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A TWO SIDED APPROXIMATION THEOREM FOR 2-SPHERES

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The side approximation theorem proved by R. H. Bing and later improved by F. M. Lister states that a sphere S topologically embedded in Euclidean three space can be ε-approximated with polyhedral spheres $g(S)$ and $h(S)$ such that $g(S-\cup G_i)\subset \operatorname{Int} S$, $g(G_i)\cap S\subset G_i$, $h(S-\cup H_i)\subset \operatorname{Ext} S$, and $h(H_i)\cap S\subset H_i$ where $\{G_i\}$ and $\{H_i\}$ are respectively finite collections of disjoint ε-disks in S. In this article the theorem is strengthened by showing that the sets $\cup G_i$ and $\cup H_i$ may also be taken to be disjoint.

The theorem is a generalization of R. H. Bing's side approximation theorem [2], [3] and is used [7], [8] to distinguish certain decomposition spaces from 3-manifolds.

The proof is divided into two main parts. In §2, Theorem 2.1 is reduced to Corollary 6.1 of §6 using techniques that are consequences of Bing's side approximation theorem. The proof of Corollary 6.1 is the main topic of this paper. The proof depends on Lemma 5.15 and Theorem 5.16 of §5 which give the combinatorial structure of collections of components associated with the general position of two 2-spheres in a 3-manifold. The proofs of Lemma 5.15 and Theorem 5.16 depend in turn on the concepts of separation complexes and winding functions introduced, respectively, in §§3 and 4.

2. Reduction to geometry. The main result of this paper is Theorem 2.1 stated below. The purpose of this section is to reduce the proof of Theorem 2.1 to Corollary 6.1 of §6.

THEOREM 2.1. *If S is a 2-sphere in E^3 and ε is a positive number then there exists a finite collection $\{G_1,\cdots,G_n,H_1,\cdots,H_n\}$ of disjoint ε-disks in S and ε-homeomorphisms $g,h\colon S\to E^3$ such that*

(2.1.1) $g(S-\cup \operatorname{Int} G_i)\subset \operatorname{Int} S$,

(2.1.2) $g(G_i)\cap S\subset \operatorname{Int} G_i$,

(2.1.3) $h(S-\cup \operatorname{Int} H_i)\subset \operatorname{Ext} S$, *and*

(2.1.4) $h(H_i)\cap S\subset \operatorname{Int} H_i$.

Furthermore, $g(S)$ and $h(S)$ may be chosen to be polyhedral and disjoint.

Proof. It follows from F. M. Lister's approximation theorem [9] that there exists a disjoint collection of $(\varepsilon/3)$-disks $C_1,\cdots,C_n$ in S and $(\varepsilon/3)$-homeomorphisms $g_0,h_0\colon S\to E^3$ such that $g_0(S-\cup \operatorname{Int} C_i)\subset$

$\operatorname{Int} S$, $h_0(S - \cup \operatorname{Int} C_i) \subset \operatorname{Ext} S$, $g_0(C_i) \cap S \subset \operatorname{Int} C_i$, $h_0(C_i) \cap S \subset \operatorname{Int} C_i$, and $g_0(S)$, $h_0(S)$ are polyhedral. Let $V_1, \cdots, V_n$ be a collection of disjoint open ε-subsets of E^3 such that $C_i \cup g_0(C_i) \cup h_0(C_i) \subset V_i$ There is no loss in generality [4] in assuming that $\operatorname{Bd} C_i$ is tame.

The required disks G_i, H_i will be disjoint subdisks of C_i and the required homeomorphisms g, h will have the properties that

$$g \mid S - \cup C_i = g_0 \mid S - \cup C_i,\ g(C_i) \subset V_i,\ h \mid S - \cup C_i = h_0 \mid S - \cup C_i$$

and $h(C_i) \subset V_i$. All of the adjusting is done in the V_i's one-at-a-time and independently. Let i be fixed and to simplify notation let $C = C_i$, and $V = V_i$. Let $E_0 = h_0(C)$ and $E_1 = g_0(C)$. By standard scissor-and-paste techniques we may assume that $E_0 \cap E_1 = \emptyset$.

By the polyhedral approximation theorem [1] we may polyhedrally approximate C to obtain a disk E' such that $E' \cap (S - \operatorname{Int} C) = \operatorname{Bd} E' = \operatorname{Bd} C$, $E' \subset V$, $E' \cap h_0(S) \subset \operatorname{Int} E_0$, and $E' \cap g_0(S) \subset \operatorname{Int} E_1$. We may assume that E' and $E_0 \cup E_1$ are in general position. Let E'' be the component of $E' - (E_0 \cup E_1)$ that contains $\operatorname{Bd} E'$. By filling in the holes of E'' with disks in $E_0 \cup E_1$ pushed slightly to the sides of $E_0 \cup E_1$, we obtain a polyhedral disk E such that $E \cap (S - \operatorname{Int} C) = \operatorname{Bd} E = \operatorname{Bd} C$, $E \subset V$, and $E \cap h_0(S) = E \cap g_0(S) = \emptyset$.

Let $\delta = (1/9)\rho(E, E_0 \cup E_1)$. By [5, Theorem 9.1] there exists a tame Sierpinski curve $X \subset C$ such that $\operatorname{Bd} C \subset X$ and the components $X_1, X_2, \cdots$ of $C - X$ each have diameter less than δ. By the polyhedral approximation theorem [1], there exist disks Y_i such that $\operatorname{Bd} Y_i = \operatorname{Bd} X_i$, $D = X \cup (\cup Y_i)$ is a disk, $\operatorname{Diam} Y_i < \delta$, $Y_i \subset V$, $X_j \cap Y_i = \emptyset$ if $i \neq j$, and Y_i is locally polyhedral modulo $\operatorname{Bd} Y_i$. It follows from [5] that D is tame. Keeping points of $E^3 - V$, $g_0(S) - \operatorname{Int} E_1$, $S - \operatorname{Int} E$, and $h_0(S) - \operatorname{Int} E_0$ fixed and moving no point as far as δ, we first move $E_0 \cup E \cup E_1$ into general position with respect to D with $\operatorname{Bd} E = \operatorname{Bd} D$ and then with a move "parallel" to D, we push $((E_0 \cup E \cup E_1) \cap D) - \operatorname{Bd} E$ to the inaccessible part of X (see [6] for more details).

The preceding adjustments enable us assume without loss of generality that E_0, E, E_1 are polyhedral and disjoint, $E_0 \cup E \cup E_1$ is in general position with respect to D with $((E_0 \cup E \cup E_1) \cap D) - \operatorname{Bd} E$ in the inaccessible part of X and

(2.2) no $\operatorname{Cl} X_i$ intersects more than one of E_0, E, E_1.

In order to apply Corollary 6.1 we add the ideal point ∞ to E^3, let T be the 2-sphere $(S - C) \cup D$, U_i be the component of $S^3 - T$ containing $\operatorname{Bd} E_i$, and let E, E_0, E_1 of Corollary 6.1 be as above. Let D_0, D_1 be the singular disks of the conclusion of Corollary 6.1.

Since $(E_0 \cup E_1) \cap D$ lies in the inaccessible part of X there is a natural open disk-with-holes B^* in S associated with each $B \in \mathscr{D}_i$; namely, let B^* be obtained from B by replacing each $Y_j \subset B$ with $\mathrm{Cl}\, X_j$. If $\mathscr{D}_i^* = \{B^* | B \in \mathscr{D}_i\}$ then $D_i^* = \cup \{\mathrm{Cl}\, B | B \in \mathscr{D}_i^* \cup \mathscr{E}_i\}$ are clearly singular disks, and $D_0^* \cap D_1^* = \varnothing$ since $D_0 \cap D_1 = \varnothing$ and each X_j intersects at most one of E_0 and E_1. Some of the X_j's may pass through elements of $\mathscr{E}_i$; thus, D_i^* may not lie in the closure of a complementary domain of S. In the next paragraph we adjust the D_i^*'s so that each does lie in the closure of a complementary domain of S while retaining the property $D_0^* \cap D_1^* = \varnothing$. We first observe that no element of $\mathscr{D}_{1-i}^*$ intersects an element of $\mathscr{E}_i$. For suppose that $X_j \subset B^* \in \mathscr{D}_{1-i}^*$, $A \in \mathscr{E}_i$ and $X_j \cap A \neq \varnothing$ then, since X_j is connected and $X_j \cap Y_k = \varnothing$ if $k \neq j$, there exists a component K of $X_j \cap U_i$ such that $K \cap A \neq \varnothing$ and $Y_j \cap \mathrm{Cl}\, K \neq \varnothing$. Since $Y_j \subset B$ and E separates A from B in $\mathrm{Cl}\, U_i$ by (6.1.2), $\mathrm{Cl}\, K$ intersects E. Thus, $\mathrm{Cl}\, X_j$ intersects both E and E_i and this contradicts (2.2).

