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NOTES ON RELATED STRUCTURES OF A UNIVERSAL ALGEBRA

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The related structures of a universal algebra $\mathfrak{A}$ that are studied here are the subalgebra lattice of $\mathfrak{A}$, the congruence lattice of $\mathfrak{A}$, the automorphism group of $\mathfrak{A}$, and the endomorphism semigroup of $\mathfrak{A}$. Characterizations of these structures known, and E. T. Schmidt proved the independence of the automorphism group and the subalgebra lattice. It has been conjectured that the first three of the structures listed above are independent, i.e., that the congruence lattice, subalgebra lattice, and automorphism group are independent. One result in this paper is a proof of a special case of this conjecture. Various observations concerning the relationship between the endomorphism semigroup and the congruence lattice are also in this paper. In the last section a problem of G. Grätzer is solved, namely that of characterizing the endomorphism semigroups of simple unary algebras. (An algebra is simple when the only congruences are the trivial ones.)

The characterizations of the various related structures are as follows: the congruence lattice is an arbitrary algebraic lattice the subalgebra lattice is an arbitrary algebraic lattice; the automorphism group is an arbitrary group; the endomorphism semigroup is an arbitrary semigroup with identity. The "independence of the automorphism group and the subalgebra lattice" is more precisely phrased as: for each pair $\langle \mathfrak{G}, \mathfrak{L} \rangle$, where $\mathfrak{G}$ is a group and $\mathfrak{L}$ is an algebraic lattice with more than one element, there is an algebra $\mathfrak{A}$ with $\mathfrak{G}$ isomorphic to the automorphism group of $\mathfrak{A}$ and with $\mathfrak{L}$ isomorphic to the subalgebra lattice of the same algebra $\mathfrak{A}$. All statements about the independence of related structures will be phrased in this way.

Mentioned above was a proof of a special case of the independence of the triple consisting of the automorphism group, the subalgebra lattice, and the congruence lattice. As a corollary one gets a proof of a special case of the independence of the pair consisting of the automorphism group and the congruence lattice. E. T. Schmidt published what was supposed to be a proof of the independence of this pair of structures. But, his proof [10] was incorrect. (See e.g. Exercise 31 of chapter 2 of [2]). The author has just completed a proof of the independence of this pair [8].

The terminology essentially conforms to that in [2]. ω(or ω_A) will denote the equality relation on the set A, and ι(or ι_A) will denote

the total relation. $\Theta(a_0, a_1)$ will represent the smallest congrucence collapsing a_0 and a_1. $\mathfrak{L} = \langle L, \wedge, \vee \rangle$ will denote a lattice. $\mathfrak{C}(\mathfrak{A}) = \langle \mathscr{C}(\mathfrak{A}); \subseteq \rangle$ will denote the congruence lattice of $\mathfrak{A}$. $\mathfrak{S}(\mathfrak{A}) = \langle \mathscr{S}(\mathfrak{A}); \subseteq \rangle$ will denote the subalgebra lattice of $\mathfrak{A}$. $\mathfrak{G}(\mathfrak{A}) = \langle G(\mathfrak{A}); \circ \rangle$ will denote the automorphism group of $\mathfrak{A}$. $\mathfrak{E}(\mathfrak{A}) = \langle E\mathfrak{A}); \circ \rangle$ will denote the endomorphism semigroup of $\mathfrak{A}$.

An important algebra for dealing with endomorphism semigroup and automorphism group problems is the algebra of left multiplications $\mathfrak{L}(\mathfrak{S})$ of the semigroup $\mathfrak{S}$. The operations are all left multiplication maps and the endomorphisms are all right multiplication maps. As in Cayley's Theorem, the semigroup of right multiplications of $\mathfrak{S}$ is isomorphic to $\mathfrak{S}$.

Many of the details of the proofs which are left out can be found in the author's dissertation [6]. The various characterizations mentioned above can be found in [1], [2], [3]. E. T. Schmidt's result on the independence of the automorphism group and subalgebra lattice is found in [11].

1. The property restricting the representation of $\langle \mathfrak{G}, \mathfrak{L}_0, \mathfrak{L}_1 \rangle$ as $\langle \mathfrak{G}(\mathfrak{A}), \mathfrak{S}(\mathfrak{A}), \mathfrak{C}(\mathfrak{A}) \rangle$.

Let $\mathfrak{A} = \langle A; F \rangle$ be an algebra. The lattice $\mathfrak{L}$ is assumed to be an algebraic lattice. Let $a \in L$, and let $(x_i \mid i \in I)$ be a family of elements of L.

Essentially the property mentioned in the heading is: there exist $a_0, a_1 \in A$ such that for any $x \neq a_0$ and for any congruence Θ, if $a_0 \equiv x(\Theta)$, then $a \equiv a_1(\Theta)$. We will give a generalization of this property and a property of the congruence lattice equivalent to the more general property. Also, the class of algebraic lattices having the equivalent property will be discussed.

Let $a_0, a_1 \in A$ with $a_0 \neq a_1$.

(**) There exists a partition $\{A_0, A_1\}$ of A such that $a_i \in A_i$ and for any $\langle x, y \rangle \in A_0 \times A_1$, $\Theta(a_0, a_1) \leqq \Theta(x, y)$.

(*) If $a \leqq \bigvee (x_i \mid i \in I)$, then $a \leqq x_i$ for some i.

Notice that the originally stated condition is a special case of (**) where $A_0 = \{a_0\}$. Obviously, if an element a of $\mathfrak{L}$ has property (*), then a is complete-join irreducible. Also, a has property (*) if and only if a's dual ideal is completely prime.

PROPOSITION 1. *If property* (**) *is satisfied for* $\langle a_0, a_1 \rangle$, *then* $\Theta(a_0, a_1 \rangle$ *satisfies property* (*) *in the congruence lattice of* $\mathfrak{A}$.

REMARK. This statement was first observed by G. Grätzer.

Proof. Suppose that $(\Phi_i \mid i \in I)$ is a family of congruences and that $\Theta(a_0, a_i) \subseteq \bigvee (\Phi_i \mid i \in I)$. There exists a sequence $a_0 = z_0, \cdots, z_n, = a_1$, with $z_j \in A$ such that $z_j \equiv z_{j+1}(\Phi_{i_j})$ for some $i_j \in I$. Since $a_0 \in A_0$, $a_1 \in A_1$, and $\{A_0, A_1\}$ is a partition of A, there is a k such that $z_k \in A_0$ and $z_{k+1} \in A_1$. So $\Theta(a_0, a_1) \subseteq \Theta(z_k, z_{k+1}) \subseteq \Phi_{i_k}$.

PROPOSITION 2. *If there is a congruence Θ different from ω having property (*), then $\Theta = \Theta(a_0, a_1)$ for some a_0, a_1 in A with $a_0 \neq a_1$ and property (**) is satisfied for $\langle a_0, a_1 \rangle$.*

Proof. Always $\Theta = \bigvee (\Theta(x, y) \mid x \equiv y(\Theta))$. Since Θ has property (*), $\Theta = \Theta(x, y)$ for some $x, y \in A$. Fix a_0, a_1 such that $\Theta = \Theta(a_0, a_1)$. Set

$$B_0 = \{x \mid \Theta(x, a_0) \not\supseteq \Theta(a_0, a_1)\},$$
$$B_1 = \{y \mid \Theta(y, a_1) \not\supseteq \Theta(a_0, a_1)\},$$
$$B_2 = \{z \mid \Theta(a_0, z) = \Theta(a_1, z)\}.$$

Set $A_0 = B_0$ and $A_1 = B_1 \cup B_2$. It follows that $A_0 \cap A_1 = \varnothing$. Clearly, $a_0 \in A_0$ and $a_1 \in A_1$. Also $A = A_1 \cup A_2$.

Let $x_0 \in A_0$ and $x_1 \in A_1$ and consider $\Theta(x_0, x_1)$. First suppose that $x_1 \in B_1$. Thus, $\Theta(x_0, a_0) \not\supseteq \Theta(a_0, a_1)$ and $\Theta(x_1, a_1) \not\supseteq \Theta(a_0, a_1)$. Now, since $\Theta(a_0, a_1) \subseteq \Theta(x_0, a_0) \vee \Theta(x_0, x_1) \vee \Theta(x_1, a_1)$ and since $\Theta(a_0, a_1)$ has (*), we have that $\Theta(a_0, a_1) \subseteq \Theta(x_0, x_1)$. Now suppose that $x_1 \in B_2$. So $\Theta(a_0, a_1) \subseteq \Theta(a_0, x_1) \subseteq \Theta(a_0, x_0) \vee \Theta(x_0, x_1)$, and thus, $\Theta(a_0, a_1) \subseteq \Theta(x_0, x_1)$.

Combining these two propositions with the congruence lattice characterization theorem, we get the following statement.

