note that an element of R_0 gives an endomorphism of each B_m and thus has an m by m matrix representation for each m. Since we are writing endomorphisms on the right, the m by m matrix corresponding to an element φ of R_0 is the matrix whose ith row is the sequence of coefficients used to represent $(c_{mi})\varphi$ where $0 \leqq i < m$. This means that in an m by m matrix we are numbering the rows and columns from 0 to $m - 1$.

LEMMA 8. *Given an element φ in R_0, there is a small endomorphism σ of B and an element $\psi \in R_0$ such that $\varphi = \psi + \sigma$ and ψ is a finite Z-linear combination of the form*

(*) $$\psi = \sum k_i P_i + \sum m_{ij} E_i P_j + \sum m'_{ij} F_i P_j.$$

Furthermore, if $\varphi \doteq 0$ on $B[p]$, then each of the coefficients in (*) is divisible by p.

Proof. We first observe that by restricting attention to the subgroup B_m and applying elementary matrix theory, we obtain:

(2) $\qquad P_j E_k = E_{k-j} P_j$, $\qquad\qquad 0 \leqq k < m, 0 \leqq k - j < m$.

(3) $\qquad P_j E_k = F_{j-k} P_j$, $\qquad\qquad 0 \leqq k < j < m$.

(4) $\qquad P_j F_k = F_{k+j} P_j$, $\qquad\qquad 0 < k < m, 0 < k + j < m$.

(5) $\qquad P_j F_k = E_{-(k+j)} P_j$, $\qquad\qquad 0 < k \leqq -j < m$.

Furthermore, formulas (2) and (3) cover the cases in which $0 \leqq k < m/2$, $0 \leqq |j| < m/2$, while (4) and (5) cover the cases $1 \leqq k < m/2, 0 \leqq |j| < m/2$. Thus, by choosing M sufficiently large φ is equal to an element of the form of ψ on $\sum \{B_m \,|\, m > M\}$, whence $p^M(\varphi - \psi) = 0$ and $\varphi = \psi + \sigma$ for some $\sigma \in E_s(B)$. Now, if $\varphi \doteq 0$, then $\psi \doteq 0$, so that for some integer N, $\psi = 0$ on $B_m[p]$ for every $m > N$. But observe that $E_i P_j$ is simply the matrix with 1 in the $(i, i \oplus j)$ entry and 0 elsewhere, $F_i P_j$ has 1 in the $(m - i, (m - i) \oplus j)$ entry and 0 elsewhere, while P_j has 1 in the $(k, k \oplus j)$ entry for $k = 0, 1, \cdots, (m - 1)$ and 0 elsewhere. Thus, by choosing m sufficiently large, we conclude that if $\psi = 0$ on $B_m[p]$, then all of the coefficients in the expression for ψ must be 0 modulo p.

Now by Lemma 8 we can apply Theorem 1 to the ring R_0 to

obtain a pure subgroup G of C containing B such that

$$E(G) = E_s(G) + (E_s(B) + R_0)^- \ .$$

Then by Theorem 2 we get a pure subgroup G' of C' containing B', having G as a large subgroup and such that $E(G') = E_s(G') + R'$ where $R' = \{\alpha \in E(B') \mid (B \mid \alpha) \in (E_s(B) + R_0)^-\}$.

LEMMA 9. *If an endomorphism φ of B_m has a matrix representation which is upper triangular, then φ does not diminish B'-height on B_m. If φ is an endomorphism of B_m which does not diminish B'-heights, then the matrix representation of φ on $B_m[p]$ is upper triangular.*

Proof. According to Lemma 7, $B_m = \bigoplus \sum_{j=0}^{m-1} (c_{mj})$ where $c_{mj} = p^{r(j)} b'_{s(j)}$ for strictly increasing sequences $\{r(j)\}_0^{m-1}$ and $\{s(j)\}_0^{m-1}$. Now if the matrix representation of φ is upper triangular, we have $(c_{mj})\varphi = \sum_{i=j}^{m-1} w_{ji} c_{mi}$ and, writing an element x of B_m as $x = \sum_{j=0}^{m-1} u_j p^{k(j)} c_{mj}$ with $p \nmid u_j$, we obtain $(x)\varphi = \sum_{j=0}^{m-1} (\sum_{i=0}^{j} w_{ij} u_i p^{k(i)+r(j)}) b'_{s(j)}$. It is clear that $h_{B'}(x)$ is the minimum value of $\{k(j) + r(j)\}_0^{m-1}$, so that for some j_0,

$$k(i) + r(i) \geqq k(j_0) + r(j_0) = h_{B'}(x) \ .$$

But in view of the fact that the sequence $\{r(i)\}_0^{m-1}$ is strictly increasing, we then infer that for $m > j \geqq i \geqq 0$, $k(i) + r(j) \geqq k(j_0) + r(j_0)$, from which we conclude that $h_{B'}((x)\varphi) \geqq k(j_0) + r(j_0) = h_{B'}(x)$. For the partial converse suppose that φ is a B'-height preserving endomorphism of B_m and $(c_{mj})\varphi = \sum_{i=0}^{m-1} w_i c_{mi}$. Then

$$(p^{r(j)} b'_{s(j)})\varphi = \sum_0^{j-1} w_i p^{r(i)} b'_{s(i)} + \sum_j^{m-1} w_i p^{r(i)} b'_{s(i)} \ .$$