For $i = 0, 1$, let f_1^* be a map on E_i with the property that $f_1^*(E_i) = D_1^*$ and $f_i^* | \mathrm{Bd} E_i = 1$. Let E_0^* be the component of

$$E_0 \cap (f_0^*)^{-1}(\mathrm{Cl}\,(\mathrm{Ext}\, S) \cap D_0^*)$$

that contains $\mathrm{Bd} E_0$ and let E_1^* be the component of

$$E_1 \cap (f_0^*)^{-1}(\mathrm{Cl}\,(\mathrm{Int}\, S) \cap D_1^*)$$

that contains $\mathrm{Bd} E_1$. By the observations of the previous paragraph there exist subsequences $\{X_k^0\}$, $\{X_j^1\}$, of $\{X_i\}$ such that $\mathrm{Cl}\,(\cup X_k^0) \cap \mathrm{Cl}(\cup X_j^1) = \varnothing$ and $f_0^*((\mathrm{Bd} E_0^*) - \mathrm{Bd} E_0) \subset \mathrm{Cl}(\cup X_k^0)$, $f_1^*((\mathrm{Bd} E_1^* - \mathrm{Bd} E_0) \subset \mathrm{Cl}\,(\cup X_j^1)$. Also, if C_0 is a component of $E_0 - E_0^*$ then there exists a unique X_k^0 such that $f_0^*(\mathrm{Bd} C_0) \subset X_k^0$. Let $E_0^k = \mathrm{Cl}\,(\cup \{C_0 | f_0^*(\mathrm{Bd} C_0) \subset X_k^0\})$. We apply the Tietze-Extension-Theorem to normal space E_0^k, closed set $\mathrm{Bd} E_0^k$, map $f_0^* | \mathrm{Bd} E_0^k$, and disk $\mathrm{Cl}\, X_k^0$ to extend $f_0^* | E_0^*$ to $E_0^* \cup E_0^k$. Since $\{\mathrm{Cl}\, X_k^0\}$ is a null-sequence, extending $f_0^* | E_0^*$ to each E_0^k in the above manner and piecing together the resulting maps yeilds a map $f_0'\colon E_0 \to \mathrm{Cl}\,(\mathrm{Ext}\,\mathrm{S})$. In the same way, we construct a map $f_1'\colon E_1 \to \mathrm{Cl}\,(\mathrm{Int}\, S)$. The maps $f_0'\colon E_0 \to \mathrm{Cl}\,(\mathrm{Ext}\, S), f_1'\colon E_1 \to \mathrm{Cl}\,(\mathrm{Int}\, S)$ have the following properties $f_i' | \mathrm{Bd} E_i = 1$, $f_0'(E_0 - \mathrm{Bd} E_0) \cap h_0(S - \mathrm{Int}\, C) = \varnothing$, $f_1'(E_1 - \mathrm{Bd} E_1) \cap g_0(S - \mathrm{Int}\, C) = \varnothing, f_i'(E_i) \subset V$, and most importantly $f_0'(E_0) \cap f_1'(E_1) = \varnothing$.

By the results of [5], for each positive number δ there exist δ-maps $\alpha\colon \mathrm{Cl}\,(\mathrm{Int}\, S) \to \mathrm{Cl}\,(\mathrm{Int}\, S)$, $\beta\colon \mathrm{Cl}\,(\mathrm{Ext}\, \delta) \to \mathrm{Cl}\,(\mathrm{Ext}\, S)$ such that $S \cap \alpha(\mathrm{Cl}\,(\mathrm{Int}\, S))$, $S \cap \beta(\mathrm{Cl}\,(\mathrm{Int}\, S))$ are 0-dimensional and $\alpha | (\mathrm{Int}\, S) - N = 1$, $\beta | (\mathrm{Ext}\, S) - N = 1$ where N is an arbitrary neighborhood of S. By choosing δ and N appropriately, in particular δ less than $(1/2)\rho(f_0'(E_0), f_1'(E_1))$, we select maps α, β so that $\alpha f_1'(E_1) \subset \mathrm{Cl}\,(\mathrm{Int}\, S)$, $\alpha f_1' | \mathrm{Bd} E_1 = 1$, $\alpha f_1'(E_1 - \mathrm{Bd} E_1) \cap g_0(S - \mathrm{Int}\, C) = \varnothing$, $\alpha f_1'(E_1) \subset V$, $\beta f_0'(E_0) \subset \mathrm{Cl}\,(\mathrm{Ext}\, S)$,

$\beta f_0'|\mathrm{Bd}E_0 = 1$, $\beta f_0'(E_0 - \mathrm{Bd}E_0) \cap h_0(S - \mathrm{Int}\, C) = \varnothing$, $\beta f_0'(E_0) \subset V$, and $\alpha f_0'(E_1) \cap S$, $\beta f_0'(E_0) \cap S$ are disjoint compact 0-dimensional subsets of $\mathrm{Int}\, C$.

Let G, H be disjoint disks in $\mathrm{Int}\, C$ such that $\alpha f_1'(E_1) \cap S \subset \mathrm{Int}\, G$ and $\beta f_0'(E_0) \cap S \subset \mathrm{Int}\, H$. We apply Dehn's lemma [10] to each of $\alpha f_1'(E_1)$ and $\beta f_0'(E_0)$ in a small neighborhood and each to obtain real disks E_1' and E_0', respectively. We obtain the maps g, h of the conclusion by adjusting g_0, h_0 in V so that $g_0(G) = E_1'$ and $h_0(H) = E_0'$.

3. Separation complexes. In §2 we reduced Theorem 2.1 to Corollary 6.1 of Theorem 5.16 in §§6 and 5, respectively. In this section we develop the concept of a separation complex which is used extensively in §§4, 5, and 6. We begin by reminding the reader of the definition of a collar and then defining a separation complex.

DEFINITION 3.1. If M is a PL manifold and N is a PL manifold in $\mathrm{Bd}M$ then a *collar of N in M* is a PL embedding h of $N \times I$ into M such that $h(y \times 0) = y$, and $h(N \times (0, 1]) \subset \mathrm{Int}\, M$.

DEFINITION 3.2. If N is a compact, connected, orientable 2-manifold in S^3 with nonempty boundary and U is a side of N (i.e., U is the interior of a compact, connected 3-manifold M in S^3 such that $N \subset \mathrm{Bd}M$) then a *collar of N to the U side of N* is a PL embedding h of $N \times I$ into $\mathrm{Cl}\, U$ such that $h(y \times 0) = y$, and $h(N \times (0, 1]) \subset U$.

DEFINITION 3.3. A *separation complex* is a finite collection $K = K_1 \cup K_2 \cup K_3$ of sets in S^3 such that

(3.3.1) K_1 is a collection of disjoint PL simple closed curves, called *separation 1-simplices*,

(3.3.2) K_2 is a collection of compact, connected, orientable, PL 2-manifolds with nonempty boundary, call *separation 2-simplices*, such that

(3.3.2.1) if $\sigma \in K_2$ and $\dot\sigma$ is the collection of separation 1-simplices of K intersecting σ, then $\mathrm{Bd}\sigma = \bigcup \{\tau \mid \tau \in \dot\sigma\}$ and each element of $\dot\sigma$ is called a *1-face* of σ,

(3.3.2.2) if $\sigma_1, \sigma_2 \in K_2$ then $\sigma_1 \cap \sigma_2$ is a union of faces of each,

(3.3.3) K_3 is a collection of compact, connected, PL 3-manifolds with nonempty boundary, called *separation 3-simplices*, such that

(3.3.3.1) if $\sigma \in K_3$ and $\dot\sigma$ is the collection of separation 2-simplices of K intersecting σ, then $\mathrm{Bd}\sigma = \bigcup \{\tau \mid \tau \in \dot\sigma\}$ and each element τ of $\dot\sigma$ is called a *2-face* of σ; each separation 1-simplex which intersects σ lies in $\mathrm{Bd}\sigma$

and is called a *1-face* of σ,

(3.3.3.2) if $\sigma_1, \sigma_2 \in K_3$ then $\sigma_1 \cap \sigma_2$ is a union of 1 and 2 faces of each.

DEFINITION 3.4. A *subcomplex* of a separation complex K is a subcollection $L \subset K$ that is also a separation complex.

LEMMA 3.5. *If K is a separation complex then $L \subset K$ is a subcomplex of K if and only if $\sigma \in L$ implies $\dot{\sigma} \subset L$.*

NOTATION. If K is a separation complex then $|K| = \bigcup_{\sigma \in K} \sigma$, and if $\sigma \in K$ then $\bar{\sigma}$ denotes the subcomplex of K consisting of σ and its faces.

DEFINITION 3.6. If $K = K_1 \cup K_2 \cup K_3$, $L = L_1 \cup L_2 \cup L_3$ are separation complexes, then a function $f: |K| \to |L|$ is a *separation isomorphism from K to L* if f is one-to-one onto, $\sigma \in K_i$ implies $f(\sigma) \in L_i$, and $\sigma \in K$ implies $f(\mathrm{Bd}\sigma) = \mathrm{Bd} f(\sigma)$. Note that the function $f | \sigma \in K$ need not be a homeomorphism or even continuous. In general σ and $f(\sigma)$ are necessarily homeomorphic only if $\sigma \in K_1$, in which case they are both simple closed curves.

DEFINITION 3.7. If K, K' are separation complexes then K' is a *subdivision of K* if $|K'| = |K|$ and each element of K is a union of elements of K'.

Suppose K, L are separation complexes, f is a separation isomorphism from K to L, and K' is a subdivision of K. Does there exist a subdivision L' of L such that K' and L' are separation isomorphic? In general the answer is no as Figure 1 indicates. However, for suitably restricted subdivisions K' the answer is yes.

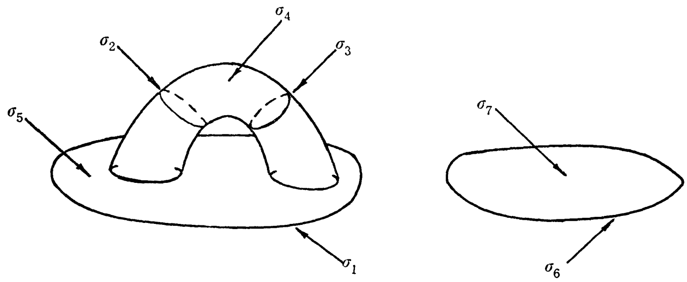
DEFINITION 3.8. If $K = K_1 \cup K_2 \cup K_3$ is a separation complex then a subdivision $K' = K_1' \cup K_2' \cup K_3'$ of K is a *separation subdivision of K* if

(3.8.1) if $\sigma \in K_1'$ then there exists $\tau \in K_2$ such that $\sigma \in \dot{\tau}$ or $\sigma \subset \mathrm{Int}\,\tau$ and separates τ,

(3.8.2) if $\sigma \in K_2'$ then (a) there exists a $\tau \in K_2$ such that $\sigma \subset \tau$ or (b) there exists a $\tau \in K_3$ such that $\sigma \subset \tau$, $\sigma \cap \mathrm{Bd}\tau = \mathrm{Bd}\sigma$, and σ separates τ.