PROPOSITION 3. *If $\mathfrak{L}$ is an algebraic lattice, the following are equivalent:*

(i) *there exists $a \neq 0$, $a \in L$, such that a has property (*);*

(ii) *there exists an algebra $\mathfrak{A} = \langle A; F \rangle$ with $\mathfrak{C}(\mathfrak{A})$, the congruence lattice of $\mathfrak{A}$, isomorphic to $\mathfrak{L}$, and there are $a_0, a_1 \in A$, $a_0 \neq a_1$, such that (**) is satisfied for $\langle a_0, a_1 \rangle$;*

(iii) *for any algebra $\mathfrak{A} = \langle A; F \rangle$ with $\mathfrak{C}(\mathfrak{A})$ isomorphic to $\mathfrak{L}$, there are $a_0, a_1 \in A$, $a_0 \neq a_1$, such that (**) is satisfied for $\langle a_0, a_1 \rangle$.*

Let $\mathscr{K}$ be the class of algebraic lattices having an $a \neq 0$ with property (*). Several simple observations can be made. The five element modular non-distributive lattice is not in $\mathscr{K}$ since none of the dual ideals generated by a nonzero element is prime. Every distributive algebraic lattice with a complete-join irreducible element is in $\mathscr{K}$. If $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are algebraic lattices, then $\mathfrak{L}_1 + \mathfrak{L}_2 \in \mathscr{K}$ (where +

denotes ordinal sum). (The zero of $\mathfrak{L}_2$ is a nonzero element in $\mathfrak{L}_1 + \mathfrak{L}_2$ having (*)). Every algebraic lattice $\mathfrak{L}$ is both a complete sublattice of and a homomorphic image of a member of $\mathscr{K}$ since $\mathfrak{C}_1 + \mathfrak{L} \in \mathscr{K}$. ($\mathfrak{C}_n$ denotes the n-element chain.) Also, observe that for a family $(\mathfrak{L}_i \mid i \in I)$ of algebraic lattices, $\Pi(\mathfrak{L}_i \mid i \in I) \in \mathscr{K}$ if and only if there exists at least one $j \in I$ with $\mathfrak{L}_j \in \mathscr{K}$.

2. **The construction for representing $\langle \mathfrak{G}, \mathfrak{L}_0, \mathfrak{L}_1 \rangle$ as $\langle \mathfrak{G}(\mathfrak{A}), \mathfrak{S}(\mathfrak{A}), \mathfrak{E}(\mathfrak{A}) \rangle$.** First we need some notation. Let $\mathfrak{A} = \langle A; F \rangle$ be an algebra and $X \subseteq A$. Set $F(\mathfrak{A}, X) = \{\varphi \mid \varphi$ is an endomorphism of $\mathfrak{A}, \{x\} = x\varphi^{-1}$ for all $x \in X\}$, and set $\mathfrak{F}(\mathfrak{A}, X) = \langle F(\mathfrak{A}, X); \circ \rangle$. In other words, an endomorphism φ is in $F(\mathfrak{A}, X)$ if $(A - X)\varphi \subseteq A - X$ and $x\varphi = x$ for $x \in X$. Clearly, $\mathfrak{F}(\mathfrak{A}, X)$ is a nonempty semigroup with identity.

$\mathscr{S}(\mathfrak{A})$ is the subalgebra system of $\mathfrak{A}$. Recall that $\mathfrak{S}(\mathfrak{A})$ is the subalgebra lattice, that $\mathfrak{C}(\mathfrak{A})$ is the congruence lattice, and that $\mathfrak{E}(\mathfrak{A})$ is the endomorphism semigroup.

THEOREM 1. *Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are algebras, that $\mathfrak{A}$ is simple, that there is a $U \subseteq A$, $|U| = 2$, $U \subseteq D$ for every $D \in \mathscr{S}(\mathfrak{A})$, and that there is an $\langle a_0, a_1 \rangle \in B^2$ with $a_0 \neq a_1$ for which property (**) is satisfied. There exists an algebra $\mathfrak{A}'$ such that:*

(i) $\mathfrak{S}(\mathfrak{A}')$ *is isomorphic to* $\mathfrak{S}(\mathfrak{A})$;

(ii) $\mathfrak{C}(\mathfrak{A}')$ *is isomorphic to* $\mathfrak{C}(\mathfrak{B})$;

(iii) $\mathfrak{E}(\mathfrak{A}')$ *is isomorphic to* $\mathfrak{F}(\mathfrak{A}, U)$.

Proof. Let $\mathfrak{A} = \langle A; F \rangle$ and $\mathfrak{B} = \langle B; G \rangle$ and $U = \{u_0, u_1\}$ and let $\langle a_0, a_1 \rangle \in B^2$ have (**) and let $a_0 \neq a_1$. Assume that A and B are disjoint. For each $x \in B \cup U$ define a nullary operation f_x whose value is x. Let $\{A_0, A_1\}$ be a partition of B for satisfying (**). Define four unary operations as follows:

$$g_1(x) = \begin{cases} u_0, & x \in A \cup A_0 \\ u_1, & x \in A_1 \end{cases};$$

$$g_2(x) = \begin{cases} a_0, & x = u_0 \\ a_1, & \text{otherwise} \end{cases};$$

$$g_3(x) = \begin{cases} a_0, & x \in A \\ a_1, & x \in B \end{cases};$$

$$g_4(x) = \begin{cases} x, & x \in A - U \\ u_0, & x \in B \cup \{u_1\} \\ a_0, & x = u_0 \end{cases}.$$

For $x \in A'$, set $\hat{x} = x$ if $x \in B$ and set $\hat{x} = a_0$ if $x \in A$. Let $x_i \in A'$.

Extend the operations of F and G to A' as follows: For $f \in G$ set $f(a_0, \cdots, a_{n-1}) = f(\hat{a}_0, \cdots, \hat{a}_{n-1})$. For $f \in F$, if all $a_i \in A$, then keep the value of f in $\mathfrak{A}$, and set $f(a_0, \cdots, a_{n-1}) = u_0$, otherwise. Set $F' = F \cup G \cup \{f_x | x \in B \cup U\} \cup \{g_i | i = 1, 2, 3, 4\}$. Set $\mathfrak{A}' = \langle A'; F' \rangle$.

For each $D \in \mathscr{S}(\mathfrak{A})$, set $\bar{D} = D \cup B$. For each $\varphi \in F(\mathfrak{A}, U)$, define $\bar{\varphi}$ by letting $x\bar{\varphi} = x\varphi$ if $x \in A$ and $x\bar{\varphi} = x$ if $x \in B$. For $\Theta \in \mathscr{C}(\mathfrak{A})$ define Θ^* by letting $\Theta^* = \omega_A$, the equality relation on A, if $\Theta(a_0, a_1) \not\subseteq \Theta$ and $\Theta^* = \iota_A \cup \{\langle x, b\rangle | x \in A, b \equiv a_0(\Theta)\} \cup \{\langle b, x\rangle | x \in A, b \equiv a_0(\Theta)\}$ in case $\Theta(a_0, a_1) \subseteq \Theta$. Now set $\bar{\Theta} = \Theta \cup \Theta^*$. To complete the proof one shows that $D \to \bar{D}$, $\varphi \to \bar{\varphi}$, and $\Theta \to \bar{\Theta}$ are isomorphisms. The lengthy, but routine, calculations are left to the reader.

In the proof above the operation g_4 guarantees that an endomorphism σ of $\mathfrak{A}'$ has the property that $A\sigma \subseteq A$. The operations g_1, g_2, g_3 guarantee that $a_0 \equiv a_1$ iff $u_0 \equiv u_1$ iff $a_0 \equiv a_1 \equiv u_0 \equiv u_1$. That $\mathfrak{A}$ is simple guarantees that if $x, y \in A$ and $x \neq y$ and $x \equiv y$ then $x \equiv u_0 \equiv u_1$. Finally g_3 guarantees that if $x \in A$ and $y \in B$ and $x \equiv y$ then $a_0 \equiv a_1$.

3. Representing $\langle \mathfrak{G}, \mathfrak{L}_0, \mathfrak{L}_1 \rangle$ as $\langle \mathfrak{G}(\mathfrak{A}), \mathfrak{S}(\mathfrak{A}), \mathfrak{C}(\mathfrak{A}) \rangle$.

LEMMA 1. *If $\mathfrak{A} = \langle A; F \rangle$ is an algebra, then there is an algebra $\mathfrak{A}' = \langle A; F \rangle$ such that:*

(i) *$\mathfrak{A}'$ is simple;*
(ii) *D is a subalgebra of $\mathfrak{A}$ iff D is a subalgebra of $\mathfrak{A}'$;*
(iii) *$E(\mathfrak{A}) = \{\varphi | \varphi \in E(\mathfrak{A}), \varphi$ is $1 - 1$ or φ is constant$\}$.*

REMARKS. Roughly (iii) says $E(\mathfrak{A}')$ is as big as is possible given (i) and (ii).

