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BOCHNER'S THEOREM IN INFINITE DIMENSIONS

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1. Introduction. Let G be a locally compact abelian group. A well-known theorem of Bochner ([1], [11]) states that a mapping ψ of G into C is positive definite and continuous if and only if there is a unique nonnegative finite regular Borel measure m_ψ on $\hat{G}$ (the dual group of G) such that $\psi(g) = \int_{\hat{G}} (\gamma, g) dm_\psi$ where (γ, g) denotes the action of the character γ on g. An alternate version of the theorem ([9]) states that if A is a semi-simple, self-adjoint, commutative Banach algebra and ψ is a linear functional on A, then ψ is positive and extendable if and only if there is a finite positive Baire measure ν_ψ on $\mathscr{M}$ (the maximal ideal space of A) such that $\psi(\alpha) = \int_{\mathscr{M}} \hat{\alpha}(M) d\nu_\psi$ where $\hat{\alpha}$ is the Gelfand transform of $\alpha \in A$. Here we shall extend these theorems to mappings takings values in a Banach space X. Our results generalize the extension of Bochner's theorem made in [5].

We shall, in fact, first prove that if A is a self-adjoint, commutative Banach algebra and ψ is a linear map of A into the Banach space X, then ψ is positive[1] and "almost" extendable if and only if there is a weak-*-regular, finite, positive set function ν_ψ^{**} mapping $\Sigma(\mathscr{M})$ (the Borel field of $\mathscr{M}$) into X^{**} such that $\psi(\alpha) = \int_{\mathscr{M}} \hat{\alpha}(M) d\nu_\psi^{**}$ (where $\psi(\alpha)$ is viewed as an element of X^{**}). We next show that if the mapping $\hat{\psi}$ of $\hat{A}$ into X given by $\hat{\psi}(\hat{\alpha}) = \psi(\alpha)$ is weakly compact[2], then ν_ψ^{**} can be viewed as a weakly regular positive vector measure ν_ψ mapping $\Sigma(\mathscr{M})$ into X and, conversely, if

$$\psi(\alpha) = \int_{\mathscr{M}} \hat{\alpha}(M) d\nu_\psi$$

where ν_ψ is a weakly regular positive vector measure on $\Sigma(\mathscr{M})$ to X, then ψ is positive and "almost" extendable and $\hat{\psi}$ is weakly compact. In the case where $A = L_1(G, C)$, these results lead to a representation of ψ by an element p_ψ of $L_\infty(G, X)$ i.e. $\psi(\alpha) = \int_G \alpha(g) p_\psi(g) d\mu$ where μ is the Haar measure on G. We then develop an extended Bochner's theorem for maps p in $L_\infty(G, X)$. Finally, we use some particular Banach spaces to illustrate the theory.

The general results obtained here are combined with the transform theory on $L_1(G, X)$ to develop an inversion theorem and a Plancherel theorem in [7]. These theorems are also applied to the solution of convolution equations in Hilbert spaces in [8]. The convolution

[1] Positivity is with respect to a suitable cone in X.

[2] This means that $\hat{\psi}$ maps bounded sets in A into weakly compact sets in X.

equations arise in the study of problems relating to the stability and control of systems described by parabolic partial differential equations.

2. **Positive functions.** Let X be a Banach space and let X^* and X^{**} be the dual spaces of X and X^*, respectively. If φ is an element of X^*, then the operation of φ on x is denoted by (x, φ). The notion of positivity that we use is based on a cone of "positive" elements contained in X. We assume that the cone is defined by a family of elements of X^*. More precisely, we have

DEFINITION 2.1. Let Φ be a subset of X^*. The subset K_Φ (or simply K when Φ is fixed by the context) of X given by

$$K_\Phi = \{x \in X: (x, \varphi) \geqq 0 \text{ for all } \varphi \text{ in } \Phi\} \tag{2.2}$$

is called the cone determined by Φ.

Now let A be a Banach algebra with an involution given by $\alpha \rightarrow \alpha^*$, $\alpha \in A$, and let ψ be a linear mapping of A into X. We then have

DEFINITION 2.3. The mapping ψ is positive with respect to the cone K_Φ (or Φ-positive) if $\psi(\alpha\alpha^*) \in K_\Phi$ for all α in A.

We observe that ψ is Φ-positive if and only if the mappings $(\psi(\cdot), \varphi)$ of A into C are positive functionals for all φ in Φ. Note also that if ψ is Φ-positive, then, for any φ in Φ, the functional $B_\varphi(\alpha, \beta)$ given by

$$B_\varphi(\alpha, \beta) = (\psi(\alpha\beta^*), \varphi) \tag{2.4}$$

is a symmetric bilinear form satisfying the Cauchy inequality

$$|B_\varphi(\alpha, \beta)|^2 \leqq B_\varphi(\alpha, \alpha) B_\varphi(\beta, \beta) \tag{2.5}$$

for α, β in A.

DEFINITION 2.6. The mapping ψ is symmetric with respect to Φ (or simply symmetric) if $(\psi(\alpha), \varphi) = \overline{(\psi(\alpha^*), \varphi)}$ for all φ in Φ and α in A.

If A has a unit e, then every Φ-positive mapping is symmetric since $(\psi(\alpha), \varphi) = (\psi(\alpha e), \varphi) = B_\varphi(\alpha, e) = \overline{B_\varphi(e, \alpha)} = \overline{(\psi(\alpha^*), \varphi)}$ for all φ. If A does not have a unit, then A can be imbedded in an algebra $\tilde{A} = A \oplus C$ with a unit in a natural way. Letting e be the unit in $\tilde{A}$, we can extend ψ to a linear mapping $\tilde{\psi}_{x_0}$ of $\tilde{A}$ into X by setting

$\tilde{\psi}_{x_0}(\alpha + ce) = \psi(\alpha) + cx_0$ for a given x_0 in X. Clearly ψ is symmetric if and only if $\tilde{\psi}_{x_0}$ is. We now have

DEFINITION 2.7. *A Φ-positive mapping ψ is almost extendable if* (i) ψ is symmetric, (ii) ψ is continuous, and (iii) $|(\psi(\alpha), \varphi)|^2 \leqq d\|\psi\| \|\varphi\|(\psi(\alpha\alpha^*), \varphi)$ for all φ in Φ and α in A where d is a constant with $d \geqq 1$.

DEFINITION 2.8. *A Φ-positive mapping ψ is extendable if ψ is symmetric and if there is an x_0 in X such that*

$$|(\psi(\alpha), \varphi)|^2 \leqq (x_0, \varphi)(\psi(\alpha\alpha^*), \varphi)$$

for all φ in Φ and α in A.

If A has a unit e, then any Φ-positive mapping is extendable. ($x_0 = \psi(ee^*)$). If A does not have a unit, then we have

PROPOSITION 2.9. *A Φ-positive mapping ψ is extendable if and only if there is an extension $\tilde{\psi}$ of ψ to $\tilde{A}$ which is Φ-positive.*

Proof. If $\tilde{\psi}$ is a Φ-positive extension of ψ and e is the unit in $\tilde{A}$, then, letting $x_0 = \tilde{\psi}(e)$, we deduce immediately that $|(\psi(\alpha), \varphi)|^2 = |(\tilde{\psi}(\alpha), \varphi)|^2 \leqq (x_0, \varphi)(\tilde{\psi}(\alpha\alpha^*), \varphi) = (x_0, \varphi)(\psi(\alpha\alpha^*), \varphi)$ (by 2.5) and that ψ is symmetric.

