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# ON A GENERALIZATION OF $\Sigma$-SPACES

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**In order to simultaneously generalize the class of $M$-spaces and $\sigma$-spaces, K. Nagami introduced $\Sigma$-spaces. Subsequently, E. Michael defined a class of $\Sigma^{\#}$-spaces. In this paper we will discuss the class of $\Sigma^{*}$-spaces which lies between $\Sigma$-spaces and $\Sigma^{\#}$-spaces and which contains all images of $\Sigma$-spaces under closed continuous maps.**

1. Introduction. Recently K. Nagami [6] has investigated a new class of spaces, called $\Sigma$-spaces, containing two different classes of generalized metric spaces; i.e. the class of $M$-spaces (cf. [4]) as well as the class of $\sigma$-spaces (cf. [5], [7]).

If $\mathscr{K}$ is a cover of a space $X$, then a cover $\mathscr{A}$ is called a *(mod $\mathscr{K}$)-network* for $X$ if, whenever $K \subset U$ with $K \in \mathscr{K}$ and $U$ open in $X$, then $K \subset A \subset U$ for some $A \in \mathscr{A}$. According to K. Nagami [6], $X$ is a *$\Sigma$-space* if it has a $\sigma$-locally finite closed (mod $\mathscr{K}$)-network for some cover $\mathscr{K}$ of $X$ by countably compact sets.

E. Michael [2] has pointed out that the image of a paracompact, $T_2$ $\Sigma$-space under a closed continuous map need not be a $\Sigma$-space and also that replacing "$\sigma$-locally finite" by "$\sigma$-closure-preserving" in the definition of a $\Sigma$-space leads to a strictly larger class of spaces, which are called *$\Sigma^{\#}$-spaces*.

We say that a space $X$ is a *$\Sigma^{*}$-space* if it satisfies the definition of a $\Sigma$-space with "$\sigma$-locally finite" weakened to "$\sigma$-hereditarily closure-preserving", where we say that a collection $\mathscr{A} = \{A_\lambda : \lambda \in \Lambda\}$ is *hereditarily closure-preserving* if any collection $\{B_\lambda : \lambda \in \Lambda\}$ with $B_\lambda \subset A_\lambda$ is closure-preserving (cf. [3]).

Clearly, every $\Sigma$-space is a $\Sigma^{*}$-space and every $\Sigma^{*}$-space is a $\Sigma^{\#}$-space. Since the image of a locally finite closed cover of the domain under a closed continuous onto map is a hereditarily closure-preserving closed cover of the range, we can easily see that the image of a $\Sigma$-space by a closed continuous map is always a $\Sigma^{*}$-space. As a matter of fact, E. Michael [2] has pointed out that a paracompact, $T_2$ $\Sigma^{*}$-space need not be a $\Sigma$-space, in general. Hence this fact arouses our interest in studying $\Sigma^{*}$-spaces comparing with $\Sigma$-spaces as well as $\Sigma^{\#}$-spaces.

In this paper we will investigate some relationship between above spaces and obtain the following results:

(A) Any image of a $\Sigma^{*}$-space under a closed continuous map is a $\Sigma^{*}$-space.

(B) Any inverse image of a $\Sigma^{\#}$-space by a perfect map (i.e. a closed continuous map whose fibre at each point is compact) is a $\Sigma^{\sharp}$-space, while this is not true for a $\Sigma^*$-space.

(C) Every Lindelöf, $T_2$, $\Sigma^*$-space is a $\Sigma$-space, while this is not true for a $\Sigma^{\sharp}$-space.

(D) A $\Sigma^*$-space $X$ is a $\Sigma$-space if every open set of $X$ is an $F_\sigma$.

(E) For a paracompact, $T_2$ space $X$ the following conditions are equivalent:

(1) $X$ is a $\Sigma$-space.

(2) $X \times I$ is a $\Sigma$-space, where $I$ denotes the unit closed interval with usual topology.

(3) $X \times I$ is a $\Sigma^*$-space.

According to the first half of (B), the product of a $\Sigma^{\sharp}$-space with $I$ is a $\Sigma^{\sharp}$-space. On the other hand, as noted above there exists a paracompact, $T_2$, $\Sigma^*$-, non $\Sigma$-space. Hence statement (E) shows that the product of a paracompact, $T_2$, $\Sigma^*$-, non $\Sigma$-space $X$ with $I$ is a $\Sigma^{\sharp}$-, non $\Sigma^*$-space. Since the projection from $X \times I$ to $I$ is perfect, this is an example for the later half of (B). Also, this shows that the class of $\Sigma^{\sharp}$-spaces is strictly larger than the class of $\Sigma^*$-spaces.

Concerning (D), it raises the following question:

Is (D) true for $\Sigma^{\sharp}$-spaces?

§2 is concerned with hereditarily closure-preserving closed covers of a countably compact, $T_2$ space, a Lindelöf, $T_2$ space and a $T_2$ space whose open sets are $F_\sigma$'s. As an immediate consequence of 2.1 and 2.3 we have the simple facts that every hereditarily closure-preserving closed cover of a countably compact, $T_2$ space (resp. a Lindelöf, $T_2$ space) has a finite (resp. a countable) subcover. In §3 we will prove main results.