Suppose $\mathfrak{S}$ is a semigroup in which every element is right cancellative or a right zero. Every endomorphism of $\mathfrak{L}(\mathfrak{S})$ is $1 - 1$ or constant. By applying this lemma to $\mathfrak{L}(\mathfrak{S})$ we get an easier proof that $\langle \mathfrak{S}; \mathfrak{C}_2 \rangle$ is representable. (See [3].)

Proof. Add an additional operation g defined as follows:

$$g(x, y, u, v) = \begin{cases} u, & \text{if } x \neq y \\ v, & \text{if } x = y \,. \end{cases}$$

Suppose $x \neq y$ and Θ is any congruence of $\mathfrak{A}'$ with $x \equiv y(\Theta)$. Let $u, v \in A$. Thus, $u = g(x, y, u, v) \equiv g(y, y, u, v) = v(\Theta)$. So $\Theta = \iota$, and (i) is established.

The rest is routine.

The operation used in the above lemma was used in [5] in a different context, but in each case the purpose of the operation is to "fill out" subalgebras in a direct power. This 4-ary function is equivalent to the ternary discriminator function [12] [9] in that each can be expressed as a polynomial in the other.

A modification of the above 4-ary function is used in Lemma 6. It does not appear that the modified 4-ary function is equivlent to a ternary function.

LEMMA 2. *If* $\mathfrak{A}$ *is any algebra, then there is an algebra* $\mathfrak{A}' = \langle A'; F' \rangle$ *and* $U \subseteq A'$ *with* $|U| = 2$ *such that:*

(i) $\mathfrak{S}(\mathfrak{A})$ *is isomorphic to* $\mathfrak{S}(\mathfrak{A}')$;

(ii) $U \subseteq D$ *for all* $D \in \mathscr{S}(\mathfrak{A}')$;

(iii) $\mathfrak{F}(\mathfrak{A}, U)$ *is isomorphic to* $\mathfrak{S}(\mathfrak{A})$.

Proof. Add two elements u_0, u_1. Let u_0 and u_1 each be the value of a nullary operation. Extend every operation f of $\mathfrak{A}$ by setting $f(x_0, \cdots, x_{n-1}) = u_0$ if $x_j \in U$. The rest is obvious.

The next lemma is a theorem due to E. T. Schmidt [11]. Recall that $\mathfrak{G}(\mathfrak{A})$ is the automorphism group of $\mathfrak{A}$.

LEMMA 3. *If* $\mathfrak{G}$ *is any group and* $\mathfrak{L}$ *is any algebraic lattice with* $|L| > 1$, *then there is an algebra* $\mathfrak{A}$ *with* $\mathfrak{G}$ *isomorphic to* $\mathfrak{G}(\mathfrak{A})$ *and* $\mathfrak{L}$ *isomorphic to* $\mathfrak{S}(\mathfrak{A})$.

THEOREM 2. *If* $\mathfrak{G}$ *is any group, if* $\mathfrak{L}_0$ *and* $\mathfrak{L}_1$ *are algebraic lattices such that* $|L_0| > 1$, *and if there is an* $a \neq 0, a \in L_1$, *with property* (*), *then there is an algebra* $\mathfrak{A}$ *such that:*

(i) $\mathfrak{G}$ *is isomorphic to* $\mathfrak{G}(\mathfrak{A})$;

(ii) $\mathfrak{L}_0$ *is isomorphic to* $\mathfrak{S}(\mathfrak{A})$;

(iii) $\mathfrak{L}_1$ *is isomorphic to* $\mathfrak{C}(\mathfrak{A})$.

REMARKS. A best possible representation theorem would, of course, have the restriction that $|L_1| > 1$. Also, if $|L_0| = 1$, then it is necessary that $|G| = 1$. Of course any triple of the form $\langle 1, \mathfrak{C}_1, \mathfrak{L} \rangle$ is representable. ($\mathfrak{C}_1$ is the one element chain.)

Proof. Let $\mathfrak{B}$ be the algebra given by Lemma 3 when applied to $\mathfrak{G}$ and $\mathfrak{L}_0$. Let $\mathfrak{B}'$ be the algebra given by Lemma 2 applied to $\mathfrak{B}$. Let $\mathfrak{B}''$ be the algebra given by Lemma 1. Let $\mathfrak{C}$ be the algebra constructed in the proof of the congruence lattice characterization theorem [2], [4] or [7]. Let $\mathfrak{A}$ be the algebra given by Theorem 1 when applied to $\mathfrak{B}''$ and $\mathfrak{C}$. The rest is routine.

COROLLARY 1. *If $\mathfrak{G}$ is any group and $\mathfrak{L}$ is any algebraic lattice with $a \neq 0, a \in L$, having property (*), then there is an algebra $\mathfrak{A}$ with $\mathfrak{G}$ isomorphic to $\mathfrak{G}(\mathfrak{A})$ and $\mathfrak{L}$ isomorphic to $\mathfrak{C}(\mathfrak{A})$.*

COROLLARY 2. *If $\mathfrak{L}_0$ is any algebraic lattice and $\mathfrak{L}_1$ is an algebraic lattice with an $a \neq 0 (a \in L_1)$ such that a has property (*), then there is an algebra $\mathfrak{A}$ with $\mathfrak{S}(\mathfrak{A})$ isomorphic to $\mathfrak{L}_0$ and $\mathfrak{C}(\mathfrak{A})$ isomorphic to $\mathfrak{L}_1$.*

4. **Necessary conditions for $\langle \mathfrak{S}, \mathfrak{L} \rangle$ to be representable as $\langle \mathfrak{E}(\mathfrak{A}), \mathfrak{C}(\mathfrak{A}) \rangle$.** Recall that if $\mathfrak{S}$ is a semigroup, $\mathfrak{L}(\mathfrak{S})$ is the algebra of left multiplications of $\mathfrak{S}$. $\mathfrak{A} = \langle A; F \rangle$ is some universal algebra. The basic thing established in this section is a relationship between $\mathfrak{C}(\mathfrak{L}(\mathfrak{E}(\mathfrak{A})))$ and $\mathfrak{E}(\mathfrak{A})$. If φ is an endomorphism, then set $x \equiv y(\varepsilon_\varphi)$ iff $x\varphi = y\varphi$. ε_φ is a congruence.

Let $\mathfrak{S} = \langle S; \cdot \rangle$ be a semigroup with identity, and let $x, s \in S$. The right multiplication map for s is defined by $x\rho_s = xs$.

Thus, if $\varphi \in E(\mathfrak{A})$, then we have the congruence ε_φ on $\mathfrak{A}$ and the mapping ρ_φ on $E(\mathfrak{A})$. So we have the equivalence relation $\varepsilon_{\rho_\varphi}$ on $E(\mathfrak{A})$. Observe that since ρ_φ is an endomorphism of $\mathfrak{L}(\mathfrak{E}(\mathfrak{A}))$, $\varepsilon_{\rho_\varphi}$ is a congruence of $\mathfrak{L}(\mathfrak{E}(\mathfrak{A}))$.

The proof of the next lemma involves only routine calculations.

LEMMA 4. *If $\varepsilon_\psi = \bigcap (\varepsilon_{\varphi_i} \mid i \in I)$, then $\varepsilon_{\rho_\psi} = \bigcap (\varepsilon_{\rho_{\varphi_i}} \mid i \in I)$.*

COROLLARY. *$\varepsilon_\varphi \to \varepsilon_{\rho_\varphi}$ is a mapping, and this mapping preserves arbitrary existing meets. In particular, it is order preserving.*

This mapping need not be $1 - 1$.

LEMMA 5. *If $\varepsilon_{\rho_\varphi} = \iota$ (φ is a right zero), then $\varepsilon_\psi \subseteqq \varepsilon_\varphi$ for every endomorphism ψ.*

Proof. Trivial.

$\varepsilon_{\rho_\varphi}$ can be ι and ε_φ need not be ι. $\varepsilon_{\rho_\varphi} = \iota$ means φ is a right zero in $\mathfrak{E}(\mathfrak{A})$, but φ need not be a constant map. But φ is a constant map iff $\varepsilon_\varphi = \iota$. On the other hand, if $\varepsilon_\varphi = \iota$, then $\varepsilon_{\rho_\varphi} = \iota$ (i.e., if φ is a constant map, then φ is a right zero). Also, there is a φ with $\varepsilon_\varphi = \omega$ and $\varepsilon_{\rho_\varphi} = \omega$ (the identity map).

To summarize we state the following theorem.