On the other hand, if ψ is extendable, then let $\tilde{\psi}(\alpha + ce) = \tilde{\psi}_{x_0}(\alpha + ce) = \psi(\alpha) + cx_0$. Since $(\tilde{\psi}([\alpha + ce][\alpha + ce]^*), \varphi) = (\psi(\alpha\alpha^*), \varphi) + 2Re\, \bar{c}(\psi(\alpha), \varphi) + |c|^2(x_0, \varphi)$, we have

$$\begin{aligned}(\tilde{\psi}([\alpha + ce][\alpha + ce]^*), \varphi) &\geqq (\psi(\alpha\alpha^*), \varphi) - 2|c|\ |(\psi(\alpha), \varphi| + |c|^2(x_0, \varphi) \\ &\geqq \{(\psi(\alpha\alpha^*), \varphi)^{1/2} - |c|(x_0, \varphi)^{1/2}]^2 \geqq 0\end{aligned}$$

(as ψ is extendable). Thus, $\tilde{\psi}$ is Φ-positive.

PROPOSITION 2.10. *If there is an approximate identity $\{e_n\}$ in A, then a continuous Φ-positive mapping ψ is almost extendable.*

Proof. Since $(\psi(\alpha^*), \varphi) = \lim_{n\to\infty} (\psi(e_n\alpha^*), \varphi) = \lim_{n\to\infty} \overline{(\psi(\alpha e_n^*), \varphi)} = \overline{(\psi(\alpha), \varphi)}$, ψ is symmetric, and since $|B_\varphi(e_n, \alpha)|^2 \leqq B_\varphi(e_n, e_n)B_\varphi(\alpha, \alpha) \leqq \|\psi\| \|\varphi\|B_\varphi(\alpha, \alpha) = \|\psi\| \|\varphi\|(\psi(\alpha\alpha^*), \varphi)$, ψ is almost extendable.

In order to prove the extension of Bochner's theorem, we require a condition on the family Φ defining the cone of "positive" elements. As we shall see, the essential point is to deduce an estimate of the form $\|\psi(\alpha)\|^2 \leqq k\|\psi(\alpha\alpha^*)\|$ from estimates of the form $|(\psi(\alpha), \varphi)|^2 \leqq$

$d\|\varphi\|^2\|\psi\|\,\|\psi(\alpha\alpha^*)\|$ (ψ almost extendable) or

$$|(\psi(\alpha), \varphi)|^2 \leqq \|\varphi\|^2\|x_0\|\,\|\psi(\alpha\alpha^*)\|$$

(ψ extendable). The following definition allows us to do this.

DEFINITION 2.11. The family Φ is full if there is a $\rho > 0$ such that

$$\|x\| \leqq \rho \sup_{\substack{\varphi\in\Phi\\ \varphi\neq 0}} \{|(x, \varphi)|/\|\varphi\|\} \tag{2.12}$$

for all x in X.[3]

We now have

LEMMA 2.13. *If A has a unit e, if the involution on A is continuous, and if Φ is full, then every Φ-positive mapping ψ is continuous and almost extendable.*

Proof. Suppose first that α is a Hermitian element of A with $\|\alpha\| \leqq 1$. The binomial series $(1 - t)^{1/2} = 1 - t/2 - t^2/2^2 2! - \cdots$ converges absolutely for $|t| \leqq 1$ and so the series $e - \alpha/2 - \alpha^2/2^2 2! - \cdots$ converges absolutely in A. Since the involution is continuous, the sum β of this series is a Hermitian element of A with $\beta\beta^* = \beta^2 = e - \alpha$. It follows that $(\psi(e - \alpha), \varphi) = (\psi(\beta\beta^*), \varphi) \geqq 0$ and hence, that $(\psi(e), \varphi) \geqq (\psi(\alpha), \varphi)$. Replacing α by $-\alpha$, we have $(\psi(e), \varphi) \geqq (\psi(-\alpha), \varphi)$. But $(\psi(\alpha), \varphi)$ is real (since α is Hermitian) and so $|(\psi(\alpha), \varphi)| \leqq \|\varphi\|\,\|\psi(e)\|$. Since Φ is full, $\|\psi(\alpha)\| \leqq \rho\|\psi(e)\|$.

Now, if α is any element of A, then $\alpha = 1/2(\alpha + \alpha^*) - i/2(i(\alpha - \alpha^*))$. Since the involution is continuous, there is a $c > 0$ such that $\|\alpha^*\| \leqq c\|\alpha\|$ and so, if $\|\alpha\| \leqq 2/c+1$, then $\|(\alpha+\alpha^*)/2\| \leqq 1$ and $\|i(\alpha-\alpha^*)/2\| \leqq 1$. It follows that $\|\psi(\alpha)\| \leqq 2\rho\|\psi(e)\|$ for all α in A with $\|\alpha\| \leqq 2/c + 1$. Thus, ψ is bounded and therefore continuous.

Since $|(\psi(\alpha), \varphi)|^2 \leqq (\psi(e), \varphi)(\psi(\alpha\alpha^*), \varphi) \leqq \|\psi\|\,\|\varphi\|(\psi(\alpha\alpha^*), \varphi)$, ψ is almost extendable.

COROLLARY 2.14. *If the involution on A is continuous, if Φ is full, and if ψ is Φ-positive and extendable, then ψ is continuous and almost extendable.*

Proof. Apply Proposition 2.9 and the lemma.

[3] This could be replaced by the following: Φ is full relative to ψ if there is a $\rho > 0$ such that $\|\psi(\alpha)\| \leqq \rho \sup_{\substack{\varphi\in\Phi\\ \varphi\neq 0}} \{|\psi(\alpha), \varphi)|/\|\varphi\|\}$ for all α in A.

Let G be a σ-finite locally compact abelian group and let $A = L_1(G, C)$. The involution on $L_1(G, C)$ is given by $\alpha^*(g) = \overline{\alpha(-g)}$ and is continuous since $L_1(G, C)$ is semi-simple. Observe that if Φ is full and ψ is a Φ-positive mapping of $L_1(G, C)$ into X, then ψ is continuous and almost extendable if ψ is extendable (Corollary 2.14) and conversely, ψ is almost extendable if ψ is continuous (Proposition 2.10).

Now let us introduce the following

DEFINITION 2.15. Let p be an element of $L_\infty(G, X)$. The mapping p is Φ-positive definite if

$$\sum_{n=1}^{N}\sum_{m=1}^{N} c_n\bar{c}_m(p(g_n - g_m), \varphi) \geqq 0 \tag{2.16}$$

for any integer N, any $c_1, \cdots, c_N$ in C, any $g_1, \cdots, g_N$ in C, and all φ in Φ. The mapping p is integrally Φ-positive definite if

$$\left(\int_G\int_G \alpha(g)\overline{\alpha(g')}p(g - g')d\mu d\mu, \varphi\right) \geqq 0 \tag{2.17}$$

for all α in $L_1(G, C)$ and all φ in Φ.

We then have

PROPOSITION 2.18. *Let p be a continuous element of $L_\infty(G, X)$. Then p is Φ-positive definite if and only if p is integrally Φ-positive definite.*

Proof. If p is Φ-positive definite, then p is integrally Φ-positive definite by a result of Naimark ([10], p. 397). Conversely, if p is integrally Φ-positive definite, then there is a continuous positive definite function f_φ mapping G into C such that $f_\varphi(g) = (p(g), \varphi)$ locally almost everywhere on G ([10], p. 397) for each φ in Φ. Since $(p(\cdot), \varphi)$ is continuous, $f_\varphi(\cdot) = (p(\cdot), \varphi)$ everywhere and hence, p is Φ-positive definite.