One easily verifies the equivalence of Definition 3.8 with the following:

DEFINITION 3.9. If $K = K_1 \cup K_2 \cup K_3$ is a separation complex, then a subdivision $K' = K_1' \cup K_2' \cup K_3'$ is an *elementary separation subdivision* of K if there are separation simplices $\sigma \in K_i'$ ($i = 1$ or 2),



$K' = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$ $L = \{\sigma_6, \sigma_7\}$
$K = \{\sigma_1, \sigma_4 \cup \sigma_5\}$

FIGURE 1

$\tau_1, \tau_2 \in K'_{i+1}$, and $\tau \in K_{i+1}$ such that

(3.9.1) $K' = (K - \{\tau\}) \cup \{\tau_1, \tau_2, \sigma\}$

(3.9.2) $\tau = \tau_1 \cup \tau_2$, and

(3.9.3) $\sigma = \tau_1 \cap \tau_2$.

(Note that if $i = 2$, then each face of σ lies not only in K'_1 but also in K_1.) A subdivision K'' of K is a *separation subdivision* if there is a sequence $K = K_0, K_1, \cdots, K_n = K''$ such that, for each $i > 0$, K_i is an elementary separation subdivision of K_{i-1}.

REMARK. The conditions of Definition 3.8 are often easier to check then those of Definition 3.9. On the other hand, the conditions of Definition 3.9 are easier to use in proofs about separation subdivisions (e.g., see the proof of Lemma 3.10). Separation subdivisions will be obtained in this paper by the methods outlined in Lemmas 3.10, 3.13, and 3.14.

LEMMA 3.10. *Suppose $K = K_1 \cup K_2 \cup K_3$, $L = L_1 \cup L_2 \cup L_3$ are separation complexes, f is a separation isomorphism from K to L, and $K' = K'_1 \cup K'_2 \cup K'_3$ is a separation subdivision of K. Then there exists a separation subdivision $L' = L'_1 \cup L'_2 \cup L'_3$ of L and a separation isomorphism f' of K' to L' such that $f'(\sigma) = f(\sigma)$ if $\sigma \in K$.*

Proof. By induction on the number of elementary subdivisions required to change K to K', it suffices to prove Lemma 3.10 when K' is an elementary separation subdivision of K. Let $\sigma,$, τ_1, τ_2, and τ be as (3.9) with (Case A) $\sigma \in K'_1$ or (Case B) $\sigma \in K'_2$.

Case A. Let M be a collar of $f((\mathrm{Bd}\tau_1) - \sigma)$ in $f(\tau)$ and let M' be a 2-manifold-with-boundary in $f(\tau)$ obtained by connecting up the components of M with 2-dimensional tubes in $\mathrm{Int} f(\tau)$. The boundary of M' is $f((\mathrm{Bd}\tau_1) - \sigma) \cup J$ where J is a PL simple closed curve in $\mathrm{Int} f(\tau)$. Let $L' = L'_1 \cup L'_2 \cup L'_3$ be defined by $L'_1 = L_1 \cup \{J\}$, $L'_2 =$

$(L_2 - \{f(\tau)\}) \cup \{M', \mathrm{Cl}\,(f(\tau) - M')\}$, and $L_3' = L_3$. Let f' be defined by $f' \,|\, |K| - \mathrm{Int}\,\tau = f \,|\, |K| - \mathrm{Int}\,\tau$, $f' | \sigma$ is any one-to-one function onto J, $f' | \mathrm{Int}\,\tau_1$ is any one-to-one function onto $\mathrm{Int}\,M'$, and $f' | \mathrm{Int}\,\tau_2$ is any one-to-one function onto $(\mathrm{Int}\,f(\tau)) - M'$.

Case B. This case is handled in much the same way as Case A. Let M be a collar of $f((\mathrm{Bd}\tau_1) - \mathrm{Int}\,\sigma)$ in $f(\tau)$ and let M' be a 3-manifold-with-boundary in $f(\tau)$ obtained by connecting up the components of M with 3-dimensional tubes in $\mathrm{Int}\,f(\tau)$. The boundary of M' is $f((\mathrm{Bd}\tau_1) - \sigma) \cup J$ where J is a connected PL 2-manifold in $f(\tau)$ such that $J \cap \mathrm{Bd}f(\tau) = \mathrm{Bd}J = f(\mathrm{Bd}\sigma)$. Let $L' = L_1' \cup L_2' \cup L_3'$ be defined by $L_1' = L_1$, $L_2' = L_2 \cup \{J\}$, and $L_3' = (L_3 - \{f(\tau)\}) \cup \{M', \mathrm{Cl}\,(f(\tau) - M')\}$. Let f' be defined by $f' \,|\, |K| - \mathrm{Int}\,\tau = f \,|\, |K| - \mathrm{Int}\,\tau$, $f' | \mathrm{Int}\,\sigma$ is any one-to-one function onto $\mathrm{Int}\,J$, $f' | \mathrm{Int}\,\tau_1$ is any one-to-one function onto $\mathrm{Int}\,M'$, and $f' | \mathrm{Int}\,\tau_2$ is any one-to-one function onto $(\mathrm{Int}\,f(\tau)) - M'$.

NOTATION. If K is a separation complex, L subcomplex of K and K' a subdivision of K then L' denotes a subdivision of L induced by K' given by $L' = \{\sigma \in K' \,|\, \sigma \subset |L|\}$. Note that if K' is a separation subdivision of K then L' is separation subdivision of L.

We are greatly indebted to the referee for the notation K mod W in definition 3.12. This concept has simplified our earlier exposition considerably.

DEFINITION 3.11. Let $K = K_1 \cup K_2 \cup K_3$ denote a separation complex in S^3 and W a compact PL 2-manifold-with-boundary in S^3. We shall say that K and W are in general position if the following conditions are satisfied:

(3.11.1) $W \cap |K_1| = \varnothing$.

(3.11.2) If $\tau \in K_2$ and J is a component of $W \cap |\tau|$, then either J is a component of $\mathrm{Bd}\,W$, or J is a simple closed curve in $\mathrm{Int}\,W$ and τ and W locally cross each other at J.

(3.11.3) If $\sigma \in K_3$, then $\mathrm{Bd}\,W \cap \mathrm{Int}\,\sigma = \varnothing$ and each component of $W \cap |\sigma|$ intersects $\mathrm{Bd}\sigma$.

DEFINITION 3.12. If K and W are in general position, then we define K mod $W = K_1' \cup K_2' \cup K_3'$ as follows:

(3.12.1) $K_1' = K_1 \cup \{t \,|\, \text{there exists a } \tau \in K_2 \text{ such that } t \text{ is a component of } W \cap |\tau|\}$.

(3.12.2) $K_2' = \{\mathrm{Cl}\,Y \,|\, \text{there exists a } \sigma \in K_3 \text{ such that } Y \text{ is a component of } W \cap \mathrm{Int}\,\sigma\} \cup \{\mathrm{Cl}\,Y \,|\, \text{there exists a } \tau \in K_2 \text{ such that } Y \text{ is a component of } |\tau| - W\}$.

(3.12.3) $K_3' = \{\mathrm{Cl}\,Z \,|\, \text{there exists a } \sigma \in K_3 \text{ such that } Z \text{ is a component of } |\sigma| - W\}$.

It is not necessarily true that K mod W is even a separation complex (a 2-simplex might intersect the interior of a 3-simplex or a 1-simplex the interior of a 2-simplex). However, we do have the following.

LEMMA 3.13. *Let K denote a separation complex in S^3 and W a compact PL 2-manifold-with-boundary in S^3 such that K and W are in general position. Then K mod W is a separation subdivision of K if and only if the following two conditions are satisfied:*

(3.13.1) If $\tau \in K_2$ *and J is a (nonempty) component of $W \cap |\tau|$, then J separates τ.*

(3.13.2) *If $\sigma \in K_3$ and Y is a (nonempty) component of $W \cap \operatorname{Int} \sigma$, then Cl Y separates σ.*

Proof. First check that K mod W is a separation complex. Conditions (3.8.1) and (3.8.2) follow respectively from (3.13.1) and (3.13.2).

LEMMA 3.14. *Suppose the following are given:*

(3.14.1) *Q is a PL 2-sphere in S^3.*

(3.14.2) *U is a component of $S^3 - Q$.*

(3.14.3) *K is a separation complex such that $|K_1| \subset Q$ and $\cap\{\operatorname{Int} \sigma \mid \sigma \in K_3\} \subset U$.*

(3.14.4) *W is a compact PL 2-manifold-with-boundary such that $W \subset U$ and W and K are in general position* (3.12).

Then K mod *W is a separation subdivision of K if condition* (3.13.1) *is satisfied.*

Proof. We check that (3.13.2) is also satisfied. Let $\sigma \in K_3$, Y be a nonempty component of $W \cap \operatorname{Int} \sigma$, and J be a nonempty component of Bd Y. By (3.13.1) J separates the member τ of K_2 that includes J. Thus, let $\tau(J)$ be the closure of a component of $\tau - J$. By (3.14.3), $(\operatorname{Bd} \tau(J)) - J \subset Q$. For each component K of $(\operatorname{Bd} \tau(J)) - J$ we add to $\tau(J)$ a disk in Q bounded by K, thus we obtain a singular 2-manifold $M(J)$ with boundary J. By (3.14.3), $M(J) \cap \operatorname{Int} \sigma = \emptyset$ for each component J of Bd Y. Thus, since the singular 2-manifold $(\cup\{M(J) \mid J \text{ is a component of Bd } Y\}) \cup Y$ separates Int σ, Y separates Int σ.

4. Winding functions.

In this section we discuss the concept of a winding function which is used extensively in §§5 and 6. We begin by introducding notation that we use later for labeling abstract trees.

