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*-ACTIONS IN A^*-ALGEBRAS

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Let U be the open unit disk in the complex plane and f a function defined on U. We show that if A is an infinite dimensional dual B^*-algebra, then f defines a *-action in A if and only if f is continuous at zero and $f(0) = 0$. We also obtain that if A is commutative, then f defines a continuous action in A if and only if f is continuous on U and $f(0) = 0$.

Actions in Banach algebras were introduced and studied recently by Gulick in [1]. Most of her main results were obtained for certain subalgebras of the algebra of all completely continuous operators on a Hilbert space. By using a different approach, we generalize some results in [1].

2. Preliminaries and notation. For any set S in an algebra A, let $L_A(S)$ and $R_A(S)$ denote the left and right annihilators of S in A. A Banach algebra A is called a dual algebra if, for every closed left ideal I and every closed right ideal J, we have $I = L_A(R_A(I))$ and $J = R_A(L_A(J))$. For each element $x \in A$, $Sp_A(x)$ will denote the spectrum of x in A.

Let B be a commutative Banach algebra and X_B its carrier space. For each $x \in B$, we let $x \to \hat{x}$ be the Gelfand map on B defined by $\hat{x}(\alpha) = \alpha(x)$ for all $\alpha \in X_B$.

All algebras under consideration are over the complex field C. Definitions not explicitly given are taken from Rickart's book [5].

3. Lemmas. In this section, we give two lemmas which are useful in § 4.

LEMMA 3.1. *Let A be an A^*-algebra. If there exists a maximal commutative *-subalgebra B of A which is finite dimensional, then A is finite dimensional.*

Proof. Since B is finite dimensional, B has an identity element e such that $e = \sum_{i=1}^n e_i$, where $\{e_i, i = 1, \cdots, n\}$ is the maximal orthogonal family of hermitian minimal idempotents in B. We claim that e is an identity element of A. In fact, for each $a \in A$, let $b = a(1 - e)$. It is straightforward to show that $b^*b \in B$ and $b^*b = 0$. Therefore $b = 0$ and so $a = ae$. Similarly we can show that $a = ea$. Hence e is an identity element of A. Clearly $A = \sum_{i=1}^n \sum_{j=1}^n e_iAe_j$. To complete the proof, it suffices now to show that e_iAe_j is one dimensional. We may assume $e_iAe_j \neq (0)$. Then there exists an element $x \in A$ such that $e_ixe_j \neq 0$ and so

$$0 \neq (e_ixe_j)(e_ixe_j)^* = e_ixe_jx^*e_i = \lambda e_i,$$

where $\lambda \in C$. Now for each $y \in A$, we have

$$e_iye_j = \lambda^{-1}e_ixe_jx^*e_iye_j = \lambda^{-1}e_ix(\lambda' e_j) = \lambda^{-1}\lambda' e_ixe_j ,$$

where $\lambda' \in C$. Hence e_iAe_j is one dimensional and this completes the proof.

LEMMA 3.2. *Let A be an A^*-algebra. If the spectrum of every hermitian element of A is finite, then A is finite dimensional.*

Proof. Let B be a maximal commutative $*$-subalgebra of A. It follows easily from [5, p. 111, Theorem (3.1.6)] that every element of B has a finite spectrum and therefore B is finite dimensional (see [3, p. 376, Lemma 7]). Hence by Lemma 3.1, A is finite dimensional.

4. A^*-algebras and $*$-actions.

In this section, the symbol U denotes the open unit disk in the complex. For a given Banach $*$-algebra A, we let A_1^* be the set $\{x \in A: xx^* = x^*x \text{ and } Sp_A(x) \subset U\}$. A function f on U is said to define a $*$-action in A if there exists a mapping $x \rightarrow f'(x)$ of A_1^* into A such that whenever B is a maximal commutative $*$-subalgebra of A and $x \in B \cap A_1^*$, then $f'(x) \in B$ and $\widehat{f'(x)} = f \circ \hat{x}$ on the carrier space X_B of B.