We will use the following notations in §2 and §3:

For a cover $\mathscr{F}$ of a space $X$ and a point $x$ of $X$ we put

$$C(x, \mathscr{F}) = \cap \{F : x \in F \in \mathscr{F}\},$$

and for a sequence $\{\mathscr{F}_n : n = 1, 2, \cdots\}$ of covers of $X$ and a point $x$ of $X$ we put

$$C(x) = \bigcap_{n=1}^{\infty} C(x, \mathscr{F}_n).$$

Throughout this paper we assume that all spaces are $T_2$ and all maps are continuous.

## 2. Some properties of a hereditarily closure-preserving closed cover.

THEOREM 2.1. *Let $\mathscr{F} = \{F_\lambda : \lambda \in \Lambda\}$ be a hereditarily closure-preserving closed cover of a space $X$ and $C$ a countably compact set* *of $X$. Then $\mathscr{F}$ is locally finite at almost all points of $C$;* i.e. *there exist $x_1, \cdots, x_n$ in $C$ such that $\mathscr{F}$ is locally finite at any $x \in C - \{x_1, \cdots, x_n\}$, and only finitely many members of $\mathscr{F}$ meet $C - \{x_1, \cdots, x_n\}$.*

*Proof.* On the contrary, suppose $\mathscr{F}$ is not locally finite at infinitely many points of $C$. Since any closure-preserving, point-finite collection of closed sets is locally finite, $\mathscr{F}$ is not point-finite at infinitely many points of $C$. Then we can choose, step by step, countably many points $x_1, x_2 \cdots$ in $C$ and countably many $\lambda_1, \lambda_2, \cdots$ in $\Lambda$ such that $x_n \in F_{\lambda_n}$ for $n = 1, 2, \cdots$. Since $\mathscr{F}$ is hereditarily closure-preserving, $\{x_1, x_2, \cdots\}$ must be discrete in $X$. On the other hand, since $C$ is countably compact, $\{x_1, x_2, \cdots\}$ must have a cluster point in $C$. This is a contradiction. Hence $\mathscr{F}$ is locally finite at all points of $C$ but finitely many points $x_1, \cdots, x_n$.

To complete the proof of 2.1, assume that $D = C - \{x_1, \cdots, x_n\}$ is infinite. If infinitely many members of $\mathscr{F}$ meet $D$, then we can again obtain a sequence $\{p_1, p_2, \cdots\}$ in $D$ and a sequence $\{F_{\lambda_1}, F_{\lambda_2}, \cdots\}$ in $\mathscr{F}$ with $p_i \in F_{\lambda_i}$ for $i = 1, 2, \cdots$ by noting that $\mathscr{F}$ is point-finite at any point of $D$. Since $\mathscr{F}$ is hereditarily closure-preserving, $\{p_1, p_2, \cdots\}$ must be discrete in $X$, therefore, in $C$, which is a contradiction. Hence only finitely many members of $\mathscr{F}$ meet $D$. This completes the proof.

As an immediate corollary of 2.1 we have:

COROLLARY 2.2. *Every hereditarily closure-preserving closed cover of a countably compact space contains a finite subcover.*

REMARK. 2.2 does not necessarily hold for a closure-preserving closed cover even if a space is compact and metrizable; for example, let $X = \{1/n: n = 1, 2, \cdots\} \cup \{0\}$ be a subspace of real line and put $\mathscr{F} = \{\{0, 1/n\}: n = 1, 2, \cdots\}$. Then $X$ is a compact, metric space and $\mathscr{F}$ is a closure-preserving closed cover of $X$, but we cannot choose any finite subcover.

THEOREM 2.3. *Let $\mathscr{F} = \{F_\lambda: \lambda \in \Lambda\}$ be a hereditarily closure-preserving closed cover of a Lindelöf space $X$. Then the set*

$$X_0 = \{x \in X: \Lambda(x) = \{\lambda \in \Lambda: x \in F_\lambda\} \text{ is uncountable}\}$$

*is countable, and the set*

$$\Lambda' = \{\lambda \in \Lambda: F_\lambda \cap (X - X_0) \neq \emptyset\}$$

*is countable if $X - X_0$ is uncountable.*

*Proof*. On the contrary, suppose $X_0$ is uncountable. Then $X_0$ contains a subset $\{x_\alpha: \alpha < \omega_1\}$, where $\omega_1$ denotes the least uncountable ordinal. For each $\alpha < \omega_1$, by transfinite induction we can obtain $x_\alpha$ in $X_0$ and a $\lambda_\alpha \in \Lambda(x_\alpha)$ with $x_\alpha \in F_{\lambda_\alpha}$ and such that $\alpha \neq \beta$ implies $x_\alpha \neq x_\beta$ and $\lambda_\alpha \neq \lambda_\beta$, because for each $x \in X_0$ $\Lambda(x)$ is uncountable. Since $\mathscr{F}$ is hereditarily closure-preserving, $\{x_\alpha: \alpha < \omega_1\}$ must be discrete in $X$. This contradicts the assumption that $X$ is Lindelöf, and hence the first half of 2.3 is proved.