THEOREM 3. *Suppose $\mathfrak{S} = \langle S; \cdot \rangle$ is a semigroup with identity and $\mathfrak{L} = \langle L; \vee, \wedge \rangle$ is an algebraic lattice. Set $\mathscr{K} = \{\varepsilon_{\rho_s} \mid s \in S\}$. If*

$\langle \mathcal{S}, \mathfrak{L} \rangle$ is representable, then there is a subset H of L and there are two mappings α from S onto H and β from H onto $\mathscr{K}$ such that the following hold:

(i) $(s\alpha)\beta = \varepsilon_{\rho_s}$ *for all* $s \in S$;

(ii) β *preserves arbitrary existing meets;*

(iii) *if* $\varepsilon_{\rho_s} = \iota$, *then* $s\alpha$ *is the maximum element of* H *and* $|\iota\beta^{-1}| = 1$;

(iv) $0 \in H$ *(and* $0\beta = \omega$*);*

(v) *if* $1 \in H$, *then* $\iota \in \mathscr{K}$.

COROLLARY. *If $\langle \mathcal{S}, \mathfrak{C}_n \rangle$ is representable and $\mathfrak{C}_n$ is the n-element chain, then $\mathscr{K} \cup \{\iota\}$ is a chain of length $\leqq n$.*

5. **More on the class of representable pairs.** Throughout this section, $\mathcal{S} = \langle S; \cdot \rangle$ will be a semigroup with identity and $\mathfrak{L}$ will be an algebraic lattice. The ordinal sum of the lattices will be denoted by $+$. $\mathfrak{C}_n$ is the n element chain. $\mathfrak{A} = \langle A; F \rangle$ is an algebra.

In the preceding section, a necessary condition for $\langle \mathcal{S}, \mathfrak{L} \rangle$ to be representable as $\langle \mathfrak{E}(\mathfrak{A}), \mathfrak{C}(\mathfrak{A}) \rangle$ was given. Roughly the condition states that $\mathcal{S}$ gives a lower bound on the cardinality of L, namely, $|\mathscr{K}|$, and an upper bound on the meet struture of part of $\mathfrak{L}$. This suggests that one could take a representable pair and expand the lattice and expect the result to be a representable pair. A few such expansions are given here.

Sort of a multiplication formula for members of the class of all representable pairs is given.

One could question whether or not there exist a semigroup with identity and an algebraic lattice which are in some vague sense completely "incompatible." Theorem 4 gives a negative answer.

First we will state the theorems, and then we will give sketches of their proofs.

THEOREM 4. *If $\mathcal{S}$ is any semigroup with identity and $\mathfrak{L}$ is any algebraic lattice, then there is an algebra $\mathfrak{A}$ with $\mathcal{S}$ isomorphic to $\mathfrak{E}(\mathfrak{A})$ and $\mathfrak{L}$ isomorphic to a sublattice of $\mathfrak{C}(\mathfrak{A})$.*

This follows from Theorem 7.

THEOREM 5. *If $\langle \mathcal{S}; \mathfrak{L} \rangle$ is representable, then $\langle \mathcal{S}; \mathfrak{L} + \mathfrak{C}_1 \rangle$ is representable.*

COROLLARY 1. *If $\langle \mathcal{S}; \mathfrak{C}_k \rangle$ is representable, then $\langle \mathcal{S}; \mathfrak{C}_n \rangle$ is representable for any $n \geqq k$.*

COROLLARY 2. *If every member of $\mathcal{S}$ is right cancellative or is*

a right zero and $n \geqq 2$, *then* $\langle \mathfrak{S}, \mathfrak{C}_n \rangle$ *is representable.*

See [3], or see the remarks after Lemma 1.

THEOREM 6. *If* $\langle \mathfrak{S}_0, \mathfrak{L}_0 \rangle$ *is representable and* $\mathfrak{L}_1$ *is any algebraic lattice, then both* $\langle \mathfrak{S}_0, \mathfrak{L}_0 + \mathfrak{L}_1 + \mathfrak{C}_1 \rangle$ *and* $\langle \mathfrak{S}_0, \mathfrak{L}_1 + \mathfrak{L}_0 + \mathfrak{C}_1 \rangle$ *are representable.*

THEOREM 7. *If* $\langle \mathfrak{S}_0, \mathfrak{L}_0 \rangle$ *is representable and* $\mathfrak{L}_1$ *is any algebraic lattice, then* $\langle \mathfrak{S}_0, (\mathfrak{L}_0 \times \mathfrak{L}_1) + \mathfrak{C}_1 \rangle$ *is representable.*

This is a special case of Theorem 8.

THEOREM 8. *If* $\langle \mathfrak{S}_0, \mathfrak{L}_0 \rangle$ *and* $\langle \mathfrak{S}_1, \mathfrak{L}_1 \rangle$ *are representable, then* $\langle \mathfrak{S}_0 \times \mathfrak{S}_1, (\mathfrak{L}_0 \times \mathfrak{L}_1) + \mathfrak{S}_1 \rangle$ *is representable.*

Note that each of the "$+ \mathfrak{C}_1$"'s gives us a nonzero element in the resulting lattice that has property (*). (See §1.)

In Theorem 6 one can easily do without the "$+\mathfrak{C}_1$" in the first pair (i.e., one can show $\langle \mathfrak{S}_0, \mathfrak{L}_0 + \mathfrak{L}_1 \rangle$ is representable) in case $\mathfrak{L}_1$ already had a non-zero element satisfying property (*). A similar comment can be made for the other pair in Theorem 6 in case $\mathfrak{L}_0$ already had a non-zero element satisfying (*). To do the same for Theorem 7 or 8 would seem to require that both $\mathfrak{L}_0$ and $\mathfrak{L}_1$ have such an element.

Proof of Theorem 5. Let $\mathfrak{A}$ represent $\langle \mathfrak{S}; \mathfrak{L} \rangle$. Let $U = \{u, v\}$ be a two element set disjoint from A. Set $A' = A \cup U$. Extend each $f \in F$ to A' by setting $f(x_0, \cdots, x_{n-1}) = u$ if there is an $x_i \in U$. Let u, v each be the value of a nullary operation. Define a unary operation p and a binary operation g as follows:

$$p(x) = \begin{cases} x, & \text{if } x \in A\ ; \\ v, & \text{if } x = u\ ; \\ u, & \text{if } x = v\ ; \end{cases}$$

$$g(x, y) = \begin{cases} x, & \text{if } x, y \in A \text{ or if } y = u\ ; \\ y, & \text{if } x = u\ ; \\ v, & \text{if } x \text{ or } y = v\ . \end{cases}$$

Let $\mathfrak{A}' = \langle A'; F \cup \{p, g, u, v\} \rangle$. For each $\varphi \in E(\mathfrak{A})$ define $\bar{\varphi}$ on A' by $x\bar{\varphi} = x\varphi$ if $x \in A$ and $x\bar{\varphi} = x$ if $x \in U$. For each $\Theta \in \mathscr{C}(\mathfrak{A})$ set $\bar{\Theta} = \Theta \cup \omega_U$.

$\varphi \to \bar{\varphi}$ is an isomorphism from $\mathfrak{E}(\mathfrak{A})$ onto $\mathfrak{E}(\mathfrak{A}')$. $\Theta \to \bar{\Theta}$ is an

embedding of $\mathfrak{C}(\mathfrak{A})$ into $\mathfrak{C}(\mathfrak{A}')$. $\mathscr{C}(\mathfrak{A}') = \{\bar{\Theta} \mid \Theta \in \mathscr{C}(\mathfrak{A})\} \cup \{\iota_{A'}\}$. The details are almost identical to the details in [3].

Proof of Theorem 8. Let $\mathfrak{A}_0 = \langle A_0; F_0\rangle$ and $\mathfrak{A}_1 = \langle A_1; F_1\rangle$ be algebras with $\mathfrak{E}(\mathfrak{A}_i)$ isomorphic to $\mathfrak{S}_i$ and $\mathfrak{C}(\mathfrak{A}_i)$ isomorphic to $\mathfrak{L}_i$. Assume $A_0 \cap A_1 = \varnothing$. Let $A_2 = A_0 \cup A_1 \cup \{u, v\}$ where $u \neq v$ and $u, v \notin A_0 \cup A_1$. Let $x_0, \cdots, x_{n-1} \in A_2$, and let $f \in F_i$. Extend f to A_2 by setting $f(x_0, \cdots, x_{n-1}) = u$ if there exists $x_j \notin A_i$. Let $a_i \in A_i$ and define two unary operations g_0 and g_1 by $g_0(a_0) = g_0(u) = u$ and $g_0(a_1) = g_0(v) = v$ and $g_1(a_i) = a_i$ and $g_1(u) = v$ and $g_1(v) = u$. Define a binary g_2 on A_2 by setting $g_2(x, v) = g_2(v, x) = x$ and $g_2(x, y) = u$ otherwise. Take each of u and v as the value of a nullary operation. Let

$$\mathfrak{A}_2 = \langle A_2; F_0 \dot{\cup} F_1 \dot{\cup} \{g_0, g_1, g_2, u, v\}\rangle = \langle A_2; F_2\rangle .$$