THEOREM 4.1. *Let A be an A^*-algebra. Then A is finite dimensional if and only if any function f on U defines a $*$-action in A.*

Proof. Suppose A is finite dimensional. Let $x \in A_1^*$ and let B be a maximal commutative $*$-subalgebra of A containing x. Then B is a finite dimensional dual B^*-algebra. Hence the carrier space X_B of B consists of a finite number of elements, say $\alpha_1, \cdots, \alpha_n$. Let e_i be the element of B corresponding to the characteristic function of the point $\alpha_i (i = 1, \cdots, n)$. Then for each $x \in B$, we have $x = \sum_{i=1}^n \alpha_i(x)e_i$ (see [4, p. 21]). By [5, p. 111, Theorem (3.1.6.)],

$$Sp_B(x) = \{\alpha_i(x); i = 1, \cdots, n\} .$$

Let f be any function on U. Define

$$f'(x) = \sum_{i=1}^{n} f(\alpha_i(x))e_i .$$

Then it is easy to see that $f'(x) \in B$ and $\widehat{f'(x)} = f \circ \hat{x}$. Therefore f defines a $*$-action in A.

Conversely suppose that any function f on U defines a *-action in A. If A were not finite dimentional, then by Lemma 3.2 there would exist an element x in A_1^* such that $Sp_A(x)$ is infinite. Let B be a maximal commutative *-subalgebra of A containing x. Choose $\lambda_n \in Sp_A(x)$ such that $\lambda_n \neq 0$ $(n = 1,2, \cdots)$. Let f be any function on U such that $f(\lambda_n) = n$. Since f defines a *-action, there exists some $f'(x) \in B$ such that $\widehat{f'(x)} = f \circ \hat{x}$. But this means $n = f(\lambda_n) \in Sp_A(f'(x))$, contradicting the boundedness of $Sp_A(f'(x))$. Hence A is finite dimensional and the proof is complete.

THEOREM 4.2. *Let A be an infinite dimensional dual A^*-algebra which is a dense two-sided ideal of a B^*-algebra. If a function f on U defines a *-action in A, then f is continuous at* 0 *and* $f(0) = 0$.

Rroof. Let B be a maximal commutative *-subalgebra of A. By [4, p. 31, Theorem 19], B is a dual algebra and so its carrier space X_B is discrete. For each $\alpha \in X_B$, let e_α be the element of B corresponding to the characteristic function of α. Then $\{e_\alpha: \alpha \in X_B\}$ is a maximal orthogonal family of hermitian minimal idempotents in A. By Lemma 3.1, B is infinite dimensional and so X_B is infinite. Therefore we can choose a countable subset $\{\alpha_n\}$ of X_B such that the complement $\{\alpha_n\}'$ of $\{\alpha_n\}$ in X_B is infinite.

Let $\{a_n\}$ be a sequence in U such that $a_n \rightarrow 0$. We want to show $f(a_n) \rightarrow f(0) = 0$. By passing to a subsequence, we can assume that $|a_n| \leqq (n^2 \| e_{\alpha_n} \|)^{-1}$. Then $x = \sum_{n=1}^{\infty} e_n e_{\alpha_n}$ is defined in B. Clearly $x \in A_1^*$. Hence there exists some $f'(x) \in B$ such that $\widehat{f'(x)} = f \circ \hat{x}$ on X_B. By [4, p. 30, Theorem 16], we have

$$f'(x) = \sum_{\alpha} e_\alpha f'(x) e_\alpha = \sum_{\alpha} \alpha(f'(x)) e_\alpha \, . \tag{4.1}$$

Therefore $\alpha(f'(x)) \rightarrow 0$. Since $\alpha_n(x) = a_n$, we have $f(a_n) = \alpha_n(f'(x))$. Thus it follows that $f(a_n) \rightarrow 0$ as $n \rightarrow \infty$. For each $\alpha \in \{\alpha_n\}'$, $\alpha(x) = 0$ and so $\alpha(f'(x)) = f(\alpha(x)) = f(0)$. Since $\{\alpha_n\}'$ is infinite, it follows easily from (4.1) that $\alpha(f'(x)) = 0$ for all $\alpha \in \{\alpha_n\}'$. Hence $f(0) = 0$ and so f is continuous at 0. This completes the proof.

Theorem 4.2 is a generalization of [1, p. 668, Proposition 5.1], since $Cp(1 \leqq p < \infty)$ and their *-subalgebras are dual A^*-algebras which are dense two-sided ideals of their completions in the auxiliary norm (see [6]).

We remark that the converse of Theorem 4.2 does not hold as is shown by the following example.