To complete the proof, again suppose $\Lambda'$ is uncountale. From the definition of $X_0$, $\mathscr{F}$ must be point-countable at any $x \in X - X_0$. If $X - X_0$ is uncountable, by transfinite induction, we can choose an uncountable set $\{x_\alpha: \alpha < \omega_1\}$ in $X - X_0$ and a corresponding set $\{\lambda_\alpha: \alpha < \omega_1\}$ with $x_\alpha \in F_{\lambda_\alpha}$ for each $\alpha < \omega_1$ and so that $\alpha \neq \beta$ implies $x_\alpha \neq x_\beta$ as well as $\lambda_\alpha \neq \lambda_\beta$. Since $\mathscr{F}$ is hereditarily closure-preserving, $\{x_\alpha: \alpha < \omega_1\}$ must be an uncountable discrete set in $X$, which contradicts the assumption that $X$ is Lindelöf. Therefore $X - X_0$ is countable, and hence the proof is completed.

As an immediate consequence of 2.3 we have:

COROLLARY 2.4. *Every hereditarily closure-preserving closed cover of a Lindelöf space contains a countable subcover.*

REMARK. Example 3.4 in next section shows that 2.4 does not necessarily hold for a closure-preserving closed cover.

LEMMA 2.5. *Let* $\mathscr{F}$ *be a closure-preserving closed cover of a space* $X$. *Then the set*

$$X_1 = \{x \in X: C(x, \mathscr{F}) = \{x\}\}$$

*is discrete in* $X$.

*Proof*. Let $y \in X$ be an arbitrary point and

$$U = X - \cup\{F \in \mathscr{F}: y \notin F\} .$$

Then $U$ is an open neighborhood of $y$, because $\mathscr{F}$ is a closure-preserving closed cover. If $x \in U \cap X_1$, then we have

$$\phi \neq U \cap C(x, \mathscr{F}) = (X - \cup\{F \in \mathscr{F}: y \notin F\}) \cap (\cap\{F \in \mathscr{F}: x \in F\})$$

and hence $C(y, \mathscr{F}) \subset C(x, \mathscr{F})$. Since $x \in X_1$, $C(x, \mathscr{F}) = \{x\}$ and thus we have $y = x$. This means that $U$ contains at most one point of $X_1$, which completes the proof.

THEOREM 2.6. *Let* $X$ *be a space each of whose open sets is an*

*$F_\sigma$, and let $\mathscr{F}$ be a closure-preserving closed cover of $X$. Then the set*

$$X_n = \{x \in X : |C(x, \mathscr{F})| = n\}$$

*is $\sigma$-discrete in $X$ for $n = 1, 2, \cdots$, where we denote by $|A|$ the cardinality of $A$.*

*Proof.* We shall prove 2.6 by induction on $n$. By 2.5 $X_1$ is discrete in $X$. Assume that $X_n$ is $\sigma$-discrete in $X$ for any $n \leqq k$. We shall show that $X_{k+1}$ is also $\sigma$-discrete.

First note that $X - \bigcup_{n=1}^{k} X_n$ is open in $X$. Let $y$ be any point of $X - \bigcup_{n=1}^{k} X_n$ and let $U = X - \cup\{F \in \mathscr{F} : y \notin F\}$. Then $U$ is an open neighborhood of $y$. If $x \in X$ belongs to $U$, we have $C(y, \mathscr{F}) \subset C(x, \mathscr{F})$. Since $y$ does not belong to $\bigcup_{n=1}^{k} X_n$, $C(y, \mathscr{F})$ contains at least $k+1$ points of $X$ and thus $C(x, \mathscr{F})$ also contains at least $k+1$ points. In other words, $x \notin \bigcup_{n=1}^{k} X_n$. This shows that $U \cap (\bigcup_{n=1}^{k} X_n) = \varnothing$ and hence $X - \bigcup_{n=1}^{k} X_n$ is open in $X$.

According to hypothesis, $X - \bigcup_{n=1}^{k} X_n$ is an $F_\sigma$; i.e. $X - \bigcup_{n=1}^{k} X_n = \bigcup_{i=1}^{\infty} Y_i$, where each $Y_i$ is closed in $X$ and $Y_i \subset Y_{i+1}$ for $i = 1, 2, \cdots$. Since $X_{k+1} \subset \bigcup_{i=1}^{\infty} Y_i$, it suffices to show that $Z_i = X_{k+1} \cap Y_i$ is discrete in $X$ for $i = 1, 2, \cdots$.

Let $y \in X$ be an arbitrary point and $i$ fixed. If $y \notin Y_i$, then $X - Y_i$ is clearly the desired neighborhood of $y$. If $y \in Y_i$, put $U = X - \cup\{F \in \mathscr{F} : y \notin F\}$. Then $x \in U \cap Z_i$ implies $C(y, \mathscr{F}) \subset C(x, \mathscr{F})$ and $|C(x, \mathscr{F})| = k + 1$. Since $y$ belongs to $Y_i$, $y$ does not belong to any $X_n$ with $n \leqq k$: i.e. $|C(y, \mathscr{F})| > k$. Hence we have $C(y, \mathscr{F}) = C(x, \mathscr{F})$. This means that $x$ must be in $C(y, \mathscr{F})$ which is finite. Consequently, $U$ contains at most $k+1$ points of $Z_i$. Since $X$ is $T_1$, we obtain the desired neighborhood of $y$ by deleting finitely many points from $U$. Therefore $Z_i$ is discrete in $X$. This completes the proof.