Now it is a fact that ψ is a bounded weakly compact linear map of $L_1(G, C)$ into X with separable range if and only if there is a p in $L_\infty(G, X)$ with (essentially) weakly compact range such that

$$\psi(\alpha) = \int_G \alpha(g)p(g)d\mu \tag{2.19}$$

for all α in $L_1(G, C)$ ([2], p. 279, or [4], p. 507). Moreover, $\|\psi\| = \|p\|_\infty$. The fact that the weakly compact maps in $\mathscr{L}(L_1(G, C), X)$ are essentially the same as the functions in $L_\infty(G, X)$ with (essentially) weakly compact range will allow us to relate the notion of Φ-positivity to the notions of Φ-positive definiteness and integral Φ-

positive definiteness.

LEMMA 2.20. *Let Φ be full. If ψ is a weakly compact linear mapping of $L_1(G, C)$ into X which is Φ-positive and extendable, then there is an (essentially unique) integrally Φ-positive p in $L_\infty(G, X)$ such that*

$$\psi(\alpha) = \int_G \alpha(g)p(g)d\mu \tag{2.21}$$

for all α in $L_1(G, C)$. Conversely, if p is an integrally Φ-positive definite element of $L_\infty(G, X)$ and ψ is given by 2.21, then ψ is Φ-positive and almost extendable.

Proof. Assume that ψ is given. In view of [4], p. 507, the mapping p exists and we need only show that p is integrally Φ-positive definite. But

$$\begin{aligned} \psi(\alpha\alpha^*) &= \int_G\int_G \alpha(g - g')\overline{\alpha(-g')}p(g)d\mu d\mu \\ &= \int_G\int_G \alpha(g)\overline{\alpha(g')}p(g - g')d\mu d\mu \end{aligned} \tag{2.22}$$

by virtue of the Fubini and Tonelli theorems and the invariance of Haar measure. Conversely, given p, we simply note that $\psi(\alpha\alpha^*)$ is determined by 2.22 in order to prove that ψ is Φ-positive. Moreover, since ψ is continuous, ψ is almost extendable by Proposition 2.10.

3. Bochner's theorem for algebras. Before proving the generalization of Bochner's theorem to maps of A into X, we recall the following.

DEFINITION 3.1. Let S be a locally compact topological space and let $\Sigma(S)$ be the Borel field of S. A vector measure ν is a weakly countably additive set function taking values in X. The vector measure ν is weakly regular if the scalar measures $(\nu(\cdot), \varphi)$ are regular[4] for all φ in X^*. The vector measure ν is Φ-positive if $(\nu(E), \varphi) \geqq 0$ for all φ in Φ and E in $\Sigma(S)$. A set function ν^{**} mapping $\Sigma(S)$ into X^{**} is weak-*-regular if $(\varphi, \nu^{**}(\cdot))$ is a regular scalar measure for all φ in X^*. The set function ν^{**} is Φ-positive if $(\varphi, \nu^{**}(E)) \geqq 0$ for all φ in Φ and E in $\Sigma(S)$.

We now have

[4] A scalar measure μ is regular if given $\varepsilon > 0$ and $E \in \Sigma(S)$ with $\|\mu\|(E) < \infty$, then there is a compact $K \subseteq E$ and an open $O \supseteq E$ such that $\|\mu\|(O-K) < \varepsilon$.

THEOREM 3.2. *Let A be a self-adjoint commutative Banach algebra whose involution satisfies the condition $\widehat{(\alpha^*)} = \bar{\hat{\alpha}}$ (e.g. A semi-simple) and let Φ be a full family. If ψ is a mapping of A into X, then ψ is Φ-positive and almost extendable if and only if there is a set function ν^{**} mapping $\Sigma(\mathscr{M})$ into X^{**} such that* (i) *ν^{**} is weak-*-regular,* (ii) *ν^{**} is Φ-positive,* (iii) *ν^{**} has finite semi-variation, i.e. $\|\nu^{**}\|(\mathscr{M}) < \infty$,* (iv) *the mapping $T_{\nu^{**}}$ of X^* into the scalar measures on $\mathscr{M}$ given by $T_{\nu^{**}}(\varphi) = (\nu^{**}(\cdot), \varphi)$ is continuous in the X and $C_0(\mathscr{M})$*[5] *topologies in these spaces respectively, and* (v)

$$(\psi(\alpha), \varphi) = \int_{\mathscr{M}} \hat{\alpha}(M) d(\nu^{**}, \varphi) \tag{3.3}$$

for all α in A and all φ in X^*.

Proof. Suppose first that ψ is Φ-positive and almost extendable. Then ψ is continuous. Let $\hat{\psi}$ be the map of $\hat{A}$ into X given by $\hat{\psi}(\hat{\alpha}) = \psi(\alpha)$. Then $\|\hat{\psi}(\hat{\alpha})\| = \|\psi(\alpha)\|$ and

$$|(\psi(\alpha), \varphi)|^2 \leqq d\|\psi\| \, \|\varphi\|(\psi(\alpha\alpha^*), \varphi) \leqq d\|\psi\| \, \|\varphi\|^2 \|\psi(\alpha\alpha^*)\|$$

for all φ in Φ (since ψ is almost extendable). Since Φ is full, there is a $\rho > 0$ such that

$$\|\psi(\alpha)\| \leqq \rho \sup_{\substack{\varphi \in 0 \\ \varphi \neq \Phi}} \{|(\psi(\alpha), \varphi)|/\|\varphi\|\} .$$

Thus, there is a constant $k(=\rho^2 d\|\psi\|)$ such that

$$\|\psi(\alpha)\|^2 \leqq k\|\psi(\alpha\alpha^*)\| \tag{3.4}$$

for all α in A. It follows that

$$\|\psi(\alpha)\|^2 \leqq k\|\psi(\alpha\alpha^*)\| \leqq k^{1+1/2}\|\psi([\alpha\alpha^*]^2)\|^{1/2} \leqq \cdots \leqq k^2\|\psi\|^\circ \|\hat{\alpha}\|_\infty^2$$[6]

and hence, that $\hat{\psi}$ is a bounded linear map.

Since A is self-adjoint and commutative, $\hat{A}$ is dense in $C_0(\mathscr{M})$ and $\hat{\psi}$ can, therefore, be extended to $C_0(\mathscr{M})$. Let $\hat{\psi}_e$ denote the extension of $\hat{\psi}$ to $C_0(\mathscr{M})$. We claim that there is a weak-*-regular set function ν^{**} on $\Sigma(\mathscr{M})$ such that

$$\left(\hat{\psi}_e(f), \varphi\right) = \int_{\mathscr{M}} f(M) d(\nu^{**}, \varphi) \tag{3.5}$$

for all f in $C_0(\mathscr{M})$ and φ in X^*.

[5] If $\mathscr{M}$ is compact, then $C_0(\mathscr{M})$ is the set of all continuous complex valued functions on $\mathscr{M}$. If $\mathscr{M}$ is locally compact but not compact, then $C_0(\mathscr{M})$ is the set of all continuous complex valued functions on $\mathscr{M}$ which "vanish at infinity".

[6] We apply (3.4) repeatedly and then use the spectral radius formula.