For each $\psi = \langle \varphi_0, \varphi_1\rangle \in E(\mathfrak{A}_0) \times E(\mathfrak{A}_1)$ define a mapping $\bar{\psi}$ on A_2 by $x\bar{\psi} = x\varphi_i$ if $x \in A_i$ and $x\bar{\psi} = x$ if $x = u$ or $x = v$. For each $\Phi = \langle \Theta_0, \Theta_1\rangle \in \mathscr{C}(\mathfrak{A}_0) \times \mathscr{C}(\mathfrak{A}_1)$ set $\bar{\Phi} = \Theta_0 \cup \Theta_1 \cup \omega_U$. To complete the proof, one would show that $\mathscr{C}(\mathfrak{A}_2) = \{\bar{\Phi} \mid \bar{\Phi} \in \mathscr{C}(\mathfrak{A}_0) \times \mathscr{C}(\mathfrak{A}_1)\} \cup \{\iota_{A_2}\}$, that $\Phi \to \bar{\Phi}$ is an embedding of $\mathfrak{C}(\mathfrak{A}_0) \times \mathfrak{C}(\mathfrak{A}_1)$ into $\mathfrak{C}(\mathfrak{A}_2)$, and that $\psi \to \bar{\psi}$ is an isomorphism of $\mathfrak{E}(\mathfrak{A}_0) \times \mathfrak{E}(\mathfrak{A}_1)$ onto $\mathfrak{E}(\mathfrak{A}_2)$. A few of the details follow.

Let σ be an endomorphism of $\mathfrak{A}$. Note that $x\sigma = x$ for $x = u$ or $x = v$ since u and v are the vaules of nullary operations. Let $a_i \in A_i$. Now $g_0(a_0\sigma) = g_0(a_0)\sigma = u\sigma = u$. Thus, $a_0\sigma \in A_0$ or $a_0\sigma = u$. Suppose $a_0\sigma = u$. Then $u = a_0\sigma = g_1(a_0)\sigma = g_1(a_0\sigma) = g_1(u) = v$. Since $u \neq v$, it follows that $a_0\sigma \in A_0$. Similarly, $a_1\sigma \in A_1$. Thus, $\sigma = \overline{\langle \sigma|_{A_0}, \sigma_{A_1}\rangle}$.

Suppose $a_i \in A_i$ and $a_0 \equiv a_1(\Psi)$ and suppose $\Psi \in \mathscr{C}(\mathfrak{A}_2)$. Then $u \equiv v(\Psi)$ since $u = g_0(a_0)$, $v = g_0(a_1)$ and $g_0(a_0) \equiv g_0(a_1)(\Psi)$. For $x \in A_0 \cup A_1$ it happens that $x \equiv u(\psi)$ iff $x \equiv v$ because $g_1(x) = x$, $g_1(u) = v$ and $g_1(v) = u$. So if $x \in A_0 \cup A_1$ and $y \in \{u, v\}$ and $x \equiv y(\Psi)$, then $u \equiv v(\Psi)$. If $u \equiv v(\Psi)$, then $\Psi = \iota_{A_2}$ because for any $x \in A_2$, $x \equiv u(\Psi)$. (This is because $u = g_2(u, x)$ and $x = g_2(v, x)$.) $\Psi|_{A_i} \in \mathscr{C}(\mathfrak{A}_i)$. Thus, if $\Psi \neq \iota_{A_2}$, then $\Psi = \bar{\Phi}$ for some $\bar{\Phi} \in \mathscr{C}(\mathfrak{A}_0) \times \mathscr{C}(\mathfrak{A}_1)$, namely, for $\bar{\Phi} = \langle \Psi|_{A_0}, \Psi|_{A_1}\rangle$.

Proof (of Theorem 6). Let $\mathfrak{A}_i = \langle A_i; F_i\rangle$, for $i = 0, 1$, be algebras with $A_0 \cap A_1 = \varnothing$. We shall prove the theorem by showing that $\langle \mathfrak{E}(\mathfrak{A}_0), \mathfrak{C}(\mathfrak{A}_0) + \mathfrak{C}(\mathfrak{A}_1) + \mathfrak{C}_1\rangle$ and $\langle \mathfrak{E}(\mathfrak{A}_1), \mathfrak{E}(\mathfrak{A}_0) + \mathfrak{C}(\mathfrak{A}_1) + \mathfrak{C}_1\rangle$ are representable. First we consider the case with $\mathfrak{E}(\mathfrak{A}_0)$.

Let $u \neq v$ and $u, v \notin A_0 \cup A_1$, and let $A_2 = A_0 \dot{\cup} A_1 \dot{\cup} \{u, v\}$. For $f \in F_i$ extend f to A_2 as in Theorem 8. Define the unary operations g_0, g_1 as in the proof of Theorem 8. Define the binary operation g_2 by setting $g_2(a_0, v) = g_2(v, a_0) = a_0$ for $a_0 \in A_0$ and $g_2(x, y) = u$ otherwise. Let $a_0 \in A_0$ and $a_1, b_1 \in A_1$. Define the binary operation g_3 by $g_3(a_1, b_1) =$

v, if $a_1 \neq b_1$, and $g_3(x, y) = u$ otherwise. Define the binary operation g_4 by $g_4(a_1, y) = y$ and $g_4(x, y) = u$ otherwise. Take each member of $A_1 \cup \{u, v\}$ as the value of nullary operations. Set

$$\mathfrak{A}_2 = \langle A_2; F_0 \dot{\cup} F_1 \dot{\cup} \{g_0, \cdots, g_4\} \dot{\cup} \{u, v\} \dot{\cup} A_1 \rangle = \langle A_2; F_2 \rangle .$$

For each $\varphi \in E(\mathfrak{A}_0)$ let $\bar{\varphi}$ be defined by $x\bar{\varphi} = x\varphi$ if $x \in A_0$ and $x\bar{\varphi} = x$ if $x \notin A_0$. For each $\Theta \in \mathscr{C}(\mathfrak{U}_0) + \mathscr{C}(\mathfrak{A}_1)$ define $\bar{\Theta}$ by $\bar{\Theta} = \Theta \cup \omega_{A_1} \cup \omega_{\{u,v\}}$ if $\Theta \in \mathscr{C}(\mathfrak{A}_0)$ and $\bar{\Theta} = \Theta \cup \iota_{A_0 \cup \{u,v\}}$ if $\Theta \mathscr{C}(\mathfrak{A}_1)$.

To complete the proof for this pair, one would show that $\varphi \to \bar{\varphi}$ is an isomorphism of $\mathfrak{E}(\mathfrak{A}_0)$ onto $\mathfrak{E}(\mathfrak{A}_2)$, that $\Theta \to \bar{\Theta}$ is an embedding of $\mathfrak{C}(\mathfrak{A}_0) + \mathfrak{C}(\mathfrak{A}_1)$ into $\mathfrak{C}(\mathfrak{A}_2)$, and that $\mathscr{C}(\mathfrak{A}_2) = \{\bar{\Theta} \mid \Theta \in \mathscr{C}(\mathfrak{A}_0) + \mathscr{C}(\mathfrak{A}_1)\} \cup \{\iota_{A_2}\}$. A few of the details follow.

As in the proof of Theorem 8, for $\sigma \in E(\mathfrak{A}_2)$, $A_0\sigma \subseteq A_0$ Clearly $x\sigma = x$ for $x \in A_2 - A_0$ since every element is a nullary constant.

Let $\Theta \in \mathscr{C}(\mathfrak{A}_2)$. For $x \in A_2$, $x \equiv u$ iff $x \equiv v$ as in the proof of Theorem 8. Let $a_i, b_i \in A_i$. If $u \equiv v(\Theta)$, then $a_0 \equiv u$ because $g_2(u, a_0) = u$ and $g_2(v, a_0) = a_0$. If $a_1 \neq b_1$ and $a_1 \equiv b_1(\Theta)$, then $u \equiv v$ because $g_3(a_1, b_1) = u$ and $g_3(b_1, b_1) = v$. Let $x \in A_2 - A_1$ and let $z \in A_2$. If $a_1 \equiv x(\Theta)$, then $z \equiv u(\Theta)$ because $g_4(a_1, z) = z$ and $g_4(x, z) = u$.

We now turn to considering the case for $\langle \mathfrak{E}(\mathfrak{A}_1), \mathfrak{C}(\mathfrak{A}_0) + \mathfrak{C}(\mathfrak{A}_1) + \mathfrak{C}_1 \rangle$. We may *now* assume without loss of generality that $\mathfrak{E}(\mathfrak{A}_0)$ is the one element group and that there are no nullary operations in $\mathfrak{A}_0$. That this assumption can be made is verified in [6] and [7].