## 3. Some relations.

Let $f$ be a closed map from a space $X$ onto a space $Y$ and $\mathscr{F}$ a hereditarily closure-preserving closed cover of $X$. Then $f(\mathscr{F})$ is also a hereditarily closure-preserving closed cover of $Y$. Since the image of any countably compact space by a map is countably compact, we have the following:

THEOREM 3.1. *Any image of a $\Sigma^*$-space under a closed map is a $\Sigma^*$-space.*

Let $f$ be a perfect map from $X$ onto $Y$ and $\mathscr{A}$ a (mod $\mathscr{K}$)-network for $Y$. Then we can easily see that $f^{-1}(\mathscr{A})$ is a (mod $f^{-1}(\mathscr{K})$)-network for $X$. Since the inverse image of any countably compact space by a perfect map is countably compact, we have the following:

THEOREM 3.2. *Any inverse image of a $\Sigma^{\sharp}$-space by a perfect map is a $\Sigma^{\sharp}$-space.*

THEOREM 3.3. *Every Lindelöf $\Sigma^*$-space is a $\Sigma$-space.*

*Proof.* Let $X$ be a Lindelöf $\Sigma^*$-space having a $\sigma$-hereditarily closure-preserving closed (mod $\mathscr{K}$)-network $\mathscr{F}$ for some cover $\mathscr{K}$ of $X$ by countably compact sets. Without loss of generality, we can denote $\mathscr{F}$ by $\bigcup_{n=1}^{\infty} \mathscr{F}_n$ such that each $\mathscr{F}_n$ is a hereditarily closure-preserving closed cover of $X$. Put $\mathscr{F}_n = \{F_\lambda: \lambda \in \Lambda_n\}$ for $n = 1, 2, \cdots$.

By 2.3, for each $n$ the set

$$X_n = \{x \in X: \Lambda(x) = \{\lambda \in \Lambda_n: x \in F_\lambda\} \text{ is uncountable}\}$$

is countable. If $X - X_n$ is countable for some $n$, then $X$ is countable. Since $X$ is $T_2$, $X$ is clearly a $\Sigma$-space; more precisely, it is a cosmic space (cf. [1]). If $X - X_n$ is uncountable for $n = 1, 2, \cdots$, then again by 2.3,

$$\Lambda_n' = \{\lambda \in \Lambda_n: F_\lambda \cap (X - X_n) \neq \emptyset\}$$

is countable for $n = 1, 2, \cdots$. Put $\mathscr{H}_n = \{\{x\}: x \in X_n\} \cup \{F_\lambda: \lambda \in \Lambda_n'\}$ for $n = 1, 2, \cdots$. Then each $\mathscr{H}_n$ is countable and, therefore, $\mathscr{H} = \bigcup_{n=1}^{\infty} \mathscr{H}_n$ is still countable. Since each $\mathscr{H}_n$ covers $X$, $\mathscr{H}$ covers $X$ and thus $\mathscr{H}$ is a $\sigma$-locally finite closed cover of $X$. Furthermore, if we put $\mathscr{K}' = \{\{x\}: x \in \bigcup_{n=1}^{\infty} X_n\} \cup \{K \in \mathscr{K}: K \cap (X - X_n) \neq \emptyset \text{ for some } n\}$, then $\mathscr{K}'$ is a cover of $X$ by countably compact sets. It is easy to see that $\mathscr{H}$ is a (mod $\mathscr{K}'$)-network, and hence $X$ is a $\Sigma$-space.

EXAMPLE 3.4. We shall show that in general a Lindelöf $\Sigma^{\sharp}$-space need not be a $\Sigma$-space.

Let $X = \{x_\alpha: \alpha \in A\} \cup \{p\}$ be an uncountable set with a special point $p$. We define the topology for $X$ as follows: each $\{x_\alpha\}$ is open; $V$ is an open set containing $p$ iff $X - V$ is countable. Then we can easily see that $X$ is a regular, Lindelöf $(T_2)$ space.

Now, put $\mathscr{F} = \{\{p, x_\alpha\}: \alpha \in A\}$. Then $\mathscr{F}$ is a closure-preserving closed cover of $X$, because any subset of $X$ missing $p$ is open. If we put $\mathscr{K} = \mathscr{F}$, then $\mathscr{K}$ is a cover of $X$ by countably compact sets such that $\mathscr{F}$ is a (mod $\mathscr{K}$)-network for $X$; i.e. $X$ is a $\Sigma^{\sharp}$-space.