To verify this claim, we let $M(\mathscr{M})$ be the space of all complex valued regular measures μ on $\mathscr{M}$ for which $\|\mu\|$ is finite ([11]). Note that $C_0(\mathscr{M})^* = M(\mathscr{M})$ by the Riesz representation theorem. If $E \in \Sigma(\mathscr{M})$, then let T_E be the element of $C_0(\mathscr{M})^{**}$ defined by

$$T_E(\mu) = \mu(E),\ \mu \in M(\mathscr{M})\ . \tag{3.6}$$

Now define a set function ν^{**} of $\Sigma(\mathscr{M})$ into X^{**} by setting

$$\nu^{**}(E) = \hat{\psi}_e^{**}(T_E) \tag{3.7}$$

for E in $\Sigma(\mathscr{M})$. We show that ν^{**} is weak-*-regular. If φ is an element of X^*, then $\hat{\psi}_e^*(\varphi)$ is, by the Riesz representation theorem, a measure μ_φ in $M(\mathscr{M})$. But

$$\mu_\varphi(E) = T_E(\mu_\varphi) = T_E(\hat{\psi}_e^*(\varphi)) = \hat{\psi}_e^{**}(T_E)(\varphi) = (\nu^{**}(E), \varphi) \tag{3.8}$$

and so, the set function ν^{**} is weak-*-regular. Moreover, since $\hat{\psi}_e^*(\varphi) = (\nu^{**}(\cdot), \varphi)$ by 3.8, the mapping $T_{\nu^{**}}$ satisfies (iv). Also,

$$(\hat{\psi}_e(f), \varphi) = \hat{\psi}_e^*(\varphi)(f) = \int_{\mathscr{M}} f(M)d\mu_\varphi = \int_{\mathscr{M}} f(M)d(\nu^{**}, \varphi)$$

for f in $C_0(\mathscr{M})$ so that 3.3 is satisfied. It is easy to check that $\|\nu^{**}\|(\mathscr{M}) = \|\hat{\psi}_e\|$ ([4], p. 492) and so, (iii) is satisfied.

All that remains to establish the first half of the theorem is to prove that ν^{**} is Φ-positive. If f is an element of $C_0(\mathscr{M})$ with $f(M) \geqq 0$ for all M, then $f^{1/2}$ is in $C_0(\mathscr{M})$ and there is a sequence $\{\alpha_n\}$ in A such that $\lim_{n\to\infty} \hat{\alpha}_n = f^{1/2}$. Since

$$(\hat{\psi}(\widehat{\alpha_n\alpha_n^*}), \varphi) = \int_{\mathscr{M}} |\hat{\alpha}_n(M)|^2 d(\nu^{**}, \varphi)\ ,$$

it follows that if φ is an element of Φ, then

$$0 \leqq (\psi(\alpha_n\alpha_n^*), \varphi) = \int_{\mathscr{M}} |\hat{\alpha}_n(M)|^2 d(\nu^{**}, \varphi)$$

and hence, by taking limits, that

$$\int_{\mathscr{M}} f(M)d(\nu^{**}, \varphi) \geqq 0 \tag{3.9}$$

for all φ in Φ and all f in $C_0(\mathscr{M})$ with $f(\cdot) \geqq 0$. But $(\nu^{**}(\cdot), \varphi)$ when restricted to the Baire sets in $\mathscr{M}$ is a Baire measure, and as such, is positive. The Baire measure can be extended to a *unique* regular Borel measure ([2]) which must (by uniqueness) be $(\nu^{**}(\cdot), \varphi)$. It follows that ν^{**} is Φ-positive.

Now suppose that ν^{**} is given. Since the mapping $T_{\nu^{**}}$ is continuous in the X and $C_0(\mathscr{M})$ topologies, the linear mapping $\varphi \to$

$\int_{\mathscr{M}} f(M)d(\nu^{**}, \varphi)$ is, for each fixed f in $C_0(\mathscr{M})$, continuous in the X-topology of X^* and is, therefore, generated by an element x_f of X. Thus, the mapping $\hat{\psi}_e$ of $C_0(\mathscr{M})$ into X given by $\hat{\psi}_e(f) = x_f$ is a bounded linear map of $C_0(\mathscr{M})$ into X. If α is an element of A, then let $\psi(\alpha) = \hat{\psi}_e(\hat{\alpha})$. Since $\|\psi(\alpha)\| = \|\hat{\psi}_e(\hat{\alpha})\| \leqq \|\hat{\psi}_e\| \|\hat{\alpha}\|_\infty \leqq \|\hat{\psi}_e\| \|\alpha\|$, ψ is a continuous linear map. If φ is an element of Φ, then

$$(\psi(\alpha\alpha^*), \varphi) = \int_{\mathscr{M}} |\hat{\alpha}(M)|^2 d(\nu^{**}, \varphi) \geqq 0$$

and

$$(\psi(\alpha^*), \varphi) = \int_{\mathscr{M}} \overline{\hat{\alpha}}(M)d(\nu^{**}, \varphi) = \overline{\int_{\mathscr{M}} \hat{\alpha}(M)d(\nu^{**}, \varphi)} = (\psi(\alpha), \varphi)$$

so that ψ is Φ-positive and symmetric. Also,

$$\begin{aligned} |(\psi(\alpha), \varphi)|^2 &\leqq \left[\int_{\mathscr{M}} |\hat{\alpha}(M)|^2 d(\nu^{**}, \varphi)\right]\left[\int_{\mathscr{M}} 1^2 d(\nu^{**}, \varphi)\right] \\ &\leqq (\psi(\alpha\alpha^*), \varphi)(\nu^{**}(\mathscr{M}), \varphi) \\ &\leqq \|\nu^{**}\|(\mathscr{M})\|\varphi\|(\psi(\alpha\alpha^*), \varphi) \\ &\leqq \max\{1, \|\nu^{**}\|(\mathscr{M})/\|\psi\|\}\|\psi\| \|\varphi\|(\psi(\alpha\alpha^*), \varphi) \end{aligned}$$

so that ψ is almost extendable.

COROLLARY 3.10. *Let A be a self-adjoint commutative Banach algebra with $\widehat{(\alpha^*)} = \overline{\hat{\alpha}}$ and let Φ be a full family. If ψ is Φ-positive and almost extendable and $\hat{\psi}$ is weakly compact, then there is a weakly regular Φ-positive vector measure ν on $\Sigma(\mathscr{M})$ such that*

$$\psi(\alpha) = \int_{\mathscr{M}} \hat{\alpha}(M)d\nu \tag{3.11}$$

for all α in A. Conversely, if ν is a weakly regular Φ-positive vector measure and ψ is given by (3.11), *then ψ is Φ-positive and almost extendable and $\hat{\psi}$ is weakly compact.*

Proof. Suppose that ψ is given. Since $\hat{\psi}$ is weakly compact, $\hat{\psi}_e$[7] is weakly compact and so, $\hat{\psi}_e^{**}(C_0(\mathscr{M})^{**})$ is contained in the natural imbedding of X in X^{**}. Thus, the set function ν^{**} given by 3.7 may be identified with a mapping ν of $\Sigma(\mathscr{M})$ into X. In that case, $(\nu(\cdot), \varphi)$ is an element of $M(\mathscr{M})$ for all φ in X^*. It follows that $\nu(\cdot)$ is a weakly regular vector measure (as $\nu(\cdot)$ is weakly countably additive). Clearly ν is Φ-positive. Moreover, since

[7] We use the notation of the proof of the theorem.

$$(\psi(\alpha), \varphi) = \int_{\mathscr{M}} \hat{\alpha}(M) d(\nu, \varphi) = \left(\int_{\mathscr{M}} \hat{\alpha}(M) d\nu, \varphi\right)$$

for all φ in X^*, 3.11 is satisfied.