Let $w, r, s \notin A_2$. Let $A_3 = A_2 \cup \{w, r, s\}$. For $f \in F_0$ or F_1, change the value of $f(x_0, \cdots, x_{n-1})$ to w where in the above case it was u, i.e., in the case when there is an x_i not an element of the appropriate A_i. Extend the g_i in the following way: $g_0(r) = g_0(s) = v$; $g_0(w) = w$; $g_1(w) = w$; $g_1(r) = r$; $g_1(s) = s$; still keep $g_4(a_1, y) = y$, but let $g_4(x, y) = w$ otherwise; keep $g_3(a_1, b_1) = v$ for $a_1 \neq b_1$ and $g_3(x, y) = u$ otherwise except let $g_3(w, w) = w$; $g_2(w, q) = g_2(v, w) = g_2(w, w) = w$ and $g_2(x, y) = u$ for any other new pair. Define three new operations as follows. Let $x, y \in A_0$, $z \in A_0 \cup \{u, v, w\}$ and let $a_1 \in A_1$. Set $g_5(w, w) = w$, $g_5(w, x) = u$, $g_5(y, x) = v$, $g_5(z_1, z) = a_1$ and $g_5(z, a_1) = z$. Set $g_6(r) = s$, $g_6(s) = r$ and $g_6(x) = x$ otherwise. For $x, y \in A_3$ set $g_7(r, x) = g_7(x, r) = r$ and $g_7(x, s) = g_7(s, x) = x$, if $x \neq r$, and $g_7(x, y) = x$ otherwise. Take w, r, s as values of nullary operations but don't take $A_1 \cup \{u, v\}$ as nullaries. Set

$$\mathfrak{A}_3 = \langle A_3; F_0 \cup F_1 \dot{\cup} \{g_0, \cdots, g_7\} \dot{\cup} \{w, r, s\} \rangle .$$

For $\varphi \in E(\mathfrak{A}_1)$ define $\bar{\varphi}$ on A_3 as follows; $x\bar{\varphi} = x\varphi$ if $x \in A_1$, $x\bar{\varphi} = x$ if $x = w, r,$ or s, and for $x \in A_0 \cup \{u, v, w\}$ set $x\bar{\varphi} = x$ if φ is $1 - 1$ and $x\bar{\varphi} = w$ if not. Let $\Theta \in \mathscr{C}(\mathfrak{A}_0) + \mathscr{C}(\mathfrak{A}_1)$. For $\Theta \in \mathscr{C}(\mathfrak{A}_0)$ set $\bar{\Theta} = \Theta \cup \omega_{A_1 \cup \{u,v,w,r,s\}}$, and for $\Theta \in \mathscr{C}(\mathfrak{A}_1)$ set $\bar{\Theta} = \Theta \cup \iota_{A_0 \cup \{u,v,w\}} \cup \omega_{\{r,s\}}$. The

outline of the rest of the proof is clear. Some details follow, particularly concerning endomorphisms.

Let $\Theta \in \mathscr{C}(\mathfrak{A}_3)$. All the statements made about Θ in the previous case still hold with one change. Here if $a_1 \in A_1$ and $x \in A_3 - A_1$ and $x \equiv a_1$, then for all $z \in A_3$, $z \equiv w$ (instead of u). Some more should now be said. If $x \in A_0$ and $w \equiv x(\Theta)$, then $u \equiv v$ because $g_5(w, x) = u$ and $g_5(x, x) = v$. If $u \equiv v(\Theta)$, then one gets $w \equiv u$ using g_2. Similar to the case with u and v, for any $x \in A_3$, $x \equiv r$ iff $x \equiv r \equiv s$(use g_6). Using g_7 we have that if $r \equiv s(\Theta)$ and $z \in A_3$, then $z \equiv r(\Theta)$.

Note that there can be no constant endomorphisms because there are three nullary operations with different values. Let $\sigma \in E(\mathfrak{A}_3)$. Let $x \in A_0 \cup A_1 \cup \{w\}$, and let $y \in A_0 \cup A_1 \cup \{u, v, w\}$. Using g_1, $x\sigma \neq u$ or v, and using g_6, $y\sigma \notin \{r, s\}$. Thus, $(A_0 \cup A_1 \cup \{w\})\sigma \subseteq A_0 \cup A_1 \cup \{w\}$. Let $a_i \in A_i$. Now $w\sigma = w$. If $a_1\sigma = w$, then we would have σ is a constant endomorphism because the congruence relation induced by σ would be ι_{A_3}. So $a_1\sigma \in A_0 \cup A_1$. Now, as before, $A_1\sigma \subseteq A_1$. Similarly, one gets $(A_0 \cup \{w\})\sigma \subseteq A_0 \cup \{w\}$. Using the congruence structure and the fact that $w\sigma = w$, either $A_0\sigma \subseteq A_0$ or $(A_0 \cup \{u, v, w\})\sigma = \{w\}$. Clearly, if $A_0\sigma \subseteq A_0$, then $a_0\sigma = a_0$. When $A_0\sigma \subseteq A_0$, using the congruence structure and g_2, one gets $u\sigma = u$ and $v\sigma = v$. Finally, the congruence structure requires that if σ is not $1-1$ on A_1, then σ must be constant on $A_0 \cup \{u, v, w\}$. And if σ is constant on $A_0 \cup \{u, v, w\}$, then σ would have the value w there.

6. Concerning $\langle \mathfrak{S}, \mathfrak{C}_3 \rangle$. From §4 we know that a necessary condition for the representability of $\langle \mathfrak{S}, \mathfrak{C}_3 \rangle$ is that $|\{\varepsilon_{\rho_s} | s \in S\} \cup \{\iota\}| \leqq 3$.

A stronger condition is proved to be sufficient. The representability of $\langle \mathfrak{S}, \mathfrak{C}_2 \rangle$ has been characterized [3] (or see the remarks after Lemma 1), and $\langle \mathfrak{S}, \mathfrak{C}_2 \rangle$ is representable iff $|\{\varepsilon_{\rho_s} | s \in S\} \cup \{\iota\}| \leqq 2$.

The method for proving the next lemma is very similar to that in Lemma 1. Recall the definition of ε_φ.

LEMMA 6. *Let $\mathfrak{A} = \langle A; F \rangle$ be an algebra, and let $\Theta \neq \omega, \Theta \in \mathscr{C}(\mathfrak{A})$. There is an algebra $\mathfrak{A}' = \langle A; F' \rangle$ so that:*

(i) $\mathscr{S}(\mathfrak{A}) = \mathscr{S}(\mathfrak{A}')$;

(ii) $\mathscr{C}(\mathfrak{A}') = \{\omega, \Theta, \iota\}$;

(iii) $\varphi \in E(\mathfrak{A}')$ *iff* $\varphi \in E(\mathfrak{A})$, $\varepsilon_\varphi = \omega, \Theta$, *or* ι *and following conditions are satisfied:*

(a) *if* $\varepsilon_\varphi = \Theta$, *then* $\varepsilon_{\varphi^2} = \Theta$ *or* ι;

(b) *if* $\varepsilon_\varphi = \omega$ *and* ψ *is any map with* $\varepsilon_\psi = \Theta$, *then* $\varepsilon_{\varphi\circ\psi} = \Theta$.

REMARK. Obviously, one could not improve upon condition (a), but perhaps a proof could be given with (b) changed to read" $\cdots$, then $\varepsilon_{\varphi\circ\psi} = \omega, \Theta$ or ι." Notice that all automorphisms are kept.

Proof. Add one 4-ary operation g defined as follows:

$$g(x, y, u, v) = \begin{cases} u, \text{ if } x \not\equiv y(\Theta), u \not\equiv v(\Theta) \text{ or} \\ \quad \text{if } x \equiv y(\Theta), u \equiv v(\Theta) \text{ and} \\ \quad\quad x \neq y, u \neq v \\ v, \text{ otherwise .} \end{cases}$$

Set $\mathfrak{A}' = \langle A; F \cup \{g\}\rangle$. Clearly, (i) holds.

Proving that $\Theta \in \mathscr{C}(\mathfrak{A}')$ involves only routine calculation. So let $\Phi \in \mathscr{C}(\mathfrak{A}')$ with $\omega \neq \Phi$. So there exist x, y with $x \neq y$ and $x \equiv y(\Phi)$. Suppose $x \not\equiv y(\Theta)$. We will show that $\Phi = \iota$. Let $u \neq v$. First assume $u \not\equiv v(\Theta)$. Then $u = g(x, y, u, v) \equiv g(y, y, u, v) = v(\Phi)$. Now assume $u \equiv v(\Theta)$. Since $x \not\equiv y(\Theta)$, there is a $z \in \{x, y\}$ with $z \not\equiv u(\Theta)$ and $z \not\equiv v(\Theta)$. From above $u \equiv z(\Phi)$ and $v \equiv z(\Phi)$. Thus, $u \equiv v(\Phi)$. So $\Phi = \iota$. Now suppose for every u, v, with $u \equiv v(\Phi)$ that $u \equiv v(\Theta)$. Thus, $\Phi \subseteqq \Theta$. (We are still assuming $\Phi \neq \omega$, that $x \neq y$, and that $x \equiv y(\Phi)$.) We will show that in this case $\Phi = \Theta$. Let $u \equiv v(\Theta)$ with $u \neq v$. Then $u = g(x, y, u, v) \equiv g(y, y, u, v) = v(\Phi)$. So $\Theta \subseteqq \Phi$ and $\Theta = \Phi$. Thus $\mathscr{C}(\mathfrak{A}') = \{\omega, \Theta, \iota\}$ and (ii) holds.