Next, we shall show that $X$ is not a $\Sigma$-space. On the contrary, suppose $X$ is a $\Sigma$-space. Then there exists a $\sigma$-locally finite closed cover $\mathscr{H} = \bigcup_{n=1}^{\infty} \mathscr{H}_n$ of $X$ which is a (mod $\mathscr{K}$)-network for some cover $\mathscr{K}$ by countably compact sets. We can assume without loss of generality that $\{\mathscr{H}_n: n = 1, 2, \cdots\}$ is an increasing sequence of locally finite closed covers of $X$ and that each $\mathscr{H}_n$ is closed under finite intersections. Furthermore, in case of a $\Sigma$-space we can put $\mathscr{K} = \{C(x): x \in X\}$, where $C(x) = \bigcap_{n=1}^{\infty} C(x, \mathscr{H}_n)$ as noted in the introduction. Since $X$ is Lindelöf, each $\mathscr{H}_n$ is countable. From the definition of the topology for $X$ any member of $\mathscr{H}$ missing $p$ is a countable set. Therefore $X' = X - \cup \{H \in \mathscr{H}: p \notin H\}$ is an uncountable closed subspace of $X$, which is a $\Sigma$-space having $\mathscr{H}/X' = \{H \cap X': H \in \mathscr{H}\}$ as a $\sigma$-locally finite (mod $\mathscr{K}/X'$)-network. Consequently, we could have assumed from the beginning that each $\mathscr{H}_n$ is finite and each member of $\mathscr{H}$ contains $p$. For each $x \in X$ and $n$, let $H(x, n)$ be the smallest (as a subset) member of $\mathscr{H}_n$ containing $x$. $H(x, n)$ exists because $\mathscr{H}_n$ is closed under finite intersections. Since the compact sets of $X$ are exactly the finite sets, $C(x) = \bigcap_{n=1}^{\infty} H(x, n)$ must be finite for each $x \in X$. Furthermore, for each $x \in X$ there is an $n_x$ such that $H(x, n_x)$ is finite. To see this, suppose not. Then there is an increasing sequence $n_1 < n_2 < \cdots$ with $H(x, n_{i+1}) \subsetneqq H(x, n_i)$ for $i = 1, 2, \cdots$. Now pick a point $x_i \in H(x, n_i) - H(x, n_{i+1})$ which is distinct from $p$ and $x$. Then $F = \{x_i: i = 1, 2, \cdots\}$ is a closed set in $X$ with $F \cap C(x) = \emptyset$ but $F \cap H(x, n) \neq \emptyset$ for all $n$. This contradicts the fact that $\mathscr{H}$ forms a network around $C(x)$. Hence there exists such an $n_x$. We denote by $n(x)$ the smallest $n_x$ for which $H(x, n_x)$ is finite. Put

$$L_n = \{x \in X: n(x) \leqq n\} \quad \text{for} \quad n = 1, 2, \cdots .$$

Then $\{L_n: n = 1, 2, \cdots\}$ is an increasing cover of $X$. Since $X$ is uncountable, there exists an $n_0$ such that $L_{n_0}$ is an uncountable set containing $p$. Clearly $L_{n_0}$ is closed in $X$ and hence it is a $\Sigma$-space having $\mathscr{H} | L_{n_0}$ as a (mod $\mathscr{K} | L_{n_0}$)-network. But $\bigcup_{i=1}^{n_0} \mathscr{H}_i$ is finite and for each $x \in L_{n_0}$ there exists an $H(x, n(x))$ with $n(x) \leqq n_0$. This means that $L_{n_0}$ must be finite, which is a contradiction. Thus $X$ is not a $\Sigma$-space.

LEMMA 3.5. *If $X$ is a $\Sigma^*$-space* (resp. *a $\Sigma^\#$-space*), *then $X$ has a sequence $\{\mathscr{F}_n: n = 1, 2, \cdots\}$ of hereditarily closure-preserving* (resp. *closure-preserving*) *closed covers of $X$ such that any sequence $\{x_n: n = 1, 2, \cdots\}$ with $x_n \in C(x, \mathscr{F}_n)$ for some $x \in X$ has a cluster point. In particular, $X$ is a $\Sigma$-space iff $X$ has a sequence $\{\mathscr{F}_n: n = 1, 2, \cdots\}$ of locally finite closed covers of $X$ such that any sequence $\{x_n: n = 1, 2, \cdots\}$ with $x_n \in C(x, \mathscr{F}_n)$ for some $x \in X$ has a cluster point.*

*Proof*. Since all cases are proved similarly, we shall prove for a $\Sigma^\#$-space, only. Let $X$ be a $\Sigma^\#$-space having a $\sigma$-closure-preserving closed (mod $\mathscr{K}$)-network $\mathscr{H} = \bigcup_{n=1}^{\infty} \mathscr{H}_n$ for a cover $\mathscr{K}$ of $X$ by countably compact sets, where we can assume that each $\mathscr{H}_n$ is a closure-preserving closed cover of $X$. Put $\mathscr{F}_n = \bigcup_{k \leq n} \mathscr{H}_k$ for $n = 1, 2, \cdots$. Now we shall show that $\{\mathscr{F}_n : n = 1, 2, \cdots\}$ satisfies the required condition. On the contrary, suppose not. Then there exists a discrete sequence $\{x_n : n = 1, 2, \cdots\}$ with $x_n \in C(x, \mathscr{F}_n)$ for some $x \in X$. Since $\mathscr{K}$ covers $X$, there is a $K \in \mathscr{K}$ containing $x$. Since $\{x_n : n = 1, 2, \cdots\}$ is discrete, there exists an $n_0$ such as $\{x_n : n \geqq n_0\} \cap K = \emptyset$. Then $G = X - \{x_n : n \geqq n_0\}$ is an open set containing $K$ and thus, by the assumption, there exists an $F \in \mathscr{F}_m$ for some $m$ with $K \subset F \subset G$. Hence we have $x_i \in C(x, \mathscr{F}_i) \subset C(x, \mathscr{F}_m) \subset F \subset G$ for any $i$ with $m < i$ as well as $n_0 < i$, which is a contradiction.

