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SEQUENCES OF QUASI-SUBORDINATE FUNCTIONS

JAMES MILLER

In this paper a theorem is proved which connects bounded analytic functions in the unit disk and sequences of quasi-subordinate functions. As an application a necessary and sufficient condition for certain sequences of quasi-subordinate functions to converge is found.

Let f and F be analytic functions in $|z| < R$. If there exist two functions ϕ and ω which are analytic in $|z| < R$ and satisfy $\omega(0) = 0$, $|\phi(z)| \leqq 1$, $|\omega(z)| < R$, and $f(z) = \phi(z)F(w(z))$ for $|z| < R$, then we say that f is quasi-subordinate to F in $|z| < R$ and write $f \prec_q F$. Without loss of generality we may assume that $R = 1$. This class was introduced by Robertson [2, 3].

We note that there are two special cases of quasi-subordination which are of interest: If ϕ is the constant function one, then f is subordinate to F, and on the other hand, if ω is the identity function, then f is majorized by F.

Let B denote the class of functions θ which are analytic in $|z| < 1$ and satisfy $|\theta(z)| \leqq 1$ for $|z| < 1$. Then the functions ϕ and ω which are defined above are elements of B. In this paper we prove a theorem which connects functions in B and sequences of quasi-subordinate functions. As an application we find necessary and sufficient conditions for certain sequences of quasi-subordinate functions to converge. This is a generalization of Pommerenke's results [1] on sequences of subordinate functions.

Let $\{f_n\}$, $n = 1, 2, \cdots$, be a sequence of functions which are analytic in $|z| < 1$ such that $f_n \prec_q f_{n+1}$ for each n or $f_{n+1} \prec_q f_n$ for each n. When considering the convergence of such sequences we need to require that either the sequence $\{f_n(0)\}$ converges or the functions agree at a single point. In this paper we shall assume that the functions agree at a single point. Further we may assume that the point is $z = 0$ for if the functions f_n agree at the point $a \neq 0$ then we could consider the functions $g_n(z) = f_n((z-a)/(1-az))$. We will use $f_n(0) = 0$ for all n, otherwise the function ϕ would be identically one. The proof for the case where $\{f_n(0)\}$ is convergent is similar.

THEOREM 1. *Let $\{f_n\}$ be a sequence of functions which are analytic in $|z| < 1$ and satisfy $f_n(0) = 0$, $\alpha_n = f'_n(0) \neq 0$, and $f_n(z) \prec_q f_{n+1}$, and let $\phi_{n+1}, \omega_{n+1} \in B$ and $\omega_{n+1}(0) = 0$ be such that*

$$f_n(z) = \phi_{n+1}(z) f_{n+1}(\omega_{n+1}(z))$$

for $|z| < 1$. *If* $\sum_{n=2}^{\infty} \arg \phi_n(0)$ *converges and* $\lim_{n\to\infty} \alpha_n = \alpha$, $|\alpha| < \infty$, *then* $\prod_{n=2}^{\infty} \phi_n(0)$ *converges.*

Proof. We observe that if $m < n$, then we have $f_m \prec_q f_n$. Thus for $m < n$ there are functions $\phi_{mn}, \omega_{mn} \in B$ where $\omega_{mn}(0) = 0$ such that

$$f_m(z) = \phi_{mn}(z) f_n(\omega_{mn}(z))$$

for $|z| < 1$. Let $\phi_{n\,n+1}(z) = \phi_{n+1}(z)$. We now observe that

$$f'_m(0) = \phi_{mn}(0)\omega'_{mn}(0) f'_n(0)$$

or

$$\alpha_m = \phi_{mn}(0)\omega'_{mn}(0)\alpha_n \,. \tag{1}$$

Since $0 < |\alpha_m| \leqq |\alpha_n|$ for $m < n$ and $\alpha_n \to \alpha$, there exists an integer K such that if $n > m > K$, then

$$\left| \frac{\alpha_m}{\alpha_n} - 1 \right| < \varepsilon \,. \tag{2}$$

From (1) and (2) we see that

$$1 - \varepsilon < \left| \frac{\alpha_m}{\alpha_n} \right| = |\phi_{mn}(0)\omega'_{mn}(0)| \leqq |\phi_{mn}(0)| \leqq 1 \,.$$

We now observe that

$$\phi_{mn}(0) = \prod_{k=m+1}^{n} \phi_k(0) \,.$$

Thus we have

$$1 - \varepsilon < \left| \prod_{k=m+1}^{n} \phi_k(0) \right| \leqq 1$$

for $n > m > K$. Since $\sum_{n=2}^{\infty} \arg \phi_n(0)$ converges this says that $\prod_{k=2}^{\infty} \phi_k(0)$ converges. Further we have that $\omega'_n(0) \to 1$ and $\omega'_{mn}(0) = 1$.

In applying Theorem 1 to sequences of quasi-subordinate functions we will also need two lemmas for functions in B. The proofs of the lemmas are essentially in [1].