On the other hand, if ν is given and ψ is defined by 3.11 (note that $\hat{\alpha}(\cdot)$ is bounded and continuous), then ψ is Φ-positive and almost extendable. In fact, $\|\psi(\alpha)\| \leqq \|\hat{\alpha}\|_\infty \|\nu\|(\mathscr{M}) \leqq \|\alpha\| \, \|\nu\|(\mathscr{M})$ and $|(\psi(\alpha), \varphi)|^2 \leqq (\psi(\alpha\alpha^*), \varphi)(\nu(\mathscr{M}), \varphi)$ so that ψ is extendable ($x_0 = \nu(\mathscr{M}) \in X$). Thus, to complete the proof we need only show that $\hat{\psi}$ is weakly compact.

Now, $\hat{\psi}$ is clearly linear and, since

$$\|\hat{\psi}(\hat{\alpha})\| = \|\psi(\alpha)\| \leqq \|\nu\|(\mathscr{M}) \|\hat{\alpha}\|_\infty ,$$

$\hat{\psi}$ is continuous. Let $\hat{\psi}_e$ be the mapping of $C_0(\mathscr{M})$ into X defined by $\hat{\psi}_e(f) = \int_{\mathscr{M}} f(M) d\nu$. Thus, it will be enough to prove that $\hat{\psi}_e$ is weakly compact.

If φ is an element of X^*, then $\hat{\psi}_e^*(\varphi) = (\nu(\cdot), \varphi)$ is an element of $M(\mathscr{M})$. Since the set $\{(\nu(\cdot), \varphi) : \varphi \in X^*, \|\varphi\| \leqq 1\}$ is weakly sequentially compact as a subset of the space of scalar measures and since ν is weakly regular, $\hat{\psi}_e^*$ is a weakly compact mapping. It follows that $\hat{\psi}_e$ is weakly compact and the corollary is established.

COROLLARY 3.12. *If ν satisfies the conditions of Corollary* 3.10 *and ψ is given by* 3.11, *then ψ is extendable. Conversely, if ψ is extendable (rather than almost extendable) and if the involution on A is continuous* (e.g. *A semi-simple*), *then a ν satisying the conditions of Corollary* 3.10 *exists (the other hypotheses of Corollary* 3.10, *are, of course, assumed).*

Proof. The first assertion was established in the course of the proof of Corollary 3.10. The second assertion is an immediate consequence of Corollary 2.14.

COROLLARY 1.13. *If X is weakly complete, if A and Φ satisfy the conditions of corollary* 3.10, *and if ψ is Φ-positive and almost extendable, then $\hat{\psi}$ is weakly compact.*

Proof. By the argument given in the proof of Theorem 3.2, $\hat{\psi}$ is a continuous linear map. If A has a unit, then $\mathscr{M}$ is compact. Since $\hat{A}$ is dense in $C_0(\mathscr{M})$, we may extend $\hat{\psi}$ to a continuous linear map $\hat{\psi}_e$ of $C_0(\mathscr{M})$ into X. As X is weakly complete, $\hat{\psi}_e$ is weakly compact ([4], p. 494) and a fortiori so is $\hat{\psi}$. If A does not have a unit, then we extend A to $\tilde{A} = A \oplus C$. Letting x_0 be an element of

X, we extend ψ to a mapping $\tilde{\psi}$ of $\tilde{A}$ into X by setting $\tilde{\psi}(\alpha + \lambda e) = \psi(\alpha) + \lambda x_0$. Then $\hat{\tilde{\psi}}(\hat{\alpha} + \lambda\hat{e}) = \hat{\psi}(\hat{\alpha}) + \lambda x_0$ is a bounded linear map of $\hat{\tilde{A}}$ into X. It follows that $\hat{\tilde{\psi}}$ is weakly compact and hence, that $\hat{\psi}$ is weakly compact.

4. Bochner's theorem on a group. Let G be a σ-finite locally compact abelian group and let $A = L_1(G, C)$. The involution on A is given by $\alpha^*(g) = \overline{\alpha(-g)}$ and is continuous. Let X be a Banach space and let Φ be a full family. We shall prove a generalization of Bochner's theorem for integrally Φ-positive definite mappings p in $L_\infty(G, X)$ by combining Lemma 2.20 with Theorem 3.2 and its corollaries. We have

THEOREM 4.1. (A) *If ν is a weakly regular Φ-positive vector measure defined on $\Sigma(\hat{G})$ (the Borel field of the dual group $\hat{G}$) and if*

$$p(g) = \int_{\hat{G}} \overline{(\gamma, g)}\, d\nu \tag{4.2}$$

then p is an integrally Φ-positive definite element of $L_\infty(G, X)$.

(B) *If p is an integrally Φ-positive definite element of $L_\infty(G, X)$, then there is a set function ν^{**} mapping $\Sigma(\hat{G})$ into X^{**} such that* (i) *ν^{**} is weak-*-regular, Φ-positive, and finite,* (ii) *the map $T_{\nu^{**}}$ given by $T_{\nu^{**}}(\varphi) = (\nu^{**}(\cdot), \varphi)$ is continuous in the X topology of X^* and the $C_0(\hat{G})$ topology of $M(\hat{G})$, and* (iii)

$$(p(g), \varphi) = \int_{\hat{G}} \overline{(\gamma, g)}\, d(\nu^{**}, \varphi) \tag{4.3}$$

for all φ in X^* and (*almost*) *all g in G.* (*The null set on which* (4.3) *does not hold may depend on φ.*)

Proof. (A) Let $p(\cdot)$ be given by 4.2. Suppose, for the moment, that $p(\cdot)$ is measurable. Then p is in $L_\infty(G, X)$ since $\|p(g)\| \leqq \|\nu\|(\hat{G})$ for all g. Let $\psi(\alpha) = \int_G \alpha(g)p(g)d\mu$ for α in $L_1(G, C)$. Then

$$\begin{aligned}(\psi(\alpha), \varphi) &= \int_G \alpha(g) \int_{\hat{G}} \overline{(\gamma, g)}\, d(\nu, \varphi) d\mu \\ &= \int_{\hat{G}} \int_G \alpha(g)\overline{(\gamma, g)}\, d\mu d(\nu, \varphi) \\ &= \int_{\hat{G}} \hat{\alpha}(\gamma) d(\nu, \varphi) = \left(\int_{\hat{G}} \hat{\alpha}(\gamma) d\nu, \varphi \right)\end{aligned}$$

for all φ in X^* by the Fubini and Tonelli theorems. Since $\hat{G}$ and $\mathscr{M}$ can be identified ([11] or [9]), we have $\psi(\alpha) = \int_{\mathscr{M}} \hat{\alpha}(M) d\nu$ (as ν may be viewed as a measure on $\mathscr{M}$). But then (Corollary 3.10) ψ is Φ-positive and extendable (Corollary 3.12). The result follows immediately from 2.22 of Lemma 2.20.

Thus, to complete the proof of (A), we need only show that p is measurable. To do this it will be sufficient to show that, for any set $F \subset G$ with $\mu(F) < \infty$, $P_F(\cdot) = \chi_F(\cdot)p(\cdot)$ is the limit in measure of a sequence of simple functions where χ_F is the characteristic function of F.