The 'if' part in the later half is easily seen noting that any $C(x, \mathscr{F}_n)$ could have been a member of $\mathscr{F}_n$.

THEOREM 3.6. *Let $X$ be a $\Sigma^*$-space for which every open set is an $F_\sigma$. Then $X$ is a $\Sigma$-space.*

*Proof.* Let $\mathscr{F} = \bigcup_{n=1}^\infty \mathscr{F}_n$ be a $\sigma$-hereditarily closure-preserving closed (mod $\mathscr{K}$)-network for a cover $\mathscr{K}$ by countably compact sets. We can assume that each $\mathscr{F}_n$ covers $X$ and that $\mathscr{F}_n \subset \mathscr{F}_{n+1}$ for $n = 1, 2, \cdots$. Put

$$X' = \{x \in X : |C(x, \mathscr{F}_n)| \text{ is finite for some } n\} .$$

Then $X'$ is $\sigma$-discrete in $X$ by 2.6. Denote $X'$ by $\bigcup_{n=1}^\infty P_n$, where each $P_n$ is discrete in $X$ and we can assume $P_n \subset P_{n+1}$ for $n = 1, 2, \cdots$.

We shall show that each $\mathscr{F}_n$ is locally finite at any $x \in X - X'$. On the contrary, suppose some $\mathscr{F}_{n_0}$ is not locally finite at some $x \in X - X'$. Since $\mathscr{F}_n \subset \mathscr{F}_{n+1}$ and since each $\mathscr{F}_n$ is closure-preserving, $\Lambda'_n = \{\lambda \in \Lambda_n : x \in F_\lambda\}$ must be infinite for all $n \geqq n_0$. Since $x \notin X'$, $C(x, \mathscr{F}_n)$ is infinite for all $n \geqq n_0$. We can choose a point $x_n \in C(x, \mathscr{F}_n)$ and a $\lambda_n \in \Lambda'_{n_0}$ with $x_n \in F_{\lambda_n}$ for each $n \geqq n_0$ and such that $n \neq m$ implies $x_n \neq x_m$ as well as $\lambda_n \neq \lambda_m$. By 3.5 $\{x_n : n = n_0, n_0 + 1, \cdots\}$ has a cluster point. On the other hand, it must be discrete, because each $\{x_n\} \subset F_{\lambda_n} \in \mathscr{F}_{n_0}$ and $\mathscr{F}_{n_0}$ is hereditarily closure-preserving. This contradiction shows that each $\mathscr{F}_n$ is locally finite at any $x \in X - X'$.

Next, put

$$Y_n = \{x \in X : \mathscr{F}_n \text{ is locally finite at } x\} , \qquad n = 1, 2, \cdots .$$

Then each $Y_n$ is open in $X$ and therefore an $F_\sigma$. Denote $Y_n$ by $\bigcup_{m=1}^\infty Q_{nm}$, where each $Q_{nm}$ is closed in $X$ and $Q_{nm} \subset Q_{nm+1}$ for $m, n = 1, 2, \cdots$. Further, as was seen above, we have $X - X' \subset Y_n$ for $n = 1, 2, \cdots$.

Finally, put

$$\mathscr{F}_{nm} = \{F_\lambda \cap Q_{nm}: \lambda \in \Lambda_n\} \cup \{X\} \quad \text{for} \quad n, m = 1, 2, \cdots,$$
$$\mathscr{H}_n = \{\{x\}: x \in P_n\} \cup \{X\} \quad \text{for} \quad n = 1, 2, \cdots.$$

Then each $\mathscr{F}_{nm}$ as well as $\mathscr{H}_n$ is locally finite closed cover of $X$. In order that $X$ be a $\Sigma$-space, it suffices to show that the sequence $\{\mathscr{F}_{nm}: n, m = 1, 2, \cdots\} \cup \{\mathscr{H}_n: n = 1, 2, \cdots\} = \{\mathscr{G}_i: i = 1, 2, \cdots\}$ satisfies the condition in 3.5. Let $x \in X$ be any point and $\{x_i: i = 1, 2, \cdots\}$ a sequence with $x_i \in C(x, \mathscr{G}_i)$. If $x \in X'$, then $x \in P_k$ for some $k$, and since $\{P_n: n = 1, 2, \cdots\}$ is increasing, we have $C(x, \mathscr{H}_n) = \{x\} \in \mathscr{H}_n$ for all $n \geqq k$. Hence $\{x_i: i = 1, 2, \cdots\}$ has a cluster point $x$. If $x \notin X'$, then $x \in Y_n$ for $n = 1, 2, \cdots$ and hence, for each $n$, there exists a $k_n$ with $x \in Q_{nk_n}$. Thus, for any $n$ we have $C(x, \mathscr{F}_{nk_n}) \subset C(x, \mathscr{F}_n)$. On the other hand, by 3.5 any sequence $\{p_n: n = 1, 2, \cdots\}$ with $p_n \in C(x, \mathscr{F}_n)$ has a cluster point. Hence $\{x_i: i = 1, 2, \cdots\}$ must have a cluster point. This shows by 3.5 that $X$ is a $\Sigma$-space.

