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DIFFERENTIABLE POWER-ASSOCIATIVE GROUPOIDS

JOHN P. HOLMES

Suppose H is a Banach space, D is an open set of H containing 0, and V is a function from $D \times D$ to H satisfying $V(0, x) = V(x, 0) = x$ for each x in D. If n is an integer greater than 1, denote by x^n the product of $n - x$'s associated as follows whenever the product exists.

$$x^n = V(x, V(x, \cdots V(x, x) \cdots)) .$$

Define $x^0 = 0$ and $x^1 = x$. V is said to be power associative if and only if $V(x^n, x^m) = x^{n+m}$ whenever each of n and m is a nonnegative integer and x^{n+m} exists.

THEOREM A. If H and V are as above, V is power associative and continuously differentiable in the sense of Frechet on $D \times D$ then there are positive numbers a and c such that if x is in H and $||x|| < a$ there is a unique continuous function T_x from $[0, 1]$ to the ball of radius c centered at 0 satisfying $V(T_x(s), T_x(t)) = T_x(s + t)$ whenever each of s, t, and $s + t$ is in $[0, 1]$, $T_x(0) = 0$, and $T_x(1) = x$.

Theorem A is similar to a result in [1] of Birkhoff. He showed that if H and V are as above, V is associative, V is Frechet differentiable on a neighborhood of (0, 0), and V' is continuous at (0, 0) then some neighborhood of 0 is covered by partial homomorphic images of the additive group of real numbers.

To see that Theorem A is not a special case of this result of Birkhoff, we offer the following example. Denote by E the 2-dimensional Euclidean space and define V from $E \times E$ to E by $V((x, y), (z, w)) = (x + [1 + (xw - yz)]z, y + [1 + (xw - yz)]w)$. If S is a 1-dimensional linear subspace of E and each of p and q is in S then $V(p, q) = p + q$. Thus V is power associative and 0 is an identity for V. V is not associative but V is continuously differentiable on $E \times E$.

We will now prove Theorem A. Regard $H \times H$ as a Banach space in the usual way, defining the norm of a member (x, y) of $H \times H$ by $||(x, y)|| = \max\{||x||, ||y||\}$. If c is a positive number, denote by $R(c)$ the set to which x belongs if and only if x is in H and $||x|| < c$. Finally, if B is a bounded linear transformation from $H \times H$ to H or from H to H, denote the norm of B by $|B|$.

Define f from D to H by $f(x) = V(x, x) = x^2$ for each x in D. Note f is continuously differentiable on D and if x is in D, $f'(x)(y) = V'(x, x)(y, y)$ for each y in H. Moreover, $V'(0, 0)(z, w) = z + w$ for each pair (z, w) in $H \times H$ so $f'(0) = 2I$ where I is the identity transformation on H.

Employing the inverse function theorem (for instance [2] page 268) we see that there is a positive number b and an open set U of H such that $(f \mid U)$ is a homeomorphism of U onto $R(b)$ and $g = (f \mid U)^{-1}$ is continuously differentiable on $R(b)$ with $g'(y) = [f'(g(y))]^{-1}$ for each y in $R(b)$. Hence $g'(0) = 1/2\, I$.

By continuity of g' and V' there is a positive number d and a number M such that if p is in $R(d) \times R(d)$ and x is in $R(d)$ then $|V'(p)| < M$ and $|g'(x)| < 2/3$.

Suppose each of x, y, z, and w is in $R(d)$. Then $\|V(x, y) - V(z, w)\| = \|\int_0^1 dt\ V'((z, w) + t(x - z, y - w))(x - z, y - w)\| < M\|(x - z, y - w)\|$. As special cases of this inequality we obtain

1. $\|V(x, y)\| < M\|(x, y)\|$ and
2. $\|V(x, y) - y\| < M\|x\|$.

Similarly, if each of x and y is in $R(d)$ we have $\|g(x) - g(y)\| = \|\int_0^1 dt\, g'(y + t(x - y))(x - y)\| < 2/3\, \|x - y\|$. Hence g is Lipschitz on $R(d)$ and has Lipschitz norm less than 2/3. In particular, for each x in $R(d)$ and each positive integer m we have $\|g^m(x)\| < (2/3)^m\|x\|$ where g^m is g composed with itself m times.

LEMMA 1. *Let $r = d/3M$. If x is in $R(r)$, m is a positive integer, and n is an integer in* $[0, 2^m]$ *then* $[g^m(x)]^n$ *exists and has norm less than* $M\|x\| \sum_1^m (2/3)^i$.

Proof. Note $|V'(0, 0)| = 2$ so $M > 3/2$. If x is in $R(r)$, it is clear, using inequality 1, that $g^1(x)^i$ exists for each $i = 0, 1$, or 2 and has norm less than $M\|x\|(2/3)$.

Suppose m is an integer greater than 1 and assume that for each integer k in $[1, m)$ that $g^k(x)^s$ exists for each integer s in $[0, 2^k]$ and has norm less than $M\|x\| \sum_1^k (2/3)^i$.