Obviously, if $\varphi \in E(\mathfrak{A}')$, then $\varphi \in E(\mathfrak{A})$ and $\varepsilon_\varphi \in \{\varphi, \Theta, \iota\}$. Suppose $\varepsilon_\varphi = \Theta$. Since $\varphi^2 \in E(\mathfrak{A}')$, then $\varepsilon_{\varphi^2} = \Theta$ or ι.

It is a routine calculation to show that if $\varepsilon_\varphi = \iota$ and $\varphi \in E(\mathfrak{A})$, then $\varphi \in E(\mathfrak{A}')$.

Let $\varphi \in E(\mathfrak{A})$ with $\varepsilon_\varphi = \Theta$ and with $\varepsilon_{\varphi_2} = \Theta$ or ι. Consider $g(x, y, u, v)$. There are two possibly troublesome cases. One is if $g(x, y, u, v) = u$ and $g(x\varphi, y\varphi, u\varphi, v\varphi) = v\varphi$. The other is if $g(x, y, u, v) = v$ and $g(x\varphi, y\varphi, u\varphi, v\varphi) = u\varphi$. The latter is the easiest to dispense with. If $g(x\varphi, y\varphi, u\varphi, v\varphi) = u\varphi$ and $u\varphi \neq v\varphi$, then $x\varphi \neq y\varphi$. Thus, $x \not\equiv y(\Theta)$ and $u \not\equiv v(\Theta)$. So $g(x, y, u, v) = u$ and $g(x, y, u, v)\varphi = u\varphi$. So now assume $g(x\varphi, y\varphi, u\varphi, v\varphi) = v\varphi$ and $g(x, y, u, v) = u$. Thus, either $x \not\equiv y(\Theta)$ and $u \not\equiv v(\Theta)$ or $x \equiv y(\Theta)$ and $u \equiv v(\Theta)$. Suppose $x \not\equiv y$ and $u \not\equiv v$. Then $x\varphi \neq y\varphi$ and $u\varphi \neq v\varphi$. Now $x\varphi \equiv y\varphi(\Theta)$ iff $u\varphi \equiv v\varphi(\Theta)$. Indeed, suppose that $x\varphi \equiv y\varphi(\Theta)$. Then $(x\varphi)\varphi = (y\varphi)\varphi$ and $\varepsilon_{\varphi^2} \neq \Theta$. So by assumption $\varepsilon_{\varphi^2} = \iota$. Thus, $(u\varphi)\varphi = (v\varphi)\varphi$, and $u\varphi \equiv v\varphi(\Theta)$. Similarly if $u\varphi \equiv v\varphi$, then $x\varphi \equiv y\varphi$. Thus, either $x\varphi \not\equiv y\varphi(\Theta)$, $u\varphi \not\equiv v\varphi(\Theta)$ or $x\varphi \equiv y\varphi(\Theta)$, $u\varphi \equiv v\varphi(\Theta)$, $x\varphi \neq y\varphi$, $u\varphi \neq v\varphi$. In any case, $g(x\varphi, y\varphi, u\varphi, v\varphi) = u\varphi \not\equiv v\varphi$. So $x \equiv y(\Theta)$ and $u \equiv v(\Theta)$. In this case $u\varphi = v\varphi$. Therefore, $g(x, y, u, v)\varphi = u\varphi = v\varphi = g(x\varphi, y\varphi, u\varphi, v\varphi)$. Thus, $\varphi \in E(\mathfrak{A}')$.

Suppose φ is $1 - 1$ and $\varphi \in E(\mathfrak{A}')$. Let ψ be any map with $\varepsilon_\psi = \Theta$. Consider $\varepsilon_{\varphi\circ\psi}$. Suppose $\varepsilon_{\varphi\circ\psi} = \omega$. Then, $\varepsilon_\psi \neq \iota$. So there exist x, y such that $x \not\equiv y(\Theta)$. Since $\Theta \neq \omega$, there exist u, v with $u \neq v$ and $u \equiv v(\Theta)$. Since $u \neq v$, it follows that $(u\varphi)\psi \neq (v\varphi)\psi$. Thus, $u\varphi \not\equiv v\varphi(\Theta)$. Similarly, since $x \neq y$, $x\varphi \not\equiv y\varphi(\Theta)$. This implies that

$g(x, y, u, v)\varphi = v\varphi \neq u\varphi = g(x\varphi, y\varphi, u\varphi, v\varphi)$. But since φ is an endomorphism, we have that $\varepsilon_{\varphi\circ\psi} \neq \omega$. Suppose $\varepsilon_{\varphi\circ\psi} = \iota$. By a similar argument we would get that $\varepsilon_{\varphi\circ\psi} \neq \iota$ unless $\Theta = \iota$. So $\varepsilon_{\varphi\circ\psi} = \Theta$.

Let $\varphi \in E(\mathfrak{A})$ with $\varepsilon_\varphi = \omega$. Let ψ be any map with $\varepsilon_\psi = \Theta$.

Suppose $\varepsilon_{\varphi\circ\psi} = \Theta$. Routine computation shows that $\varphi \in E(\mathfrak{A}')$. The crucial point in these computations is that the assumption $\varepsilon_{\varphi\circ\psi} = \Theta$ implies φ presves both Θ and not-Θ. Therefore (iii) holds.

Recall that if $\mathfrak{S} = \langle S; \cdot\rangle$ is a semigroup with identity, then $\mathscr{K} = \{\varepsilon_{\rho_s} | s \in S\}$. $\mathfrak{L}(\mathfrak{S})$ is the algebra of left multiplications. $E(\mathfrak{L}(\mathfrak{S})) = \{\rho_s | s \in S\}$.

THEOREM 9. *Let $\mathfrak{S} = \langle S; \cdot\rangle$ be a semigroup with identity.*

(A) *If $\langle\mathfrak{S}; \mathfrak{C}_3\rangle$ is representable, then $|\mathscr{K} \cup \{\iota\}| \leqq 3$.*

(B) *If $|\mathscr{K} \cup \{\iota\}| \leqq 3$ and if for right cancellative r and for m that is neither right cancellative nor a right zero $r\cdot m$ is also neither right cancellative nor a right zero, then $\langle\mathfrak{S}; \mathfrak{C}_3\rangle$ is representable.*

REMARK. If $|\mathscr{K} \cup \{\iota\}| = 2$, the rest of (B) holds trivially. So the sufficient condition includes all those representable pairs derived from Corollary 2 to Theorem 5.

Proof. For part (A) see the corollary to Theorem 3.

Suppose the hypotheses of (B) hold. If $|\mathscr{K} \cup \{\iota\}| = 2$, then $\langle\mathfrak{S}; \mathfrak{C}_2\rangle$ is representable. By Theorem 5, $\langle\mathfrak{S}; \mathfrak{C}_3\rangle$ is representable. Suppose then that $\mathscr{K} \cup \{\iota\} = \{\omega, \Theta, \iota\}$ and that $\omega \neq \Theta \neq \iota$. Suppose $\varepsilon_{\rho_m} = \Theta$. Since

$$\varepsilon_{(\rho_m)^2} = \varepsilon_{\rho_{m^2}},$$

it follows that

$$\varepsilon_{(\rho_m)^2} = \Theta \text{ or } \iota.$$

Let $\varepsilon_{\rho_r} = \omega$ and $\varepsilon_{\rho_m} = \Theta$. Since $r\cdot m$ is neither right cancellative nor a right zero, it follows that

$$\varepsilon_{\rho_r\circ\rho_m} = \varepsilon_{\rho_{rm}} = \Theta.$$

Now apply Lemma 6 to Θ and $\mathfrak{L}(\mathfrak{S})$.

7. $\langle\mathfrak{S}, \mathfrak{C}_2\rangle$ **for unary algebras.** In [3] G. Grätzer characterized the endomorphism semigroups of simple algebras. He also showed that not all such semigroups were isomorphic to endomorphism semigroups of simple unary algebras. Since previous representations involving congruence lattices and endomorphism semigroups had needed

only unary algebras, he raised the question, "What semigroups are isomorphic to the endomorphism semigroups of simple unary algebras?" The answer to that question is that there are hardly any such semigroups.