But $h_{B'}(p^{r(j)} b'_{s(j)}) = r(j)$, while for $i \geqq j$, $h_{B'}(w_i p^{r(i)} b'_{s(i)}) \geqq r(i) \geqq r(j)$, so that $h_{B'}(\sum_0^{j-1} w_i p^{r(i)} b'_{s(i)}) \geqq r(j)$, whence $p \mid w_i$ for $i = 0, \cdots, (j-1)$ and

$$(p^{r(j)+m-1} b'_{s(j)})\varphi = \sum_j^{m-1} w_i p^{r(i)+m-1} b'_{s(i)} \ ,$$

which says that the matrix representation of φ on $B_m[p]$ is upper triangular.

THEOREM 4. *Given a prime $p \neq 2$, there is a p-group G' with large subgroup G such that $E(G) = \{A(G)\}$ and $E(G') \neq \{A(G')\}$.*

Proof. Inasmuch as P_i, $1 + E_i$ and $1 + F_i$ are automorphisms of B, we have immediately that $R_0 \subseteqq \{A(G)\}$. However, since every element of $E(G)$ is eventually equal to an element of R_0 on $G[p]$, we

conclude from Theorem 3 that $E(G) = \{A(G)\}$.

To show that $E(G') \neq \{A(G')\}$, let us first consider an element $\psi \in E(B)$ that preserves B'-height on some $\sum \{B_m[p] \mid m > M\}$ and is of the form

$$\psi = \sum k_i P_i + m_{ij} E_i P_j + \sum m'_{ij} F_i P_j .$$

Let $D_i = \sum_0^{i-1} E_j$, $D'_i = \sum_1^i F_j$ $(i = 1, 2, \cdots)$ and $D_0 = D'_0 = 0$. Observe that for $0 \leq i < m$ the matrix representation of $(1 - D'_i)P_i$ is upper triangular, while for $-m < i < 0$, $(1 - D_{-i})P_i$ is strictly lower triangular. Thus, writing

$$\psi = \sum_{i \geq 0} k_i(1 - D'_i)P_i + \sum_{i < 0} k_i(1 - D_{-i})P_i + \sum n_{ij} E_i P_j + \sum n'_{ij} F_i P_j ,$$

and choosing M' sufficiently large, we have that for $m > M'$, ψ is upper triangular on $B_m[p]$, while each $E_i P_j$ and $F_i P_j$ is either upper or strictly lower triangular, so that $\psi \doteq \sum_{i \geq 0} k_i(1 - D'_i)P_i + s$ where s is some member of the ideal S_0 in R_0 generated by $\{E_i\}_0^\infty$ and $\{F_i\}_1^\infty$. Now if φ is another endomorphism of B of the same form as ψ such that φ eventually does not diminish B'-height on B and $\psi\varphi \doteq 1$, then $\varphi \doteq \sum_{i \geq 0} k'_i(1 - D'_i)P_i + s'$ for some $s' \in S_0$, whence

$$\sum_{i+j>0} k_i k'_j(1 - D'_i)P_i(1 - D'_j)P_j + s'' \doteq 1 - k_0 k'_0 1$$

for some $s'' \in S_0$. However, we see from formula (4) in the proof of Lemma 8 that $P_i(1 - D'_j) = (1 - D'_{i+j} + D'_i)P_i$ when $0 < i < i + j < m$, while $(1 - D'_i)(1 - D'_{i+j} + D'_i) = 1 - D'_{i+j}$, so that

$$\sum_{i+j>0} k_i k'_j(1 - D'_{i+j})P_{i+j} + s'' \doteq 1 - k_0 k'_0 1 ,$$

which, by Lemma 8 yields that $p \nmid k_0$ but $p \mid k_i$ for every $i > 0$. Thus $\psi \doteq k_0 1 + s$ where $p \nmid k_0$ and $s \in S_0$.

Now suppose that $E(G') = \{A(G')\}$. Since $(1 - D'_1)P_1$ is upper triangular on B_m for every m, it has an extension to an endomorphism of G' which is a sum of automorphisms $\{u'_i\}_1^n$ of G'. Setting $u_i = (G \mid u'_i)$, we have that u_i is an automorphism of G which, by Lemma 8 and what we showed above, is eventually equal to $k_i 1 + s_i$ on $G[p]$, where $p \nmid k_i$ and $s_i \in S_0$. Thus $(1 - D'_1)P_1 \doteq (\sum_1^n k_i)1 + \sum_1^n s_i$ which again contradicts Lemma 8 and we conclude that $E(G') \neq \{A(G')\}$.

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