DEFINITION 4.1. A *tree labeling system* is a collection Γ of finite sequences $(a_1, a_2, \cdots, a_n)$ of nonnegative integers such that

(4.1.1) $(0) \in \Gamma$ and is the only one element sequence in Γ,

(4.1.2) if $(a_1, a_2, \cdots, a_n) \in \Gamma$ then $(a_1, a_2, \cdots, a_{n-1}) \in \Gamma$ if $n \neq 1$, and $(a_1, a_2, \cdots a_{n-1}, a_n - 1) \in \Gamma$ if $a_n \neq 0$.

(4.1.3) A *maximal* element of Γ is finite sequence $(a_1, a_2, \cdots, a_n) \in \Gamma$ such that there does not exist a nonnegative integer a such that $(a_1, a_2, \cdots a_n, a) \in \Gamma$.

(4.1.4) the *lexicographical ordering* of Γ is 0, 00, 01, 02, $\cdots$, 000, 001, $\cdots$, etc.

(4.1.5) a *tree labeling system* for set B is a one-to-one function from Γ onto B.

NOTATION. If Γ is a tree labeling system for set B then we use lower case greek letters to refer to elements of Γ and denote the image of $\alpha \in \Gamma$ in B by $b_\alpha \in B$. If $\alpha = (a_1, a_2 \cdots, a_n) \in \Gamma$ and $(i_1, \cdots, i_j)$ is a sequence of nonnegative integers such that $(a_1, \cdots, a_n, i_1, \cdots, i_j) \in \Gamma$ then we write $a_1 a_2 \cdots a_n \in \Gamma$, $\alpha i_1 i_2 \cdots i_j \in \Gamma$, respectively. If $\alpha = (a_1, a_2, \cdots, a_n) \in \Gamma$ then $|\alpha| = 0$ if n is an even integer and $|\alpha| = 1$ if n is an odd integer.

In items (4.2) through (4.9), R is a PL 2-sphere in S^3, H is a disk-with-holes in R, V is a component of $S^3 - R$, and $M = h(H \times I)$ is a collar of H in $\mathrm{Cl}\, V$.

DEFINITION 4.2. A *corner* of M is $h(E \times [0, t])$ where E is a collar of $\mathrm{Bd} H$ in H and $0 < t < 1$.

DEFINITION 4.3. A compact, orientable, PL 2-manifold S in M *partitions* M if for each component Y of S we have $Y \cap \mathrm{Bd} M = Y \cap ((\mathrm{Bd} M) - H) = \mathrm{Bd}\, Y \neq \emptyset$.

The next lemma shows that a partition of M induces a special type of separation subdivision of the separation complex $\bar{M} = \{J | J$ is a component $\mathrm{Bd} H\} \cup \{H, \mathrm{Bd} M - \mathrm{Int}\, H\} \cup \{M\}$.

LEMMA 4.4. *If S partitions M then $\bar{M} \bmod S = K_1 \cup K_2 \cup K_3$ is a separation complex having the property that the intersection of each pair of elements of K_3 is either empty or a single component of S. Furthermore, there exists a tree labeling system Γ for K_3 such that $H \subset \mathrm{Bd} \sigma_0$ and $\sigma_\alpha \cap \sigma_\beta$ is nonempty if and only if either $\beta = \alpha i$ or $\alpha = \beta i$ for some nonnegative integer i.*

Proof. Let Y be component of S and J a component of $\mathrm{Bd}\, Y$. Since $\mathrm{Bd} M - \mathrm{Int}\, H$ is a disk-with-holes and $J \subset (\mathrm{Bd} M) - H$, J separates $\mathrm{Bd} M - \mathrm{Int}\, H$. Thus, by Lemma 3.14, $\bar{M} \bmod S$ is a separation complex. The other properties of $\bar{M} \bmod S$ follow by induction on the number of components of S.

DEFINITION 4.5. If S partitions M then we call $\bar{M} \bmod S$ the *separation complex supported by* S *in* M.

DEFINITION 4.6. If R is a PL 2-sphere in S^3, V is a component of $S^3 - R$, H is a disk-with-holes in R, M is a collar of H in $\mathrm{Cl}\, V$, S partitions M, and $K = K_1 \cup K_2 \cup K_3$ is the separation complex supported by S in M then function $P\colon M \to S^3$ is a *winding function with respect to* (M, S, H, R) if

(4.6.1) $P \mid H = 1$ and P is the identity on a corner of M,

(4.6.2) if $\sigma \in K$ then $\{P(\tau) \mid \tau \in \bar{\sigma}\} = \overline{P(\sigma)}$ is a separation complex and $P \mid \sigma$ is a separation isomorphism from $\bar{\sigma}$ to $\overline{P(\sigma)}$,

(4.6.3) if $\sigma \in K_3$ then $P(\sigma - (S \cup H)) \subset S^3 - R$ and $P(\sigma \cap (S \cup H)) \subset R$,

(4.6.4) if $\sigma_0, \sigma_1 \in K_3$ and $\sigma_0 \cap \sigma_1$ is a component of S then $P(\sigma_0 - (S \cup H))$, $P(\sigma_1 - (S \cup H))$ lie in different components of $S^3 - R$, and

(4.6.5) if $\sigma_0, \sigma_1 \in K_3$, $\sigma_0 \neq \sigma_1$, but $P(\sigma_0 - (S \cup H)) \cap P(\sigma_1 - (S \cup H)) \neq \emptyset$ then for $i = 0$, or 1, $P(\sigma_i - (S \cup H)) \subset \mathrm{Int}\, P(\sigma_{1-i})$ and $P(\sigma_i \cap (S \cup H)) \subset \mathrm{Int}\, P(\sigma_{1-i} \cap (S \cup H))$.

We now establish several properties of a winding function.

LEMMA 4.7. *If $P\colon M \to S^3$ is a winding function with respect to (M, S, H, R), $K = K_1 \cup K_2 \cup K_3$ is the separation complex supported by S in M, and Γ is a tree labeling system for K_3 satisfying Lemma 4.4. Then it is impossible to have $P(\sigma_\beta) \subset P(\sigma_\alpha)$ where $\alpha = \beta i_1 \cdots i_n$ for some positive integer n. In particular, $P(\sigma_0) \subset P(\sigma_\alpha)$ implies $\alpha = 0$.*

Proof. Suppose the contrary, then there exist a smallest positive integer n with the property that there exist $\alpha, \beta \in \Gamma$ such that $\alpha = \beta i_1 \cdots i_n$ and $P(\sigma_\beta) \subset P(\sigma_\alpha)$. By (4.6.3), (4.6.4), and (4.6.5) either $P(\sigma_{\beta i_1}) \subset P(\sigma_{\beta i_1 \cdots i_{n-1}})$ or there exists a nonnegative integer i_{n+1} such that $P(\sigma_{\beta i_1}) \subset P(\sigma_{\beta i_1 \cdots i_n i_{n+1}})$. Suppose $P(\sigma_{\beta i_1}) \subset P(\sigma_{\beta i_1 \cdots i_{n-1}})$ then since n is minimal, we must have $n = 2$ and $\sigma_{\beta i_1} = \sigma_{\beta i_1 \cdots i_{n-1}}$. But by (4.6.5) $P(\sigma_\beta \cap \sigma_{\beta i_1}) \subset \mathrm{Int}\, P(\sigma_{\beta i_1 i_2} \cap \sigma_{\beta i_1})$, thus we have distinct 2-faces of $\sigma_{\beta i_1}$ pinched together by P contradicting (4.6.2). Hence we must have $P(\sigma_{\beta i_1}) \subset P(\sigma_{\beta i_1 \cdots i_{n+1}})$.

The above argument is repeated to $P(\sigma_{\beta i_1}) \subset P(\sigma_{\beta i_1 \cdots i_{n+1}})$ to find a nonnegative integer i_{n+2} such that $P(\sigma_{\beta i_1 i_2}) \subset P(\sigma_{\beta i_1 \cdots i_{n+1} i_{n+2}})$, and then repeated over again. Eventually a nonnegative integer i_{n+p} is found such that $P(\sigma_{\beta i_1 i_2 \cdots i_p}) \subset P(\sigma_{\beta i_1 \cdots i_{n+p}})$ and $\beta i_1 \cdots i_{p+n}$ is maximal in Γ. One more application of the argument yields a contradiction.

LEMMA 4.8. *If P is a winding function with respect to (M, S, H, R) then*

(4.8.1) *$H - P(M - H)$ separates $P(M) - P(\mathrm{Bd}M)$ from $S^3 - P(M)$ in $S^3 - P(\mathrm{Bd}M - H)$, and*

(4.8.2) *$P(M)$ contains a collar of $P(\mathrm{Bd}M - \mathrm{Int}\, H)$ to the side of $P(\mathrm{Bd}M - \mathrm{Int}\, H)$ that contains a corner of M.*

Proof. Suppose pq is an arc from $p \in S^3 - P(M)$ to $q \in P(M) - P(\mathrm{Bd}M)$ that lies in $S^3 - P(\mathrm{Bd}M - H)$. Let r be the first point of pq that intersect the compact set $P(M)$. Since r is a boundary point of the 3-manifold $P(\sigma)$ for some $\sigma \in K_3$ and $pq \cap P(\mathrm{Bd}M - H) = \varnothing$ it follows that $r \in \mathrm{Int}\, P(S \cup H) \subset R$. By (4.6.2) and (4.6.4) since r is the first point of intersection with $P(M)$, $r \notin P(S)$. Hence $r \in P(H) - P(S) = H - P(M - H)$ by (4.6.1) and (4.6.3), and the proof of (4.8.1) is complete.