EXAMPLE. Let A be an infinite dimensional proper H^*-algebra. Then A is a dual A^*-algebra which is a dense two-sided ideal of its completion in an auxiliary norm (see [4, p. 31]). Let B be a maximal commutative *-subalgebra of A and let $\{e_\alpha : \alpha \in X_B\}$ be the maximal orthogonal family of hermitian minimal idempotents given in the proof of Theorem 4.2. Let $\{e_{\alpha_n} : \alpha_n \in X_B\}$ be a countable subset of $\{e_\alpha : \alpha \in X_B\}$ and let $a_n = (n \| e_{\alpha_n} \|)^{-1}$. Then $x = \sum_{n=1}^{\infty} a_n e_{\alpha_n}$ is defined in B and $\| x \|^2 = \sum_{n=1}^{\infty} n^{-2}$. Define a function f on U by $f(z) = (\sqrt{n} \, \| e_{\alpha_n} \|)^{-1}$ if $z = a_n$ and $f(z) = 0$ otherwise. Then f is continuous at 0. If f defines a *-action in A, then there exists an element $f'(x) \in B$ such that $\widehat{f'(x)} = f \circ \hat{x}$. But

$$\| f'(x) \|^2 = \sum_{n=1}^{\infty} | f(a_n) |^2 \| e_{\alpha_n} \|^2 = \sum_{n=1}^{\infty} n^{-1} .$$

This is a contradiction. Therefore f does not define a *-action in A.

THEOREM 4.3. *Let A be an infinite dimensional dual B^*-algebra. Then a function f on U defines a *-action in A if and only if f is continuous at* 0 *and* $f(0) = 0$.

Proof. Suppose f is continuous at 0 and $f(0) = 0$. Let $x \in A_1^*$ and let B be a maximal commutative *-subalgebra of A containing x. By the proof of Theorem 4.2, $x = \sum_{n=1}^{\infty} \alpha_n(x) e_{\alpha_n}$, where $\alpha_n \in X_B$ and e_{α_n} is the element of B corresponding to the characteristic function of α_n. Since $\alpha_n(x) \to 0$, $f(\alpha_n(x)) \to 0$. For any two positive integers m, $n (m \leqq n)$, it follows easily from the commutativity of B that

$$\left\| \sum_{i=m}^{n} f(\alpha_i(x)) e_{\alpha_i} \right\| = \max \{ | f(\alpha_i(x)) | : i = m, \cdots, n \} .$$

Therefore $\sum_{n=1}^{\infty} f(\alpha_n(x)) e_{\alpha_n}$ is defined in B. Now let $f'(x) = \sum_{n=1}^{\infty} f(\alpha_n(x)) e_{\alpha_n}$. Then $\widehat{f'(x)} = f \circ \hat{x}$. Hence f defines a *-action in A. The converse of the theorem follows from Theorem 4.2 and the proof is complete.

Since the algebra of all completely continuous operators on a Hilbert space is a dual B^*-algebra, Theorem 4.3 generalizes [1, p. 668, Theorem 5.2].

THEOREM 4.4. *Let A be an infinite dimensional commutative dual B^*-algebra and f a function on U. Then f defines a continuous action in A (see* [2, p. 109, *Definition* 5.1]) *if and only if f is a continuous function on U and* $f(0) = 0$.

Proof. Suppose f is continuous and $f(0) = 0$. Then by Theorem 4.3, f defines an action in A. Let x_n and $x \in A_1^*$ such that $x_n \to x$ in A. By the proof of Theorem 4.2, we have

$$x = \sum_{\alpha} \alpha(x_n) e_\alpha \quad \text{and} \quad x = \sum_{\alpha} \alpha(x) e_\alpha ,$$

where $\{e_\alpha : \alpha \in X_A\}$ is the maximal orthogonal family of hermitian minimal idempotents in A. Since A is commutative, we have

$$||x_n - x|| = \sup\{|\alpha(x_n) - \alpha(x)| : \alpha \in X_A\}$$

and

$$||f(x_n) - f(x)|| = \sup\{|f(\alpha(x_n)) - f(\alpha(x))| : \alpha \in X_A\} .$$

Therefore it is now easy to see that $f(x_n) \to f(x)$ in A. Hence f defines a continuous action in A. The converse of the theorem follows from [2, p. 109, Proposition 5.2] and Theorem 4.3.

REMARK. If A is noncommutative, then Theorem 4.4 is not true as is shown in [2, p. 110, Example 5.3].

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