THEOREM 3.7. *Let $X$ be a paracompact space. Then the following conditions are equivalent.*

(1) *$X$ is a $\Sigma$-space.*

(2) *$X \times I$ is a $\Sigma$-space.*

(3) *$X \times I$ is a $\Sigma^*$-space.*

*Proof.* Since the property of being a paracompact $\Sigma$-space is countably productive (cf. [6]), we have (1) ⇒ (2). From the definition clearly (2) ⇒ (3).

(3) ⇒ (1). Let $\mathscr{F} = \bigcup_{n=1}^\infty \mathscr{F}_n$ be a $\sigma$-hereditarily closure-preserving (mod $\mathscr{K}$)-network for some cover $\mathscr{K}$ of $X \times I$ by countably compact sets. We assume that $\mathscr{F}_n \subset \mathscr{F}_{n+1}$ for $n = 1, 2, \cdots$.

At first we shall construct by induction on $n$ a collection $\{V(\alpha_1, \cdots, \alpha_n): \alpha_1 \in A_1, \cdots, \alpha_n \in A_n; n = 1, 2, \cdots\}$ of open sets of $X$ and a corresponding collection

$$\{I(\alpha_1, \cdots, \alpha_n): \alpha_1 \in A_1, \cdots, \alpha_n \in A_n; n = 1, 2, \cdots\}$$

of subsets of $I$ satisfying the following conditions:

(i) $\{V(\alpha_1, \cdots, \alpha_n): \alpha_1 \in A_1, \cdots, \alpha_n \in A_n\}$ is a locally finite open cover of $X$ for $n = 1, 2, \cdots$.

(ii) $V(\alpha_1, \cdots, \alpha_n, \alpha_{n+1}) \subset V(\alpha_1, \cdots, \alpha_n)$ for $\alpha_1 \in A_1, \cdots, \alpha_n \in A_n$, $\alpha_{n+1} \in A_{n+1}$; $n = 1, 2, \cdots$.

(iii) If $V(\alpha_1, \cdots, \alpha_n)$ is nonempty, then $I(\alpha_1, \cdots, \alpha_n)$ is a closed interval.

(iv) $I(\alpha_1, \cdots, \alpha_n, \alpha_{n+1}) \subset I(\alpha_1, \cdots, \alpha_n)$ for $\alpha_1 \in A_1, \cdots, \alpha_n \in A_n, \alpha_{n+1} \in A_{n+1}$; $n = 1, 2, \cdots$.

(v) $\overline{V(\alpha_1, \cdots, \alpha_n)} \times I(\alpha_1, \cdots, \alpha_n)$ meets only finitely many members of $\mathscr{F}_n$ for $\alpha_1 \in A_1, \cdots, \alpha_n \in A_n$; $n = 1, 2, \cdots$.

Assume that such collections are constructed for all $n \leqq k$ and consider $n = k + 1$.

Fix $\alpha_1 \in A_1, \cdots, \alpha_k \in A_k$ with $V(\alpha_1, \cdots, \alpha_k) \neq \emptyset$. For any point $x \in \overline{V(\alpha_1, \cdots, \alpha_k)}$, since $\{x\} \times I(\alpha_1, \cdots, \alpha_k)$ is compact and $\mathscr{F}_{k+1}$ is hereditarily closure-preserving, by 2.1 $\mathscr{F}_{k+1}$ is locally finite at all but finitely many points of $\{x\} \times I(\alpha_1, \cdots, \alpha_k)$. Let $\{p_1, \cdots, p_m\}$ be those points of $\{x\} \times I(\alpha_1, \cdots, \alpha_k)$ at which $\mathscr{F}_{k+1}$ is not locally finite. Let $I_x$ be a closed subinterval of $I(\alpha_1, \cdots, \alpha_k)$ missing $p_1, \cdots, p_m$. Since $\{x\} \times I_x$ is compact, there exists an open neighborhood $U_x$(in $\overline{V(\alpha_1, \cdots \alpha_k)}$) of $x$ such that

$$\bar{U}_x \times I_x \subset X \times I - \cup\{F \in \mathscr{F}_{k+1}: F \cap (\{x\} \times I_x) = \emptyset\} .$$

Since $\overline{V(\alpha_1, \cdots, \alpha_k)}$ is paracompact, there is a locally finite open cover $\{V_\lambda: \lambda \in \Lambda(\alpha_1, \cdots, \alpha_k)\}$ of $\overline{V(\alpha_1, \cdots, \alpha_k)}$ which refines $\{U_x: x \in \overline{V(\alpha_1, \cdots, \alpha_k)}\}$. Let

$$\varphi: \Lambda(\alpha_1, \cdots, \alpha_k) \longrightarrow \overline{V(\alpha_1, \cdots, \alpha_k)} \subset X$$

be a function which satisfies $V_\lambda \subset U_{\varphi(\lambda)}$ for $\lambda \in \Lambda(\alpha_1, \cdots, \alpha_k)$.