Let Γ be the tree labeling system for K_3 in Lemma 4.4. The collar required in (4.8.2) is constructed in $P(M)$ by induction on the elements of Γ. The set $P(\sigma_0) - (\cup P(\sigma_{0\alpha}))$ contains a collar N_0 of $\mathrm{Cl}((\mathrm{Bd}P(\sigma_0)) - R)$ in $P(\sigma_0)$ such that N_0 contains a corner of M by (4.6.1), (4.6.2) and Lemma 4.7. We may assume that $N_0 \cap P(\sigma_0 \cap S)$ is a collar of $\mathrm{Bd}P(\sigma_0 \cap S)$ in $P(\sigma_0 \cap S)$. Let N_{0i} be a collar of $\mathrm{Cl}\,((\mathrm{Bd}P(\sigma_{0i})) - R)$ in $P(\sigma_{0i}) - (\bigcup_{\alpha \in \Gamma_{0i}} P(\sigma_\alpha))$ where $\alpha \in \Gamma_{0i}$ if and only if $P(\sigma_\alpha) - R \subset \mathrm{Int}\, P(\sigma_{0i})$ by (4.6.2) and (4.6.5). We may assume that $N_{0i} \cap P(\sigma_{0i} \cap S)$ is a collar of $\mathrm{Bd}P(\sigma_{0i} \cap S)$ in $P(\sigma_0 \cap S)$ and $N_0 \cap P(\sigma_0 \cap \sigma_{0i}) = N_{0i} \cap P(\sigma_0 \cap \sigma_{0i})$. The induction step from σ_α to $\sigma_{\alpha i}$ is similar. The required collar is $\cup \{N_\alpha \mid \alpha \in \Gamma\}$.

The next lemma gives a lifting property for winding functions.

LEMMA 4.9. *If $P: M \to S^3$ is a winding function with respect to (M, S, H, R), W is a compact 2-manifold such that each component of W has nonempty boundary, W is in general position with respect to R, $\mathrm{Int}\, W \subset P(M) - P(\mathrm{Bd}M - \mathrm{Int}\, H)$, and $\mathrm{Bd}\, W \subset P(H) - P(S)$. Then there exists a compact PL 2-manifold $\bar{W}$ in M and a function $P': M \to S^3$ such that*

(4.9.1) *$P' \mid H = P \mid H = 1$ and if $K = K_1 \cup K_2 \cup K_3$ is the separation complex supported by S in M then $P'(\sigma) = P(\sigma)$ if $\sigma \in K$,*

(4.9.2) *$\bar{W}$ and K are in general position and if $\bar{W}_0$ is a component of $\bar{W}$ then $\bar{W}_0 \cap \mathrm{Bd}M = \bar{W}_0 \cap H = \mathrm{Bd}\,\bar{W}_0 \neq \varnothing$,*

(4.9.3) *$K' = K \bmod \bar{W} = K_1' \cup K_2' \cup K_3'$ is a separation subdivision of K,*

(4.9.4) *if $\sigma \in K_3$ then $P' \mid \sigma$ is a separation isomorphism from $\bar{\sigma}'$ to $\{P'(\tau) \mid \tau \in \bar{\sigma}'\}$,*

(4.9.5) *$P' \mid \bar{W}$ is a separation isomorphism from $\bar{W} \bmod (S \cup H)$ to $W \bmod R$,*

(4.9.6) *if $\sigma_0, \sigma_1 \in K_3$* and *$\sigma_0 \neq \sigma_1$ but $P(\sigma_0) \subset P(\sigma_1)$ then there exists $\sigma_2 \in K_3'$ such that $\sigma_2 \subset \sigma_1$ and $P(\sigma_0) \subset P'(\sigma_2)$.*

Proof. Let Γ be the tree labeling system for K_3 of Lemma 4.4. For $\sigma \in K_3$ let $\Gamma(\sigma) = \{\alpha \in \Gamma \mid P(\sigma_\alpha)$ is a proper subset of $P(\sigma)\}$ and let $W(\sigma) = W \cap (P(\sigma) - \cup \{P(\sigma_\alpha) \mid \alpha \in \Gamma(\sigma)\})$. It is clear that $W = \cup \{W(\sigma) \mid \sigma \in K_3\}$ and $W(\sigma_\alpha) \cap W(\sigma_\beta) = \emptyset$ if $\sigma_\alpha \cap \sigma_\beta = \emptyset$ by (4.6.5). Also, $W(\sigma_\alpha) \cap W(\sigma_{\alpha i}) = W \cap (P(\sigma_\alpha \cap \sigma_{\alpha i}) - \cup \{P(\sigma_\beta) \mid \beta \in \Gamma(\sigma_\alpha)\}) = W \cap (P(\sigma_\alpha \cap \sigma_{\alpha i}) - \cup \{P(\sigma_\beta) \mid \beta \in \Gamma(\sigma_{\alpha i})\})$ by (4.6.4) and the fact that W is in general position with respect to R. By hypothesis on W, each component $W_i(\sigma)$ of $W(\sigma)$ is a compact 2-manifold with $\emptyset \neq \operatorname{Bd} W_i(\sigma) \subset \operatorname{Int} P(\sigma \cap (S \cup H)) \subset R$. The 2-manifold $W_i(\sigma)$ separates the 3-cell bounded by R and containing $W_i(\sigma)$: consequently, $W_i(\sigma)$ separates $P(\sigma)$. If t is a boundary component of $W_i(\sigma)$ then there exists a component Y of $\sigma \cap (S \cup H)$ such that $t \subset \operatorname{Int} P(Y) \subset R$. Since $P(Y)$ is a disk-with-holes, t separates $P(Y)$. It now follows by Lemma 3.13 that $L(\sigma) = L_1(\sigma) \cup L_2(\sigma) \cup L_3(\sigma) = \overline{P(\sigma)}$ mod $W(\sigma)$ is a separation subdivision of $\{P(\tau) \mid \tau \in \bar{\sigma}\} = \overline{P(\sigma)}$.

We use Lemma 3.10 to lift the separation subdivision $L(\sigma)$ of $\overline{P(\sigma)}$ to $\bar{\sigma}$. By Lemma 3.10 there exists a subdivision $\bar{\sigma}'_0$ of $\bar{\sigma}_0$ and a separation isomorphism $P' \mid \sigma_0$ from $\bar{\sigma}'_0$ to $L(\sigma_0)$ such that $(P' \mid \sigma_0)(\tau) = P(\tau)$ if $\tau \in \bar{\sigma}_0$, and $(P' \mid \sigma_0) \mid H = P \mid H$. Proceeding inductively, we assume that separation subdivision $\bar{\sigma}'_\alpha$ of $\bar{\sigma}_\alpha$ and separation isomorphism $P' \mid \sigma_\alpha$ from $\bar{\sigma}'_\alpha$ to $L(\sigma_\alpha)$ have been defined such that $(P' \mid \sigma_\alpha)(\tau) = P(\tau)$ if $\tau \in \bar{\sigma}_\alpha$. Let i be a nonnegative integer such that $\alpha i \in \Gamma$, as we have observed before $W \cap (P(\sigma_\alpha \cap \sigma_{\alpha i}) - \cup \{P(\sigma_\beta) \mid \beta \in \Gamma(\sigma_\alpha)\} = W \cap (P(\sigma_\alpha \cap \sigma_{\alpha i}) - \cup \{P(\sigma_\beta) \mid \beta \in \Gamma(\sigma_{\alpha i})\})$ so by Lemma 3.10 there exists a separation subdivision $\bar{\sigma}'_{\alpha i}$ of $\bar{\sigma}_{\alpha i}$ and a separation isomorphism $P' \mid \sigma_{\alpha i}$ from $\bar{\sigma}'_{\alpha i}$ to $L(\sigma_{\alpha i})$ such that $(P' \mid \sigma_{\alpha i})(\tau) = P(\tau)$ if $\tau \in \bar{\sigma}_{\alpha i}$ and $(P' \mid \sigma_{\alpha i}) \mid \sigma_\alpha \cap \sigma_{\alpha i} = (P' \mid \sigma_\alpha) \mid \sigma_\alpha \cap \sigma_{\alpha i}$.

Let $P' = \bigcup_{\alpha \in K_3} (P' \mid \sigma)$, $\bar{W} = \bigcup_{\alpha \in K_3} (P' \mid \sigma)^{-1}(W(\sigma))$, and $K' = \bigcup_{\alpha \in K_3} \bar{\sigma}'$. Properties (4.9.1), $\cdots$, (4.9.5) are straight forward consequences of the construction. Property (4.9.6) follows since $P(\sigma_0) \subset P(\sigma_1)$ implies that $P(\sigma_0) \cap W(\sigma_1) = \emptyset$ and the latter implies $P(\sigma_0)$ is included in an element of $L_3(\sigma_1)$ since $P(\sigma_0)$ is connected.

5. **Structure theorems.** The purpose of this section is to prove the combinatorial structure theorems (Lemma 5.15 and Theorem 5.16) for components associated with the general position of two arbitrary PL 2-spheres in E^3. Throughout this section an arbitrary PL 2-sphere R in S^3 is fixed. Also, a choice function μ on the collection of nonempty sets of simple closed curves in R is fixed. Some of the objects defined below depend on R or μ, but to save notation no special attention is called to this fact.

DEFINITION 5.1. Let $\mathscr{S}$ be the collection of all 4-tuples (X, U, q, m) such that

(5.1.1) X is a compact, connected, PL 2-manifold in S^3 without boundary and in general position with respect to R such that if t is a component (simple closed curve) of $R \cap X$ then t separates X,

(5.1.2) U is one of the two components of $S^3 - X$,

(5.1.3) q is a point of $X - R$, and

(5.1.4) m is a component (simple closed curve) of $R \cap X$.

For $a = (X, U, q, m) \in \mathscr{S}$ the existence of the following objects is clear and we use the following notation.

(5.2) A_a is the closure of the component of $X - m$ that does not contain q,

(5.3) V_a is the component of $S^3 - R$ that contains the interior of a collar of m in A_a,

(5.4) B_a is the disk in R bounded by m that contains a collar of m in $R \cap \mathrm{Cl}\, U$,

(5.5) C_a is the closure of the component of $A_a - R$ that contains m in its boundary,

(5.6) D_a is the closure of the component of $B_a - C_a$ that contains m in its boundary,

(5.7) $|a|$ is the number of components of $A_a - R$, and

(5.8) L_a is the collection of all components t of $\mathrm{Bd}C_a - \mathrm{Bd}D_a$ such that $B_b \cap D_a = \varnothing$ where $b = (X, U, q, t)$.