Every endomorphism φ induces a congruence relation which we have denoted by ε_φ. The difference with unary algebras is that every endomorphism also induces another congruence. Throughout $\mathfrak{A} = \langle A; F\rangle$ will denote a unary algebra. For $\varphi \in E(\mathfrak{A})$ and $x, y \in A$ set $x \equiv y(\Theta_\varphi)$ iff there exist natural numbers i, j such that $x\varphi^i = y\varphi^j (x\varphi^0 = x)$. Θ_φ is the "extra" congruence. To prove that the substitution property holds for Θ_φ, one needs that each operation of $\mathfrak{A}$ is unary or nullary.

LEMMA 7. *If φ is $1-1$ and $\Theta_\varphi = \omega$ or ι, then φ is onto or $A = \{a\varphi^n \mid n = 0, 1\cdots\}$ for some $a \in A$.*

Proof. $x \equiv x\varphi^n(\Theta_\varphi)$ for any natural number n (by using the numbers $n, 0$). In particular $x \equiv x\varphi(\Theta_\varphi)$. Thus, if $\Theta_\varphi = \omega$, then $x = x\varphi$, and therefore, φ is the identity map. Therefore, we can assume $\Theta_\varphi = \iota$, and this implies $x \equiv y(\Theta_\varphi)$ for any $x, y \in A$. Thus, for some i, j, $x\varphi^i = y\varphi^j$. If $i \leqq j$, then since φ is $1-1$, we have that $x = y\varphi^{i-j}$. If $j \leqq i$, then $y = x\varphi^{i-j}$. Thus, $x \in \{y\varphi^n \mid n = 0, 1, \cdots\}$ or $y \in \{x\varphi^n \mid n = 0, 1, \cdots\}$. Suppose φ is not onto. Then there is an a such that $a \neq x\varphi$ for all $x \in A (x \neq a)$. Thus, $a \notin \{x\varphi \mid n = 0, 1, \cdots\}$ for any $x \in A (x \neq a)$. Now since $x \in \{a\varphi^n \mid n = 0, 1, \cdots\}$ or $a \in \{x\varphi^n \mid n = 0, 1, \cdots\}$ for all $x \in A$, we have $x \in \{a\varphi^n \mid n = 0, 1, \cdots\}$ for all $x \in A$.

LEMMA 8. *If $\Theta_\varphi = \omega$ or ι and Θ is $1-1$ but not onto, then $\mathfrak{A}$ is not simple.*

Proof. By Lemma 7, $A = \{a\varphi^n \mid n = 0, 1\cdots\}$ for some $a \in A$. For $n > 1$, $a\varphi^n = (a\varphi^{n-1})\varphi$. Since φ is not onto $a \neq x\varphi$ for any $x \in A$. Suppose $a\varphi^i = a\varphi^j$ and $i \neq j$. We may assume $i < j$. Since φ is $1-1$, $a = a\varphi^{j-i}$. Since $j - i \geqq 1$, $a = (a\varphi^{j-i-1})\varphi$. Thus, $a\varphi^i \neq a\varphi^j$ if $i \neq j$. Now set $E = \{a\varphi^n \mid n = 0, 2, 4, \cdots\}$ and $D = \{a\varphi^n \mid n = 1, 3, 5, \cdots\}$. By the above, $D \cap E = \varnothing$. Clearly, $D \cup E = A$. Let Φ be the equivalence relation whose only two classes are D and E. Φ is a congruence. Since $\omega \neq \Phi \neq \iota$, $\mathfrak{A}$ is not simple.

For a *simple* algebra $\mathfrak{A}$ any right zero of $\mathfrak{E}(\mathfrak{A})$ is necessarily a constant mapping (unless $\mathfrak{E}(\mathfrak{A})$ is the one element group). See § 4.

COROLLARY. *If $\mathfrak{A}$ is a simple unary algebra, then $E(\mathfrak{A})$ consists of automorphisms and constant mappings.*

G. Grätzer [3] characterized the automorphism group of a simple unary algebra as a cyclic group of order p where $p = 1$ or p is a prime number. He also showed that if $p \neq 1$, then $A = \{a\alpha \mid \alpha \in G(\mathfrak{A})\}$ for any $a \in A$.

LEMMA 9. *If $\mathfrak{A}$ is simple, $|G(\mathfrak{A})| \neq 1$, and there exists a right zero in $\mathfrak{E}(\mathfrak{A})$, then $|A| = 2$ and $\mathfrak{E}(\mathfrak{A}) = \langle A^A; \circ \rangle$.*

Proof. Let $\{a\} = A\varphi$. Let f be an operation. Then $a = (f(a))\varphi = f(a\varphi) = f(a)$. If $x \in A$, then $x = a\alpha$ for some $\alpha \in G(\mathfrak{A})$. Thus, $f(x) = f(a\alpha) = f(a)\alpha = a\alpha = x$. Therefore, $E(\mathfrak{A}) = A^A$ and all equivalence relations are congruence relations. $|G(\mathfrak{A})| \geqq 2$ implies $|A| \geqq 2$. If $|A| > 2$, then there are more than two equivalence relations on A. Thus $|A| = 2$.

LEMMA 10. *If $\mathfrak{A}$ is simple and $|G(\mathfrak{A})| = 1$, then $|E(\mathfrak{A})| \leqq 2$.*

Proof. Suppose there exist two constant endomorphisms φ_0, φ_1. Let $\{a_0\} = A\varphi_0$ and $\{a_1\} = A\varphi_1$. As in the proof of Lemma 3, $f(a_0) = a_0$ and $f(a_1) = a_1$ for any operation f. If $|A|$ were two, then every operation would be the identity function and $|G(\mathfrak{A})| = 2$. Thus, $|A| > 2$. Set $x \equiv y(\Phi)$ iff $x = y$ or $x, y \in \{a_0, a_1\}$. Since every operation restricted to $\{a_0, a_1\}$ is the identity function, Φ is a congruence. Since $|A| > 2$, $\Phi \neq \iota$. Since $\Phi \neq \omega$, $\mathfrak{A}$ is not simple.

THEOREM 10. *Let $\mathfrak{S} = \langle S; \cdot \rangle$ be a semigroup with identity. $\mathfrak{S}$ is isomorphic to the endomorphism semigroup of a simple unary algebra (i.e., $\langle \mathfrak{S}, \mathfrak{C}_2 \rangle$ is representable by a unary algebra) if and only if $\mathfrak{S}$ is one of the semigroups listed below:*

(i) *the group of order p, $p = 1$ or p is a prime;*

(ii) *the two element semi-lattice;*

(iii) *a four element semigroup isomorphic to $\langle A^A; \circ \rangle$ where $|A| = 2$.*

Moreover, if $\langle \mathfrak{S}, \mathfrak{C}_2 \rangle$ is representable by a unary algebra and $|S| \neq 1$, then $\langle \mathfrak{S}, \mathfrak{C}_2 \rangle$ is representable using a unary algebra with one operation.

Proof. It follows from the corollary to Lemma 8 and Lemmas 9 and 10 that the endomorphism semigroup of a simple unary algebra is one of those listed in (i) – (iii).

To complete the proof, we will represent $\langle \mathfrak{S}, \mathfrak{C}_2 \rangle$ for each $\mathfrak{S}$ listed in (i) – (iii).

In case $\mathfrak{S}$ is the one element group, let A be a two element set. Set $\mathfrak{A} = \langle A; A^A \rangle$. Clearly, $\mathfrak{A}$ has the required properties.

In case $\mathfrak{S}$ is $\langle A^A; \circ \rangle$ where $|A| = 2$, Let $\mathfrak{A} = \langle A; F \rangle$ where f is

the identity map. Obviously, $\mathfrak{A}$ has the required properties.

In case $\mathfrak{S}$ is the two element semi-lattice, let $A = \{a, b\}$ with $a \neq b$. Set $f(a) = f(b) = b$, and set $\mathfrak{A} = \langle A; f \rangle$. Since $|A| = 2$, $\mathfrak{A}$ is simple. The endomorphisms are exactly the identity map σ and ψ where $\psi = f$. Since $\sigma \circ \psi = \psi \circ \sigma = \psi = \psi \circ \psi$, the endomorphism semigroup is the two element semi-lattice.

In case $\mathfrak{S}$ is the group of order p where p is a prime, set $A = \{0, \cdots, p-1\}$. Let $f(x) = x + 1 \pmod{p}$, and set $\mathfrak{A} = \langle A; f \rangle$. Since p is a prime it is easy to check that $\mathfrak{A}$ is simple. For $x \in A$ define the mapping φ_x by $y\varphi_x = y + x$. Clearly, $x \to \varphi_x$ is an isomorphism from the cyclic group of order p onto $\mathfrak{E}(\mathfrak{A})$.

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