Now varying $\alpha_1 \in A_1, \cdots, \alpha_k \in A_k$, put

$$A_{k+1} = \cup \{\Lambda(\alpha_1, \cdots, \alpha_k): \alpha_1 \in A_1, \cdots, \alpha_k \in A_k\}$$

and

$$V(\alpha_1, \cdots, \alpha_k, \alpha_{k+1}) = \begin{cases} V(\alpha_1, \cdots, \alpha_k) \cap V_{\alpha_{k+1}} \text{ if } V(\alpha_1, \cdots, \alpha_k) \neq \emptyset \text{ and } \alpha_{k+1} \in \Lambda(\alpha_1, \cdots, \alpha_k) \\ \emptyset \quad \text{otherwise} \end{cases} .$$

Furthermore, if $V(\alpha_1, \cdots, \alpha_k, \alpha_{k+1}) \neq \emptyset$, then from the definition we have $V(\alpha_1, \cdots, \alpha_k) \neq \emptyset$ and $\alpha_{k+1} \in \Lambda(\alpha_1, \cdots, \alpha_k)$. By inductive hypothesis $I(\alpha_1, \cdots, \alpha_k)$ is not empty. Hence we put $I(\alpha_1, \cdots, \alpha_k, \alpha_{k+1}) = I_{\varphi(\alpha_{k+1})}$, which is not empty. Otherwise we put $I(\alpha_1, \cdots, \alpha_k, \alpha_{k+1}) = \emptyset$. Then we can easily see that $\{V(\alpha_1, \cdots, \alpha_{k+1}): \alpha_1 \in A_1, \cdots, \alpha_{k+1} \in A_{k+1}\}$ and $\{I(\alpha_1, \cdots, \alpha_{k+1}): \alpha_1 \in A_1, \cdots, \alpha_{k+1} \in A_{k+1}\}$ satisfy all required conditions (i)—(v).

Consequently, for each $n$ we can construct $\{V(\alpha_1, \cdots, \alpha_n): \alpha_1 \in A_1, \cdots, \alpha_n \in A_n\}$ and $\{I(\alpha_1, \cdots, \alpha_n): \alpha_1 \in A_1, \cdots \alpha_n \in A_n\}$ satisfying (i)—(v).

Next, put

$$Y_n = \cup \{\overline{V(\alpha_1, \cdots, \alpha_n)} \times I(\alpha_1, \cdots, \alpha_n): \alpha_1 \in A_1, \cdots, \alpha_n \in A_n\}$$

and

$$Y = \bigcap_{n=1}^{\infty} Y_n .$$

Since $\{\overline{V(\alpha_1, \cdots, \alpha_n)}: \alpha_1 \in A_1, \cdots, \alpha_n \in A_n\}$ is locally finite in $X$, $Y_n$ is closed in $X \times I$ and thus $Y$ is closed in $X \times I$. Also by (v) the collection

$$\mathscr{H}_n = \mathscr{F}_n | Y = \{F \cap Y: F \in \mathscr{F}_n\}$$

is a locally finite closed cover of $Y$ for $n = 1, 2, \cdots$.

Now we show that $Y$ is a $\Sigma$-space. For this purpose it suffices to show that $\{\mathscr{H}_n: n = 1, 2, \cdots\}$ satisfies the condition in 3.5. Let $y \in Y$ be any point and $\{y_n: n = 1, 2, \cdots\}$ any sequence with $y_n \in C(y, \mathscr{H}_n)$. Since $C(y, \mathscr{H}_n) \subset C(y, \mathscr{F}_n)$ for each $n$ and since $X \times I$ is a $\Sigma^*$-space, by 3.5 $\{y_n: n, = 1, 2, \cdots\}$ has a cluster point in $X \times I$. Since $Y$ is closed in $X \times I$, $\{y_n: n = 1, 2, \cdots\}$ must have a cluster point in $Y$, which shows by 3.5 that $Y$ is a $\Sigma$-space.

Finally, let $\pi$ be the restriction to $Y$ of the projection from $X \times I$ onto $Y$. Since the projection is perfect and since $Y$ is closed in $X \times I$, $\pi$ is perfect. It remains to show that $\pi$ is onto, because a $\Sigma$-space is preserved by a perfect map (cf. [6]). Let $x$ be any point of $X$. Since $\{V(\alpha_1, \cdots, \alpha_n): \alpha_1 \in A_1, \cdots, \alpha_n \in A_n\}$ covers $X$ for $n = 1, 2, \cdots$, by (ii) we can choose a point $(\alpha_1, \alpha_2, \cdots)$ in $A_1 \times A_2 \times \cdots$ with $x \in V(\alpha_1, \cdots, \alpha_n)$ for $n = 1, 2, \cdots$. Since each $V(\alpha_1, \cdots, \alpha_n)$ is nonempty, by (iv) $\{I(\alpha_1, \cdots, \alpha_n): n = 1, 2, \cdots\}$ is a decreasing sequence of nonempty closed intervals. Hence $\bigcap_{n=1}^{\infty} I(\alpha_1, \cdots, \alpha_n) \neq \emptyset$. Pick a point $q$ in this intersection. Then $(x, q)$ belongs to $\overline{V(\alpha_1, \cdots, \alpha_n)} \times I(\alpha_1, \cdots, \alpha_n) \subset Y_n$ for $n = 1, 2, , \cdots$ and thus belongs to $Y$. Clearly $\pi((x, q)) = x$. This shows that $\pi$ is onto and hence $X$ is a $\Sigma$-space, which completes the proof.

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