Since C_a separates $\mathrm{Cl}\, V_a$ and R is a 2-sphere, the following lemma is easily established.

LEMMA 5.9 *The sets C_a, D_a, and L_a have the following properties*:

(5.9.1) $\mathrm{Bd}D_a \subset \mathrm{Bd}C_a$,

(5.9.2) *there exists a collar of* $\mathrm{Bd}D_a$ *in* $D_a \cap \mathrm{Cl}\, U$, *and*

(5.9.3) *if* $\mathrm{Bd}C_a - \mathrm{Bd}D_a \neq \varnothing$ *then* $L_a \neq \varnothing$.

The existence of the following objects associated with $a = (X, U, q, m) \in \mathscr{S}$ will be established by induction on $|a|$ in Lemma 5.15. See Figure 2.

(5.10a) H_a is a disk-with-holes in D_a such that

(5.10.1a) $m \subset \mathrm{Bd}H_a$, and $\mathrm{Bd}H_a \subset X \cap R$,

(5.10.2a) there exists a collar of $\mathrm{Bd}H_a$ in $H_a \cap \mathrm{Cl}\, U$,

(5.10.3a) $\mathrm{Bd}H_a$ is the boundary of a compact, connected, 2-manifold J_a in A_a.

(5.11a) $M_a = h(H_a \times I)$ is a collar of H_a in $\mathrm{Cl}\, V_a$ such that

(5.11.1a) $h(\mathrm{Bd}H_a \times I)$ is a collar of $\mathrm{Bd}J_a$ in J_a,

(5.11.2a) M_a intersects a component Y of $X \cap V_a$ if and only if $\mathrm{Cl}\, Y$ intersects H_a,

(5.12a) S_a is a partition of M_a and $K_a = K_1(a) \cup K_2(a) \cup K_3(a)$ is the separation complex supported by S_a in M_a,

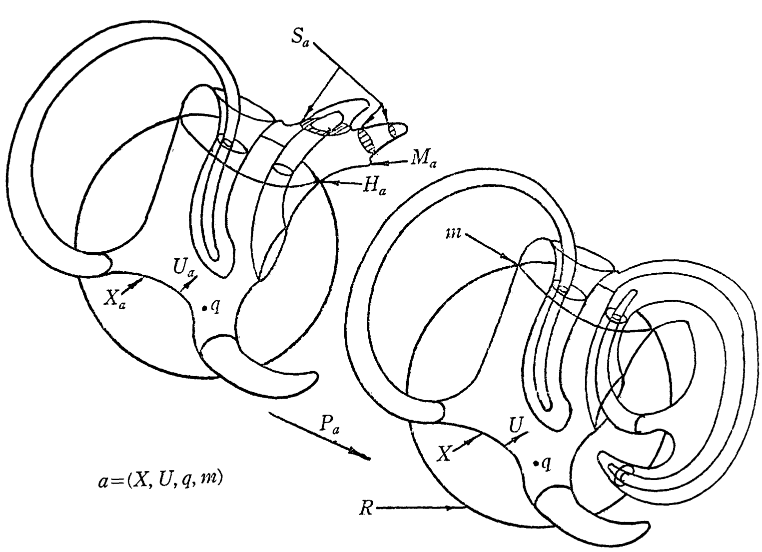


FIGURE 2

(5.13a) X_a is compact, connected, PL 2-manifold in S^3 without boundary and in general position with respect to $R \cup S_a$ such that

(5.13.1a) X_a contains $G_a = (\mathrm{Bd} M_a) - \mathrm{Int}\, H_a$,

(5.13.2a) if t is a component of $(R \cup S_a) \cap X_a$ then t separates X_a,

(5.13.3a) if t is a component of $(X_a - G_a) \cap S_a$ then t separates the component of S_a containing t.

From (5.13.3a), the fact that $(R - H_a) \cup G_a$ is a 2-sphere, and Lemma 3.14 it follows that $K'_a = K_a \bmod (X_a - \mathrm{Int}\, G_a) = K'_1(a) \cup K'_2(a) \cup K'_3(a)$ is a separation subdivision of K_a.

(5.14a) P_a is a function from S^3 to S^3 such that

(5.14.1a) $P_a | S^3 - (M_a - H_a) = 1$,

(5.14.2a) $P_a | M_a$ is a winding function with respect to (M_a, S_a, H_a, R),

(5.14.3a) $P_a | X_a$ is a separation isomorphism from $\bar{X}_a = X_a \bmod (R \cup S_a)$ to $\bar{X} = X \bmod R$.

(5.14.4a) $P_a(G_a) = J_a$,

(5.14.5a) $P_a(M_a) \cap (X_a - M_a) = \emptyset$,

(5.14.6a) if $\sigma \in K_3(a)$ then $P_a | \sigma$ is a separation isomorphism from $\bar{\sigma}'$ to $\{P_a(\tau) \,|\, \tau \in \bar{\sigma}'\}$, and

(5.14.7a) if $\sigma_0, \sigma_1 \in K_3(a)$ and $\sigma_0 \neq \sigma_1$ but $P_a(\sigma_0) \subset P_a(\sigma_1)$ then there exists $\sigma_2 \in K'_3(a)$ such that $\sigma_2 \subset \sigma_1$ and $P_a(\sigma_0) \subset P_a(\sigma_2)$.

LEMMA 5.15. *If* $a = (X, U, q, m) \in \mathscr{S}$ *where* $\mathscr{S}$ *is given by* (5.1) *then there exists a 5-tuple* $(H_a, M_a, S_a, X_a, P_a)$ *that satisfies* (5.10a), (5.11a), $\cdots$, (5.14a).

Proof. The proof is by induction on $|a|$ given by (5.7). We use the notation given by (5.1), $\cdots$, (5.8).

Suppose $a = (X, U, q, m) \in \mathscr{S}$ and $|a| = 1$. It is clear that $C_a = A_a$, $D_a = B_a$ and $\mathrm{Bd}C_a = \mathrm{Bd}D_a$. Let $H_a = D_a$, $J_a = C_a$ and let h be a PL embedding of $H_a \times I$ into $\mathrm{Cl}\, V_a$ such that $h(y \times 0) = y$, $h(H_a \times (0, 1]) \subset V_a$, $h(\mathrm{Bd}H_a \times I)$ is a collar of $\mathrm{Bd}J_a$ in J_a, and $h(H_a \times I)$ intersects a component Y of $X \cap V_a$ if and only if $\mathrm{Cl}\, Y$ intersects H_a. Let $M_a = h(H_a \times I)$ and let $S_a = \varnothing$. A one-to-one function P, given by $P|H_a = 1$, $P(G_a) = J_a$, $P(\mathrm{Int}\, M_a) = \mathrm{Int}\, L$ where L is the 3-manifold bounded by $H_a \cup J_a$ and containing a collar of J_a in $\mathrm{Cl}\, U$, is a trivial winding function with respect to (M_a, S_a, H_a, R). Let $W = X \cap (L - J_a)$ and apply Lemma 4.9 to P and W and let $X_a = \bar{W} \cup (X - L) \cup G_a$, $P_a | M_a = P'$ where $\bar{W}$, P' are as in the conclusion of Lemma 4.9. It is straight forward to check that $(H_a, M_a, S_a, X_a, P_a)$ satisfies (5.10a), $\cdots$, (5.14a).

Inductively, we suppose that the 5-tuple $(H_b, M_b, S_b, X_b, P_b)$ exists with properties (5.10b), $\cdots$, (5.14b) provided that $|b| < k$. Let $a = (X, U, q, m) \in \mathscr{S}$ with $|a| = k$. By Lemma 5.9.1 we have that $\mathrm{Bd}D_a = \mathrm{Bd}C_a$ or $\mathrm{Bd}C_a - \mathrm{Bd}D_a \neq \varnothing$ and we accordingly, break the induction step into two cases.

Case A. $\mathrm{Bd}D_a = \mathrm{Bd}C_a$. In this case we do not need the induction hypothesis and we proceed formally as above for $|a| = 1$. That is, we let $H_a = D_a$, $J_a = C_a$, M_a be a collar of H_a in $\mathrm{Cl}\, V_a$, $S_a = \varnothing$, L be the 3-manifold bounded by $H_a \cup J_a$ and containing a collar of J_a in $\mathrm{Cl}\, U$, and let X_a, P_a be obtained by the same application of Lemma 4.9 as was used above.