Since ν is weakly regular, there is a finite, positive, regular scalar measure λ such that $||\nu||(E) \to 0$ if and only if $\lambda(E) \to 0$ where $||\nu||(E)$ is the *semi-variation of* ν *on* E ([4]). Therefore, given $\eta > 0$, there is a $\xi > 0$ such that if $\lambda(\hat{G} - K)$[8] $< \xi$, then $||\nu||(\hat{G} - K) < \eta/4$ for K compact in $\hat{G}$. Since λ is finite and regular, there is a compact set $K \subset \hat{G}$ for which $\lambda(\hat{G} - K) < \xi$ and hence for which $||\nu||(\hat{G} - K) < \eta/4$. Let $\eta_1 = \eta/2||\nu||(\hat{G})$ and let

$$N(g; K, \eta_1) = \{g' \in G: |1 - (\gamma, g')| < \eta_1, r \in K\} + g\ .$$

Then $N(g; K, \eta_1)$ is an open neighborhood of g in G.

Now G is σ-finite and so there is an increasing sequence of sets G_n with $\mu(G_n) < \infty$ and $\cup G_n = G$. Moreover, since Haar measure is regular, given $\varepsilon > 0$ there is a compact set $L_n \subseteqq G_n$ such that $\mu(G_n - L_n) < \varepsilon$. The sets $N(g; K, \eta_1)$, $g \in L_n$, form an open cover of L_n. Thus there are $g_1, \cdots, g_{M_n}$ in L_n such that $L_n \subseteqq \bigcup_{i=1}^{M_n} N(g_i; K, \eta_1)$. Let $N_1^n = N(g_1; K, \eta_1)$ and $N_{i+1}^n = N(g_{i+1}; K, \eta_1) - (N(g_1; K, \eta_1) \cup \cdots \cup N(g_i; K, \eta_1))$. Then $L_n \subseteqq \bigcup_{i=1}^{M_n} N_i^n$ and the union is disjoint. Let p_0 be defined on L_n by $p_0(g) = p(g_i)$ if $g \in N_i^n$ and let $p_n^{\varepsilon,\eta}(\cdot)$ be given by

$$p_n^{\varepsilon,\eta}(g) = \begin{cases} p_0(g) & g \in L_n \\ 0 & g \notin L_n \end{cases} . \tag{4.4}$$

Then $p_n^{\varepsilon,\eta}(\cdot)$ is a simple function and we claim that

$$\mu^*(\{g \in G_n: ||p(g) - p_n^{\varepsilon,\eta}(g)|| > \eta\}) < \varepsilon \tag{4.5}$$

where $\mu^*(E) = \inf_{E_1} \mu(E_1)$, $E_1 \supseteqq E$. For if g is in L_n, then

$$\begin{aligned} ||p(g) - p_n^{\varepsilon,\eta}(g)|| &= ||p(g) - p_0(g)|| \\ &= \left|\left|\int_{\hat{G}} \overline{(\gamma, g)}[1 - \overline{(\gamma, g_i - g)}]d\nu\right|\right| \\ &\leqq \left|\left|\int_{\hat{G}-K} \overline{(\gamma, g)}[1 - \overline{(\gamma, g_i - g)}]d\nu\right|\right| \\ &\quad + \left|\left|\int_{K} \overline{(\gamma, g)}[1 - \overline{(\gamma, g_i - g)}]d\nu\right|\right| \\ &\leqq \frac{\eta}{2} + \eta_1||\nu||(\hat{G}) = \eta \end{aligned}$$

[8] Here $\hat{G}$-K is the complement of K.

(for some i) so that $\{g \in G_n: \|p(g) - p_n^{\varepsilon,\eta}(g)\| > \eta\} \subseteqq G_n - L_n$. It follows that 4.5 holds. Let $p_n(g) = p_n^{1/n,1/n}(g)$ so that p_n is a simple function.

Now suppose that a is any positive number. We show that

$$\lim_{n\to\infty} \mu^*(\{g \in F: \|p(g) - p_n(g)\| > a\}) = 0 \tag{4.6}$$

for any $F \subset G$ with $\mu(F) < \infty$. So let $\varepsilon > 0$ be given. Then there is an $n_0 \geqq \max(1/a, 2/\varepsilon)$ such that $\mu(F \cap (G - G_n)) < \varepsilon/2$ for $n \geqq n_0$. It follows that

$$\begin{aligned} &\mu^*(\{g \in F: \|p(g) - p_n(g)\| > a\}) \\ &\quad \leqq \mu^*(\{g \in F \cap G_{n'} \|p(g) - p_n(g)\| > a\}) + \varepsilon/2 \\ &\quad \leqq \mu^*(\{g \in F \cap G_n: \|p(g) - p_n(g)\| > 1/n\}) + \varepsilon/2 \\ &\quad \leqq 1/n + \varepsilon/2 \leqq \varepsilon \end{aligned}$$

for $n \geqq n_0$. In other words, p_n converges to p in measure on F. The proof of (A) is now complete.

(B) Let $\psi(\alpha) = \int_G \alpha(g)p(g)d\mu$. Then ψ is Φ-positive and almost extendable by Lemma 2.20. It follows from Theorem 3.2 that there is a set function ν^{**} on $\Sigma(\mathscr{M})$ such that (i) and (ii) are satisfied and

$$(\psi(\alpha), \varphi) = \int_{\mathscr{M}} \hat{\alpha}(M)d(\nu^{**}, \varphi) \tag{4.7}$$

for all φ in X^*. Since $\hat{G}$ and $\mathscr{M}$ can be identified, ν^{**} may be viewed as a set function on $\Sigma(\hat{G})$ and

$$(\psi(\alpha), \varphi) = \int_{\hat{G}}\left[\int_G \alpha(g)\overline{(\gamma, g)}d\mu\right]d(\nu^{**}, \varphi) \tag{4.8}$$

for all φ in X^*. Application of the Fubini and Tonelli theorems then yields

$$\int_G \alpha(g)(p(g), \varphi)du = (\psi(\alpha), \varphi) = \int_G \alpha(g)q_\varphi(g)d\mu \tag{4.9}$$

where $q_\varphi(g) = \int_{\hat{G}} \overline{(\gamma, g)}d(\nu^{**}, \varphi)$. Since $(p(\cdot), \varphi)$ and $q_\varphi(\cdot)$ are in $L_\infty(G, C)$, we have $\|(p(\cdot), \varphi) - q_\varphi(\cdot)\|_\infty = 0$ for all φ in X^*. In other words, 4.3 holds. The proof of (B) is now complete.

REMARK 4.10. Since $\overline{(\gamma, g)} = (-\gamma, g)$ and since the measure ν_1 (or the set function ν_1^{**}) given by $\nu_1(E) = \nu(-E)$ (or $\nu_1^{**}(E) = \nu^{**}(-E)$) has the same properties as ν (or ν^{**}), $p(g)$ is given by $p(g) = \int_{\hat{G}}(\gamma, g)d\nu_1$ (or satisfies $(p(g), \varphi) = \int_{\hat{G}}(\gamma, g)d(\nu_1^{**}, \varphi)$). [This agrees with convention in the scalar case.]

We observe that if the hypotheses of (A) are satisfied and $\psi(\alpha) = \int_G \alpha(g)p(g)d\mu$, then the mapping $\hat{\psi}$ of $\hat{A}$ into X given by $\hat{\psi}(\hat{\alpha}) = \psi(\alpha)$ is weakly compact (Corollary 3.10). Note also that if X is weakly complete and $p(\cdot)$ is an integrally Φ-positive definite element of $L_\infty(G, X)$, then $\hat{\psi}$ is weakly compact. This leads to

COROLLARY 4.11. *If X is weakly complete and if p is an integrally Φ-positive definite element of $L_\infty(G, X)$ then there is a weakly regular Φ-positive vector measure ν on $\Sigma(\hat{G})$ such that*

$$(p(g), \varphi) = \left(\int_{\hat{G}} (\gamma, g)d\nu, \varphi\right) \tag{4.12}$$

for all φ in X^* and (almost) all g in G. If, in addition, Φ is countable, then

$$p(g) = \int_{\hat{G}} (\gamma, g)d\nu \tag{4.13}$$

for (almost) all g in G.