LEMMA 1. *Let* $\phi \in B$, $\phi(0) = 0$, *and satisfy* $|\phi(0)| \geqq \sigma > 0$. *Then the mapping* $w = \phi(z)$ *maps the disk*

$$|z| < \rho = \frac{\sigma}{1+\sqrt{1-\sigma^2}}$$

univalently onto a region that contains $|w| < \rho^2$.

LEMMA 2. *For* $\varepsilon > 0$ *and* $0 < r < 1$, *there exists an* $\eta > 0$ $(\eta(\varepsilon, r))$ *such that if* $\phi \in B$ *satisfies* $\phi(z) = \sum_{n=0}^{\infty} \beta_n z^n$ *and* $|\beta_k - 1| \leqq \eta$, *then*

$$|\phi(z) - z^k| < \varepsilon\,, \qquad \textit{for } |z| < r\,.$$

THEOREM 2. *Let* $\{f_n\}$ *be a sequence of analytic functions in* $|z| < 1$ *such that* $f_n(0) = 0$, $f_n \prec_q f_{n+1}$, *and* $\alpha_n = f_n'(0) \neq 0$, and let $\phi_{n+1}, \omega_{n+2} \in B$ and $\omega_{n+1}(0) = 0$ be such that $f_n(z) = \phi_{n+1}(z) f_{n+1}(\omega_{n+1}(z))$ for $|z| < 1$ and $\sum_{n=2}^{\infty} \arg \phi_n(0)$ converges. *Then the sequence* $\{f_n\}$ *converges uniformly in* $|z| < r$ *for every* $0 \leqq r < 1$ *if and only if*

$$\lim_{n\to\infty} \alpha_n = \alpha\,, \qquad |\alpha| < \infty\,.$$

PROOF. If $\{f_n\}$ converges uniformly in $|z| \leqq r$ for every $0 < r < 1$ then $\alpha_n = f_n'(0)$ converges. Further since $|\alpha_n| \leqq |\alpha_{n+1}|$, $f_n(0) = 0$, and $\alpha_n \neq 0$ we see that $\lim_{n\to\infty} \alpha_n = \alpha \neq 0$ and $|\alpha| < \infty$.

Let $\omega_{n+1}, \phi_{n+1} \in B$, and $\omega_{n+1}(0) = 0$ be as defined in Theorem 2. Further for $m < n$, let $\phi_{mn}, \omega_{mn} \in B$ with $\omega_{mn}(0) = 0$ be such that

$$f_m(z) = \phi_{mn}(z) f_n(\omega_{mn}(z))\,. \tag{3}$$

Suppose that $\alpha_n \to \alpha$, $|\alpha| < \infty$. Then by Theorem 1 the product $\prod_{k=2}^{\infty} \phi_k(0)$ converges. We will first show that $\{f_n\}$ is a normal family in $|z| < 1$.

Let r, $0 < r < 1$, be fixed and σ determined by

$$\sqrt{r} = \frac{\sigma}{1+\sqrt{1-\sigma^2}}\,.$$

Since $\sigma < 1$ and $\alpha_n \to \alpha \neq 0$, there exists an integer N_1 such that

$$\left|\frac{\alpha_m}{\alpha_n}\right| > \sigma\,, \qquad \text{for } n > m > N_1\,.$$

Further, since $|\phi_{mn}(z)| \leqq 1$, we have $|\phi_{mn}(0)|^{-1} \geqq 1$. For $n > m > N_1$ we have $\omega_{mn}'(0) = \alpha_m/(\alpha_n \phi_{mn}(0))$ or

$$|\omega_{mn}'(0)| = \left|\frac{1}{\phi_{mn}(0)}\frac{\alpha_m}{\alpha_n}\right| > \sigma\,. \tag{4}$$

Thus by Lemma 1 the mapping $\zeta = \omega_{mn}(z)$ for $n < m < N_1$ maps $|z| < \sqrt{r}$ univalently onto a domain that contains $|\zeta| < r$. Let ψ_{mn} be the inverse of $\zeta = \omega_{mn}(z)$ in $|\zeta| < r$, then

$$|\psi_{mn}(\zeta)| \leqq \sqrt{r} .$$

From (3) we may write

$$f_n(\zeta) = \frac{1}{\phi_{mn}(\psi_{mn}(\zeta))} f_m(\psi_{mn}(\zeta)) , \qquad \text{for } |\zeta| < r .$$

For $|\zeta| \leqq r$ we have

$$|f_n(\zeta)| \leqq \max_{|z| \leqq \sqrt{r}} \left| \frac{f_m(z)}{\phi_{mn}(z)} \right| \leqq \frac{1}{\min_{|z| \leqq \sqrt{r}} |\phi_{mn}(z)|} \max_{|z| \leqq \sqrt{r}} |f_m(z)| .$$