Case B. $\mathrm{Bd}C_a - \mathrm{Bd}D_a \neq \varnothing$. See Figure 3. By Lemma 5.9.3, $L_a \neq \varnothing$ and we let $m_a = \mu(L_a)$ where μ is the choice function mentioned in the first paragraph of §5. Let $b = (X, U, q, m_a)$. Since $m_a \subset A_a$ we have that $A_b \subset A_a$ and $|b| < |a| = k$. By the induction hypothesis a 5-tuple $(H_b, M_b, S_b, X_b, P_b)$ exists with properties (5.10b), $\cdots$, (5.14b). By (5.6), $m \subset D_a$. by (5.10b), (5.6), (5.9), $H_b \subset D_b \subset B_b$: and by (5.8) $B_b \cap D_a = \varnothing$. Hence, $D_a \cap H_b = \varnothing$, $m \cap H_b = \varnothing$ and $M_b \cap m = \varnothing$ by (5.11b). Also, by (5.14.1b) and (5.14.3b) we have that $m \subset X_b$. Let g be a PL homeomorphism of S^3 onto S^3 that is fixed outside a thin shell neighborhood of M_b and pushes M_b across R to $S^3 - \mathrm{Cl}\, V_b$. Since $D_a \cap H_b = \varnothing$, $m \cap H_b = \varnothing$ it follows that no point near D_a is moved by g and $g(m) = m$. Let $c = (g(X_b), g(U_b), g(q_b), m)$ where U_b is the component of $S^3 - X_b$ such that $H_b \cap \mathrm{Cl}\, U_b$ contains the collar of $\mathrm{Bd}H_b$

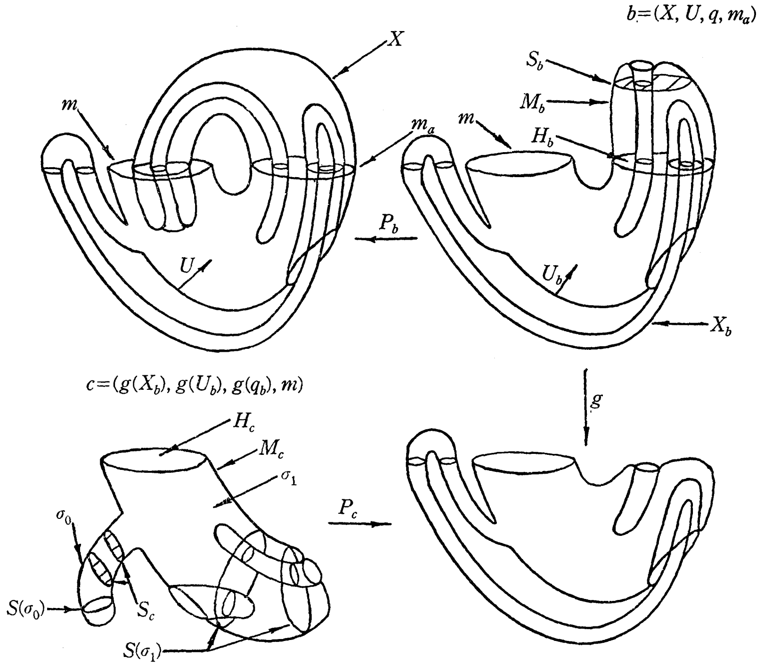


FIGURE 3

given in (5.10.2b), and $q_b = (P_b | X_b)^{-1}(q)$. Since g removes components of $R \cap X_b$, in particular m_a, and by property (5.14.3b), it follows that $|c| < |b| < |a| = k$. By the induction hypothesis, a 5-tuple $(H_c, M_c, S_c, X_c, P_c)$ exists with properties (5.10c), $\cdots$, (5.14c).

Let $H_a = H_c$, $M_a = M_c$, and $J_a = P_b g^{-1}(J_c)$. By pushing M_a into a smaller neighborhood of H_a if necessary, we may assume that (5.11.2a) is satisfied and $g^{-1}| M_a = 1$. Properties (5.10a), (5.10.1a), (5.10.2a), (5.10.3a), (5.11a), (5.11.1a) are consequences of the corresponding properties for c and the facts that $P_b | M_a = 1$ and $g^{-1} | M_a = 1$.

It follows from (4.7) that $P_c(M_c)$ misses the interior of a collar of m in $\mathrm{Cl}\,(S^3 - V_c)$. Since H_b is in the same component of $S^3 - P_c(\mathrm{Bd} M_c - H_c)$ that contains the interior of such a collar of m in $\mathrm{Cl}\,(S^3 - V_c)$, it follows from (4.8.1) and (4.7) that $H_b \cap P_c(M_c) = \emptyset$.

Let $\sigma \in K_3(c)$. Since g^{-1} is a homeomorphism that only moves points near H_b into M_b and $H_b \cap P_c(M_c) = \emptyset$ we have by (5.14.2c), (4.6.2), (4.6.3) that $g^{-1}P_c | \sigma$ is a separation isomorphism from $\bar{\sigma}$ to $\{g^{-1}P_c(\tau) | \tau \in \bar{\sigma}\} = \overline{g^{-1}P_c(\sigma)}$ such that $g^{-1}P_c(\sigma \cap S_c) \subset R - H_b$. Let $W = S_b \cup H_b$ and let Q be the 2-sphere obtained from $(R - H_b) \cup G_b$ by pushing G_b slightly away from M_b. Since $g^{-1}P_c((\mathrm{Bd}\sigma) - S_c) \subset X_b$ and X_b satisfies (5.13b), (5.13.2b), (5.13.3b) it follows that each component t of $g^{-1}P_c((\mathrm{Bd}\sigma) - S_c) \cap W$ separates the component of W containing t or $t \subset \mathrm{Bd}\, W$, and t separates the 2-face of $\overline{g^{-1}P_c(\sigma)}$ containing t. By Lemma 3.14 applied to Q and W, $\overline{g^{-1}P_c(\sigma)}$ mod W is a separation subdivision of

$\overline{g^{-1}P_c(\sigma)}$. By Lemma 3.10 we assume without loss of generality that $S(\sigma)$ is a compact PL 2-manifold in σ such that if Y is a component of $S(\sigma)$ then $Y \cap \text{Bd}\sigma = Y \cap ((\text{Bd}\sigma) - S_c) = \text{Bd}\, Y \neq \emptyset$ and $g^{-1}P_c(Y)$ is a component of $W \cap g^{-1}P_c(\sigma)$.

Let $S_a = S_c \cup (\cup \{S(\sigma) \mid \sigma \in K_3(c)\})$. It is clear that S_a satisfies (5.12) and $g^{-1}P_c \mid \sigma \in K_3(a)$ is a separation isomorphism from $\bar{\sigma}$ to $\{g^{-1}P_c(\tau) \mid \tau \in \bar{\sigma}\}$ that satisfies property (4.6.5).

Let $P = P_b g^{-1}P_c \mid M_a$. We now show that P is a winding function with respect to (M_a, S_a, H_a, R). Property (4.6.1) is satisfied since P_c satisfies (4.6.1), and g^{-1}, P_b, as pointed out previously, are the identity on a neighborhood of H_a. Property (4.6.2) is established by observing that, by the way K_a was constructed in the previous two paragraphs, if $\sigma \in K_3(a)$ then $M_b \cap \text{Int}\, g^{-1}P_c(\sigma) = \emptyset$ or $g^{-1}P_c(\sigma) = |L|$ where L is a subcomplex of $\bar{\sigma}_0'$ for some $\sigma_0 \in K_3(b)$ ($\bar{\sigma}_0'$ is the induced subdivision of $\bar{\sigma}_0$ by K_b'). In the former case, (4.6.2) follows since $P_b \mid S^3 - (M_b - H_b) = 1$. In the latter case, (4.6.2) follows by (5.14.6b). Properties (4.6.3) and (4.6.4) are easy consequences of the corresponding propertis for P_b and P_c and the way S_a was constructed in the previous two paragraphs. We now establish property (4.6.5). Suppose $\sigma_0, \sigma_1 \in K_3(a)$, $\sigma_0 \neq \sigma_1$ and $P(\sigma_0 - S_a) \cap P(\sigma_1 - S_b) \neq \emptyset$, then (a) $M_b \cap \text{Int}\, g^{-1}P_c(\sigma_i) = \emptyset$ for $i = 0, 1$ or (b) $g^{-1}P_c(\sigma_i) \subset M_b$, $M_b \cap \text{Int}\, g^{-1}P_c(\sigma_{1-i}) = \emptyset$ for $i = 0, 1$ or (c) $g^{-1}P_c(\sigma_i) \subset M_b$, $i = 0, 1$. In Case (a), (4.6.5) follows from the corresponding property for P_c and the fact that g^{-1} is a homeomorphism. In Case (b), $P(\sigma_i - S_a) \subset \text{Int}\, P(\sigma_{1-i})$ since $P(\sigma_i - S_a)$ is connected and fails to contain any point in $Bd P(\sigma_{1-i})$ by (5.14.5b). In Case (c), either (c.1) there exists a $\sigma^* \in K_3(b)$ such that $g^{-1}P_c(\sigma_i) \subset \sigma^*$ for $i = 0, 1$ or (c.2) there exists distinct $\sigma_0^*, \sigma_1^* \in K_3(b)$ such that $g^{-1}P_c(\sigma_i) \subset \sigma_1^*$. In Case (c.1), (4.6.5) follows from the corresponding property for P_c and the fact that g^{-1} is a homeomorphism and (5.14.6b). In Case (c.2), (4.6.5) follows from (5.14.7b) since we must have $P_b(\sigma_1^*) \subset P_b(\sigma_{1-i}^*)$ for $i = 0$ or 1, by property (4.6.5) for P_b.

Let $W = X \cap P(M_a) - P(\text{Bd} M_a - \text{Int}\, H_a))$. Apply Lemma 4.9 to P and W to obtain $\bar{W}$ and P' satisfying (4.9.1), $\cdots$, (4.9.6).

Let $X_a = \bar{W} \cup G_a \cup (X - P'(M_a))$ and let P_a be given by $P_a \mid M_a = P'$ and $P_a \mid S^3 - (M_a - H_a) = 1$. We now varify the remaining properties of the conclusion of the Lemma 5.15 to complete the induction step. Properties (5.14.1a), (5.14.2a) are clear by (4.9.1) applied to P'. Property (5.14.3a) follows from (4.9.5) applied to P', the definition given previously of J_a, and the fact that $P \mid S^3 - (M_a - H_a) = 1$. Property (5.14.4a) is a consequence of the definition of J_a. Property (5.14.5a) follows since $X_a - M_a = X - P'(M_a)$. Properties (5.14.6a) and (5.14.7a) are consequences of (4.9.4) and (4.9.6) respectively, applied to P'. Properties (5.13.1a), (5.13.2a), and (5.13.3a) follow from the definition of X_a and (4.9.3), (4.9.5).