As has been observed before, $g^m(x)$ exists and $\|g^m(x)^0\| = 0$. Suppose n is an integer in $(0, 2^m]$ and assume for each integer c in $[0, n)$ that $g^m(x)^c$ exists and has norm less than $M\|x\| \sum_1^m (2/3)^i$.

Then $g^m(x)^{n-1}$ exists and $\|g^m(x)^{n-1}\| < M\|x\| \sum_1^m (2/3)^i < 2M\|x\| < 2Mr = 2M\, d/3M < d$. Thus $g^m(x)^{n-1}$ is in D and $g^m(x)^n = V(g^m(x), g^m(x)^{n-1})$ exists.

If n is even, we may use power associativity and the equality $g^m(x)^2 = g^{m-1}(x)$ to obtain $g^m(x)^n = g^{m-1}(x)^{n/2}$. Hence, by the first inductive hypothesis, $\|g^m(x)^n\| < M\|x\| \sum_1^m (2|3)_i$.

If n is odd then $g^m(x)^n = V(g^m(x), g^{(m-1)}(x)^{(n-1)/2})$. Using the triangle inequality, inequality 2, and the first inductive hypothesis we obtain $\|g^m(x)^n\| \leqq \|V(g^m(x), g^{m-1}(x)^{(n-1)/2}) - g^{m-1}(x)^{(n-1)}\| + \|g^{m-1}(x)^{(n-1)/2}\| < M\|x\| \sum_1^m (2/3)^i$.

Thus we have Lemma 1.

Suppose x is in $R(r)$. Denote by E the set of dyadic rational numbers in $[0, 1]$ and define T from E to H by $T(n/2^m) = g^m(x)^n$. T exists by Lemma 1 and is well defined by power associativity. Moreover, by power associativity, $V(T(h), T(k)) = T(h + k)$ whenever each of h, k, and $h + k$ is in E. Lemma 2 will show that T has a continuous extension to all of $[0, 1]$.

LEMMA 2. *If x and T are as above, each of h and k is in E, and $|h - k| < 1/2^m$ for some positive integer m then $\|T(h) - T(k)\| < 9M\|x\|(2/3)^{m+1}$.*

Proof. Suppose $h = s/2^{m+n}$ for some nonnegative integers s and n, and u is an integer with each of $u/2^m$ and $(u + 1)/2^m$ in E so that h is in $[u/2^m, (u + 1)/2^m]$. There is a sequence $a_1, \cdots, a_n$ such that $h = u/2^m + a_1/2^{m+1} + \cdots + a_n/2^{m+n}$ and each a_i is in the set $\{0, 1\}$. Thus $T(h) = V(T(u/2^m), V(T(a_1/2^{m+1}), \cdots, V(T(a_{n-1}/2^{m+n-1}), T(a_n/2^{m+n})) \cdots))$.

Let w be defined from $\{0, 1, \cdots, n\}$ by $w_i = u/2^m + \sum_1^i a_j/2^{m+j}$. Then $w_i = w_{i-1} + a_i/2^{m+i}$ for each i in $\{1, \cdots, n\}$. Now, using the triangle inequality, we have $\|T(h) - T(u/2^m)\| \leqq \sum_1^n \|T(w_i) - T(w_{i-1})\|$. But, using inequality 2 we obtain $\|T(w_i) - T(w_{i-1})\| \leqq M\|T(a_i/2^{m+i})\| < M\|x\|(2/3)^{m+i}$. Hence $\|T(h) - T(u/2^m)\| < M\|x\| \sum_1^m (2/3)^{m+i} < 3M\|x\|(2/3)^{m+1}$.

There is an integer u such that each of $(u - 1)/2^m$ and $(u + 1)/2^m$ is in E and each of h and k is in $[(u - 1)/2^m, (u + 1)/2^m]$. Hence, by using the triangle inequality and the inequality just proved, we obtain Lemma 2.

From Lemma 2 it is clear that T has a continuous extension to all of $[0, 1]$. If each of s, t, and $s + t$ is in $[0, 1]$, choose sequences $\{a_n\}_1^\infty$ and $\{b_n\}_1^\infty$ in E converging to s and t respectively so that for each positive integer n, $d_n = a_n + b_n$ is in E. By continuity of V and T, we have $V(T(s), T(t)) = \lim_n V(T(a_n), T(b_n)) = \lim_n T(d_n) = T(s + t)$.

Choose c positive and less than r so that $R(c)$ is contained in $g(R(d))$. Let $a = c/3M$. If x is in $R(a)$ then, by Lemma 1, T_x maps into $R(c)$. Suppose F satisfies the conclusion of theorem A for x in $R(a)$. $F(1/2)$ is in $R(c)$ and hence in $g(R(d))$. $F(1/2)^2 = x$ and x is in $R(d)$ so $g(x) = F(1/2)$. Similarly $g^m(x) = F(1/2^m)$ for each positive integer m, and hence F agrees with T_x on E. Since each of F and T_x is continuous, the proof is complete.

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EMORY UNIVERSITY
AND
UNIVERSITY OF FLORIDA