From Lemma 2 with $k = 0$, given $\varepsilon > 0$, there exists an η such that if $|\beta_0 - 1| < \eta$ then $|\phi(z) - 1| < \varepsilon$ for $|z| < r$. Since $\prod_{k=2}^{\infty} \phi_k(0)$ converges by Theorem 1 and $\phi_{mn}(0) = \prod_{k=m+1}^{n} \phi_k(0)$, there exists an integer N_2 such that if $n > m > N_2$ then $|\phi_{mn}(0) - 1| < \eta$. Let $N = \max(N_1, N_2)$. Thus, by Lemma 2 we have that $|\phi_{mn}(z) - 1| < \varepsilon$ for $|z| \leqq r$ and $n > m > N$ or

$$\min_{|z| \leqq r} |\phi_{mn}(z)| \geqq 1 - \varepsilon .$$

Hence, for $n > N$ and $|\zeta| \leqq r$ we have

$$|f_n(\zeta)| \leqq \frac{1}{1 - \varepsilon} \max_{|z| \leqq \sqrt{r}} |f_{N+1}(z)| .$$

Thus there exists $M(r)$ such that

$$|f_n(z)| \leqq M(r) \tag{5}$$

for all n, that is, $\{f_n\}$ is locally uniformly bounded. Therefore $\{f_n\}$ is normal.

Let $\{f_{n_\nu}\}$ be a subsequence of $\{f_n\}$ which is uniformly convergent in $|z| \leqq r_0$, for every $r_0 < 1$. Let f be the limit function of $\{f_{n_\nu}\}$. Let $\varepsilon > 0$ and $r < 1$. Then choose ν_0 such that

$$|f_{n_\nu}(z) - f(z)| < \varepsilon/3$$

for $\nu \geqq \nu_0$ and $|z| \leqq r$. From inequality (5) we have that the sequence $\{f_n\}$ is bounded in $|z| \leqq r$ and thus equicontinuous in $|z| \leqq r$. Therefore there exists a $\delta > 0$ such that

$$|f_n(z_1) - f_n(z_2)| < \varepsilon/3$$

for $|z_1 - z_2| < \delta$, $|z_1| \leqq r + \delta$, $|z_2| \leqq r + \delta$, and for all n.

Using (4), the convergence of $\sum_{n=2}^{\infty} \arg \phi_n(0)$, and applying Lemma 2 we have that there exists an integer M_1 such that if $n \geqq m \geqq M_1$, then

$$|\omega_{mn}(z) - z| < \delta , \qquad \text{for } |z| \leqq r$$

where M_1 is chosen so that $|\omega'_{mn}(0) - 1| < \eta$ for a suitable η. Again making use of Lemma 2 we have that there exists an integer M_2 such that if $n > m > M_2$ then

$$|\phi_{mn}(z) - 1| < \varepsilon/3M(r), \quad \text{for} \quad |z| < r .$$

Let $M = \max\{M_1, M_2, n_{\nu_0}\}$. If $M \leqq k < n_\nu$ and $|z| < r$ then

$$\begin{aligned} |f_k(z) - f(z)| &\leqq |f_k(z) - f_{n_\nu}(z)| + |f_{n_\nu}(z) - f(z)| \\ &< \varepsilon/3 + |f_{n_\nu}(z) - \phi_{kn_\nu}(z) f_{n_\nu}(w_{kn_\nu}(z))| \\ &\leqq \varepsilon/3 + |f_{n_\nu}(z) - f_{n_\nu}(\omega_{kn_\nu}(z))| \\ &\quad + |f_{n_\nu}(\omega_{kn_\nu}(z))[1 - \phi_{kn_\nu}(z)]| \\ &< \varepsilon/3 + \varepsilon/3 + M(r)\,\varepsilon/3M(r) = \varepsilon \end{aligned}$$

for $|z| \leqq r$ and $k > M$. This completes the proof of Theorem 2.

THEOREM 3. *Let* $\{f_n\}$ *be a sequence of functions analytic in* $|z| < 1$ *such that* $f_n(0) = 0$, $\alpha_n = f'_n(0) \neq 0$, *and* $f_{n+1} \prec_q f_n$, *and let* $\phi_{n+1}, \omega_{n+1} \in B$ *and* $\omega_{n+1}(0) = 0$ *be such that*

$$f_{n+1}(z) = \phi_{n+1}(z) f_n(\omega_{n+1}(z))$$

for $|z| < 1$ *and* $\sum_{n=2}^{\infty} \arg \phi_n(0)$ *converges. Then the sequence* $\{f_n\}$ *converges uniformly in* $|z| \leqq r$ *for every* $r < 1$ *if the sequence* $\{\alpha_n\}$ *converges. The limit function is constant if and only if*

$$\lim_{n\to\infty} \alpha_n = 0 .$$

The proof of this theorem is similar to that of Theorem 2 and Pommerenke's Theorem 2 [1].

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TEXAS A AND M UNIVERSITY