Proof. The first assertion follows Corollary 3.12. On the other hand, if $\Phi = \{\varphi_i\}$ is countable, then there is a μ-null set N such that $(p(g) - q(g), \varphi) = 0$ for all φ in Φ and $g \notin N$ where $q(g) = \int_{\hat{G}} (\gamma, g)d\nu$. But then

$$\|p(g) - q(g)\| \leqq \rho \sup_{\substack{\varphi \in \Phi \\ \varphi \neq 0}} \{|(p(g) - q(g), \varphi)|/\|\varphi\|\} = 0 \quad \text{for} \quad g \notin N\,.$$

It follows immediately that $\|p(\cdot) - q(\cdot)\|_\infty = 0$, i.e. that 4.13 holds.

In order to state our final corollary we require

DEFINITION 4.14. Then element p of $L_\infty(G, X)$ is dominated if there exists a finite regular positive Borel measure λ such that

$$\left\|\int_G \alpha(g)p(g)d\mu\right\| \leqq \int_{\hat{G}} |\hat{\alpha}(\gamma)|\,d\lambda \tag{4.15}$$

for all α in $L_1(G, C)$, where $\hat{\alpha}$ is the Fourier transform of α.

COROLLARY 4.16. *Assume Φ is countable. Then p is a dominated integrally Φ-positive definite element of $L_\infty(G, X)$ if and only if there exists a weakly regular Φ-positive vector measure ν of finite variation mapping $\Sigma(\hat{G})$ into X such that*

$$p(g) = \int_{\hat{G}} (\gamma, g)d\nu\,. \tag{4.17}$$

Proof. We have only to note that there exists an isomorphism between the set of weakly regular vector measures $\nu\colon \Sigma(\hat{G}) \to X$ with finite variation and the set of bounded linear operators $T\colon C_0(\hat{G}) \to X$ for which there exists a finite regular positive Borel measure λ such that $\| T(f) \| \leqq \int_{\hat{G}} | f(\gamma) | d\lambda$. This isomorphism is given by $T(f) = \int_{G} f(\gamma) d\nu$, ([2], p. 380, or [8]). Now using Theorem 4.1 (B) we have, if we assume the existence of p, that $\left(\int_{\hat{G}} f(\gamma) d\nu, \varphi\right) = \int_{\hat{G}} f(\gamma) d(\nu^{**}, \varphi)$ for any f in $C_0(\hat{G})$, φ in X^*. But $C_0(\hat{G})^* = M(\hat{G})$, the space of regular complex valued measures defined on $\Sigma(\hat{G})$ of finite variation, and (φ, ν^{**}), (ν, φ) are in $M(\hat{G})$. Thus, for any E in $\Sigma(\hat{G})$, $(\nu(E), \varphi) = (\varphi, \nu^{**}(E))$. Consider $\nu(E)$ as an element of X^{**}, then $\nu(E) = \nu^{**}(E)$ and so ν^{**} is actually a measure. From the countability of Φ we derive (4.17).

The converse follows immediately from Theorem 4.1 (A).

5. Some Examples.

We now give several examples of spaces to which the theory applies.

EXAMPLE 5.1. Let $X = L_1([0, 1], C)$. Note that X is weakly complete. If $\Sigma = \Sigma([0, 1])$ is the Borel field on $[0, 1]$, then Σ is a separable metric space with respect to the usual metric $d(E, E') = \mu(E \Delta E')$ where $E \Delta E' = (E - E') \cup (E' - E)$ is the symmetric difference of E and E'. Let $\{E_i\}$ be a countable dense set in Σ with $E_1 = [0, 1]$. Let χ_i be the characteristic function of E_i and let φ_i be the element of X^* given by

$$(x(\cdot), \varphi_i) = \int_0^1 \chi_i(s) x(s) ds \; . \tag{5.2}$$

If $\Phi = \{\varphi_i\}$, then K_Φ is the cone of (essentially) nonnegative functions. Note also that $\| \varphi_i \| \leqq 1$.

Now we claim that Φ is full. Set $x^+(s) = \max\{0, x(s)\}$ and $x^-(s) = \max\{0, -x(s)\}$ for real x in X. Then $x(s) = x^+(s) - x^-(s)$ and $|x(s)| = x^+(s) + x^-(s)$. Moreover, x^+ and x^- are nonnegative. Letting

$$\int_0^1 x_0(s) ds = \max\left\{\int_0^1 x^+(s) ds, \int_0^1 x^-(s) ds\right\}$$

(i.e. $x_0 = x^+$ or x^- according to which integral is greater), we see that

$$\| x(\cdot) \|_1 \leqq 2 \int_0^1 x_0(s) ds \tag{5.3}$$

for real x in X. Now, suppose, for example, that $x_0 = x^+$. Since x^+ is measurable, $(x^+)^{-1}([0, \infty)) = E$ is in Σ and $\int_0^1 x_0(s) ds = \int_E x^+(s) ds =$

$\int_E x(s)ds$. As $\{E_i\}$ is dense in Σ, there is a sequence $\{E_{i,n}\}$ such that $d(E_{i,n}, E) \to 0$ as $n \to \infty$. But $\left| \int_{E_{i,n}} x(s)ds - \int_E x(s)ds \right| \leqq \int_{E_{i,n}\Delta E} |x(s)|ds$ and $\lim_{n\to\infty} \int_{E_{i,n}\Delta E} |x(s)|ds = 0$ as $\mu(E_{i,n}\Delta E) \to 0$ as $n \to \infty$ and x is in $L_1([0, 1], R)$. It follows that there is a sequence $\{\varphi_{i,n}\}$ such that $\lim_{n\to\infty} (x, \varphi_{i,n}) = \int_0^1 x_0(s)ds$ and hence, that $\int_0^1 x_0(s)ds \leqq \sup_\Phi |(x, \varphi)|$. Now choose any x in X. Then $x = x_1 + ix_2$ where $x_1(\cdot)$, $x_2(\cdot)$ are real valued. But $||\varphi|| \leqq 1$ for φ in Φ and so,

$$(5.4) \qquad ||x||_1 \leqq ||x_1||_1 + ||x_2||_1 \leqq 4 \sup_{\varphi\in\Phi} |(x, \varphi)| \leqq 4 \sup_{\substack{\varphi\in\Phi \\ \varphi\neq 0}} \{|(x, \varphi)|/||\varphi||\}$$

for all x in X.

EXAMPLE 5.5. $X = H$ be a separable Hilbert space. Fix an orthonormal basis $\{e_i\}$ in H. If $h \in H$, then $h = h_1 + ih_2$ where $h_1 = \sum \mathrm{Re}\{\langle h, e_i\rangle\}e_i$ and $h_2 = \sum \mathrm{Im}\{\langle h, e_i\rangle\}e_i$. An element h is *real* if $h = h_1$. Let H_0 be the set of all real elements h such that (i) $||h|| \leqq 1$, (ii) h is *positive* i.e. $\langle h, e_i\rangle \geqq 0$ for all i, (iii) h is *rational* i.e. $\langle h, e_i\rangle$ is rational for all i, and (iv) h is *finite* i.e. only a finite number of components $\langle h, e_i\rangle$ of h are not zero. Since H^* can be identified with H, we let $\Phi = H_0$. In other words, if $\varphi \in \Phi$, then $(h, \varphi) = \langle h, k\rangle$ for some k in H_0. The cone K_Φ is the set of all positive real elements of H.

We claim that Φ is full. Suppose first that $h = h_1$ is real. Then $h_1 = h_1^+ - h_1^-$ where $\langle h_1^+, e_i\rangle = \max\{0, \langle h_1, e_i\rangle\}$ and $\langle h_1^-, e_i\rangle = \max\{0, -\langle h_1, e_i\rangle\}$ for all i. Note that $||h_1||^2 = ||h_1^+||^2 + ||h_1^-||^2$. Let k_n^+ be the element of H_0 with components $\langle k_n^+, e_i\rangle$ given by

$$(5.6) \qquad \langle k_n^+, e_i\rangle = \begin{cases} r_i & i \leqq N \\ 0 & i > N \end{cases}$$

where N is chosen so that

$$(5.7) \qquad \sum_{N+1}^{\infty} |\langle h_1^+, e_i\rangle|^2 < \frac{1}{2n^2}||h_1^+||^2$$

and r_i is a nonnegative rational such that

$$(5.8) \qquad \langle h_1^+, e_i\rangle \geqq r_i||h_1^+|| \geqq \left\{\langle h_1^+, e_i\rangle - \frac{||h_1^+||}{n\sqrt{2N}}\right\}.$$

Clearly $||k_n^+ - h_1^+/||h_1^+||\,|| < 1/n$. It follows that $(h_1^+, k_n^+) \to ||h_1^+||$ as $n \to \infty$. Similarly, there is a sequence k_n^- in H_0 such that $(h_1^-, k_n^-) \to ||h_1^-||$ as $n \to \infty$. Noting that for any $h = h_1 + ih_2$ in H, $|(h, \varphi)| \geqq \max\{|(h_1, \varphi)|, |(h_2, \varphi)|\}$, we have

$$||h_1||^2 = ||h_1^+||^2 + ||h_1^-||^2 = \lim_{n\to\infty} (h_1^+, k_n^+)^2 + \lim_{n\to\infty} (h_1^-, k_n^-)^2$$
$$\leqq \overline{\lim}\, |(h_1^+, k_n^+)|^2 + \overline{\lim}|\, (h_1^+, k_n^-)|^2 \leqq 2 \sup_{\varphi\in\Phi} |(h_1, \varphi)|^2 .$$

Now, if $h = h_1 + ih_2$ is any element of H, then $||h_1||^2 = ||h||^2 + ||h_2||^2 \leqq 2 \sup_\varphi |(h_1, \varphi)|^2 + 2 \sup |(h_2, \varphi)|^2 \leqq 4 \sup_\varphi |(h, \varphi)|^2$. Since $||\varphi|| \leqq 1$ if $\varphi \in \Phi = H_0$, we deduce that

$$||h|| \leqq 2 \sup_{\substack{\varphi\in\Phi \\ \varphi\neq 0}} \{|(h, \varphi)|/||\varphi||\}$$

for all h in H. Thus, Φ is full.

EXAMPLE 5.9. Let H be a separable Hilbert space and let $X = \mathscr{L}(H, H)$ be the space of bounded linear maps of H into itself. Let H_0 be a countable dense subset of the closed unit ball in H and let $\Phi = \{\varphi \in X^*: (T, \varphi) = \langle Tk, k\rangle$, for some k in $H_0\}$. The cone K_Φ is the set of positive operators in $\mathscr{L}(H, H)$. Since $||T|| \leqq 2 \sup_{||h||\leqq 1} |\langle Th, h\rangle|$ for T in $\mathscr{L}(H, H)$ and since $||\varphi|| \leqq ||k||^2 \leqq 1$ for k in H_0, we have $||T|| \leqq 2 \sup_{\substack{\varphi\in\Phi \\ \varphi\neq 0}} \{|(T, \varphi)|/||\varphi||\}$. In other words, Φ is full.

EXAMPLE 5.10. Let $\mathscr{D}$ be a bounded domain in R^n and let $X = L_p(\mathscr{D}, C)$ where $1 < p < \infty$. Let Σ be the Borel field of $\mathscr{D}$. Then Σ is a separable metric space with respect to the usual metric $d(E, E') = \mu(E \Delta E')$. Let $\Sigma_0 = \{E_i\}$ be a countable dense set in Σ which include all hypercubes with rational vertices contained in $\mathscr{D}$ and let $Q = \{a + bi \in C: a, b \text{ rational}\}$. Let $\mathscr{Y}$ be the set of simple functions of the form $\sum_{i=1}^n q_i \chi_{E_i}$ where the q_i are in Q and the E_i are disjoint elements of Σ_0. Note that $\mathscr{Y}$ is a countable subset of $L_q(\mathscr{D}, C)$ where $1/p + 1/q = 1$. It is easy to check that $\mathscr{Y}$ is dense in $L_q(\mathscr{D}, C)$. An element $\sum_{i=1}^n q_i \chi_{E_i}$ of $\mathscr{Y}$ is positive real if q_i is a nonnegative real number for $i = 1, \cdots, n$. Let Φ be the subset of $\mathscr{Y}$ consisting of all the positive real elements. Since $X^* = L_p(\mathscr{D}, C)^* = L_q(\mathscr{D}, C)$, $\Phi \subset X^*$ and the cone K_Φ is simply the set of nonnegative functions in $L_p(\mathscr{D}, C)$. The proof that Φ is full is straightforward and is, therefore, left to the reader. Theorem 4.1, when interpreted in this context, becomes:

COROLLARY 5.11. *If p is an element of $L_\infty(G, L_p(\mathscr{D}, C))$ such that $\int_G\int_G \xi(g)\overline{\xi(g')}p(g - g')d\mu d\mu$ is a nonnegative function in $L_p(\mathscr{D}, C)$ for all $\xi(\cdot)$ in $L_1(G, C)$, then $p(g) = \int_{\hat{G}}(\gamma, g)d\nu$ where ν is a weakly regular measure on $\hat{G}$ such that $\nu(F)$ is a nonnegative function in $L_p(\mathscr{D}, C)$ for all F in $\Sigma(\hat{G})$, and conversely.*

This corollary plays a role in the study of positive solutions of certain partial differential equations.

EXAMPLE 5.12. Let H be a separable Hilbert space and let $\mathscr{C} = \mathscr{C}(H, H)$ be the closed ideal of compact operators in $\mathscr{L}(H, H)$. It is well-known ([3]) that $\mathscr{L}(H, H)^* = \mathscr{L}_1 \oplus \mathscr{C}^\perp$ where $\mathscr{C}^\perp$ is the annihilator of $\mathscr{C}$ and $\mathscr{L}_1$ is the trace class. Moreover, $\mathscr{L}_1$ is isometrically isomorphic with $\mathscr{C}^*$ and $\mathscr{C}^{**}$ is isometrically isomorphic with $\mathscr{L}_1^* = \mathscr{L}(H, H)$. Now let H_0 be a countable dense subset of the closed unit ball in H and let $\Phi = \{\varphi \in \mathscr{C}^*: (T, \varphi) = \langle Tk, k\rangle$ some k in $H_0\}$. The cone K_Φ is the set of positive compact operators and Φ is a countable full family.

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