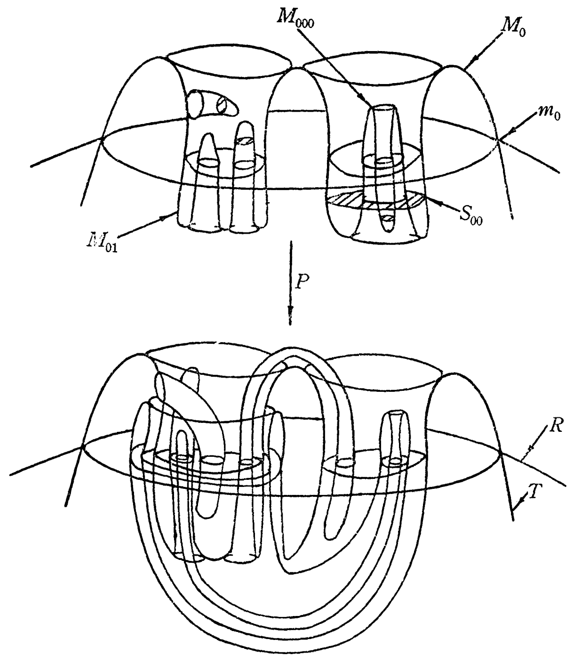


FIGURE 4

The following theorem gives the combinatorial structure of the collection of components of $R \cap T$, $R - T$, $T - R$, $S^3 - (T \cup R)$ where R and T are PL 2-spheres in general position in S^3. The reader is referred to Definition 4.1 for some of the notation used. See Figure 4 for an illustration.

Theorem 5.16. *Suppose R, T are PL 2-spheres in S^3 that are in general position and m is a simple closed curve in $R \cap T$ such that m bounds a disk D in T and a disk E in R such that $R \cap T \subset D \cap E$. Let B be the 3-cell bounded by* $\mathrm{Cl}\,(T-D) \cup \mathrm{Cl}\,(R-E)$ *such that* $\mathrm{Int}\,B \cap \mathrm{Int}\,D = \mathrm{Int}\,B \cap \mathrm{Int}\,E = \emptyset$ *and let U_0, U_1 and V_0, V_1 be respectively the components of $S^3 - T$, and $S^3 - R$ and such that* $\mathrm{Int}\,B \subset U_0$, $\mathrm{Int}\,\mathrm{B} \subset V_0$. *Then there exists a finite tree labeling system Γ, a 4-tuple $(m_\alpha, H_\alpha, M_\alpha, S_\alpha)$ for each $\alpha \in \Gamma$, and a function P from S^3 to S^3 such that*

(5.16.1) *m_α is a component (a simple closed curve) of $D \cap E$ with* $m_0 = m = \mathrm{Bd}D = \mathrm{Bd}E$,

(5.16.2) *H_α is a disk-with-holes in E such that*

(5.16.2.1) $\mathrm{Bd}H_\alpha = m_\alpha \cup m_{\alpha 0} \cup m_{\alpha 1} \cup m_{\alpha 2} \cup \cdots$,

(5.16.2.2) $E = \cup \{H_\alpha \mid \alpha \in \Gamma\}$, $H_\alpha \cap H_{\alpha i} = m_{\alpha i}$,

(5.16.2.3) *H_α contains a collar of* $\mathrm{Bd}H_\alpha$ *in* $\mathrm{Cl}\,(U_{|\alpha|})$,

(5.16.3) *M_α is a collar of H_α in* $\mathrm{Cl}\,V_{|\alpha|}$ *such that if $|\alpha| = |\beta|$ and $\alpha \neq \beta$ then $M_\alpha \cap M_\beta = \emptyset$,*

(5.16.4) *S_α is a partition of M_α that supports the separation* *complex* $K(\alpha) = K_1(\alpha) \cup K_2(\alpha) \cup K_3(\alpha)$, *and if* $D^* = \bigcup_{\alpha \in \Gamma} (\mathrm{Bd} M_\alpha - \mathrm{Int}\, H_\alpha)$, $S = \bigcup_{\alpha \in \Gamma} S_\alpha$, *and* $K = K_1 \cup K_2 \cup K_3$, $\bar{D}^*$, $\bar{D}$ *are the separation complexes given respectively by* $K_i = \bigcup_{\alpha \in \Gamma} K_i(\alpha)$, $\bar{D}^* = D^* \bmod (S \cup E)$ *and* $\bar{D} = D \bmod E$ *then*

(5.16.5) $P|S^3 - (|K| - E) = 1$, *and* P *is the identity on a corner of* M_α *for each* $\alpha \in \Gamma$,

(5.16.6) $P|D^*$ *is a separation isomorphism from* $\bar{D}^*$ *to* $\bar{D}$, *and a homeomorphism from* D^* *onto* D,

(5.16.7) *if* $\sigma \in K$ *then* $\{P(\tau) | \tau \in \bar{\sigma}\} = \overline{P(\sigma)}$ *is a separation complex and* $P|\sigma$ *is a separation isomorphism from* $\bar{\sigma}$ *to* $\overline{P(\sigma)}$,

(5.16.8) *if* $\sigma \in K_3$ *then* $P(\sigma - (S \cup E)) \subset S^3 - R$ *and* $P(\sigma \cap (S \cup E)) \subset E$,

(5.16.9) *if* $\sigma_0, \sigma_1 \in K_3$ *and* $\sigma_0 \cap \sigma_1$ *is a component of* S *then* $P(\sigma_0 - (S \cup E))$, $P(\sigma_1 - (S \cup E))$ *lie in different components of* $S^3 - R$, *and*

(5.16.10) *if* $\sigma_0, \sigma_1 \in K_3$, $\sigma_0 \neq \sigma_1$, *but* $P(\sigma_0 - (S \cup E)) \cap P(\sigma_1 - (S \cup E)) \neq \varnothing$ *then for* $i = 0$, *or* 1, $P(\sigma_i - (S \cup E)) \subset \mathrm{Int}\, P(\sigma_{1-i})$ *and* $P(\sigma_i \cap (S \cup E)) \subset \mathrm{Int}\, P(\sigma_{1-i} \cap (S \cup E))$.

Proof. The sets m_α, H_α, M_α, S_α, and function P are constructed inductively using Lemma 5.15, Lemma 3.14 and the properties of winding functions given in (4.6), (4.7), (4.8), and (4.9). Let q be a point in $T - D$ and identify the 2-sphere R of the hypothesis and the 2-sphere R of §5.

The 4-tuple $a_0 = (T, U_1, q, m)$ is in $\mathscr{S}$ of Definition 5.1. Let $(H_0, M_0, S_0, X_0, P_0)$ be the 5-tuple of the conclusion of Lemma 5.15 with $K(0) = K_1(0) \cup K_2(0) \cup K_4(0)$ the seperation complex supported by S_0 in M_0 with $K_1'(0) \cup K_2'(0) \cup K_3'(0)$ the separation subdivision of $K(0)$ of Lemma 5.15. Let $m_0 = m$, and let $m_{00}, m_{01}, m_{02}, \cdots$ be the other components of $\mathrm{Bd} H_0$. Since by (4.6.1), (4.8.1), (4.7) and (5.14.5) we have $B \cap P_0(M_0) = m$, we let U_0^0, U_1^0 be the components of $S^3 - X_0$ with $\mathrm{Int}\, B \subset U_0^0$ and let $\Gamma_0 = \{0\}$. From the conclusion of Lemma 5.15 it follows that Γ_0, 4-tuple (m_0, H_0, M_0, S_0), and function P_0 satisfy (5.16.1), (5.16.3), (5.16.4), (5.16.5), (5.16.7), (5.16.8), (5.16.9), and (5.16.10). Condition (5.16.2) is satisfied with the exception of the requirement $E = \bigcup_{\alpha \in \Gamma_0} H_\alpha$ and condition (5.16.6) is satisfied for $P|D_0^*$ where $D_0^* = X_0 - (T - D)$ and $\bar{D}_0^* = D_0^* \bmod S_0 \cup E$. The conclusion of Lemma 5.15 shows that the following properties are also satisfied.

(a_0) $P_0(M_0) \cap (X_0 - M_0) = \varnothing$,

(b_0) if $\sigma \in K_3(0)$ then $P_0|\sigma$ is a separation isomorphism from $\bar{\sigma}'$ to $\{P_0(\tau) | \tau \in \bar{\sigma}'\}$, and

(c_0) if $\sigma_0, \sigma_1 \in K_3(0)$ and $\sigma_0 \neq \sigma_1$ but $P_0(\sigma_0) \subset P_0(\sigma_1)$ then there exists $\sigma_2 \in K_3'(0)$ such that $\sigma_2 \subset \sigma_1$ and $P_0(\sigma_0) \subset P_0(\sigma_2)$.

We proceed inductively up the lexicograpical ordering of the m_α's. Suppose for each element α of a tree labeling system Γ_k, a 4-tuple

$(m_\alpha, H_\alpha, M_\alpha, S_\alpha)$, and function P_k from S^3 to S^3 exist that satisfy (5.16.1), (5.16.2) as modified above, (5.16.3), $\cdots$, (5.16.5), (5.16.6) as modified above, (5.16.7), $\cdots$, (5.16.10), and,

(a_k) $P_k(\cup\{M_\alpha|\alpha\in\Gamma_k\}\cap(X_k - \cup\{M_\alpha|\alpha\in\Gamma_k\}) = \emptyset$

(b_k) if $\sigma\in K_3(k)$ then $P_k|\sigma$ is a separation isomorphism from $\bar{\sigma}'$ to $\{P_k(\tau)|\tau\in\bar{\sigma}'\}$, and

(c_k) if $\sigma_0, \sigma_1\in K_3(k)$ and $\sigma_0\neq\sigma_1$ but $P_k(\sigma_0)\subset P_k(\sigma_1)$ then there exists $\sigma_2\in K_3'(k)$ such that $\sigma_2\subset\sigma_1$ and $P_k(\sigma_0)\subset P_k(\sigma_2)$,

where $K(k) = K_1(k)\cup K_2(k)\cup K_3(k) = \cup\{K(\alpha)|\alpha\in\Gamma_k\}$, $K'(k) = K_1'(k)\cup K_2'(k)\cup K_3'(k)$ is the subdivision of $K(k)$ given by $K(k)\bmod(X_k - \cup\{\mathrm{Bd}M_\alpha - \mathrm{Int}\,H_\alpha|\alpha\in\Gamma_k\})$, $D_k^* = X_k-(T-D)$, and $\bar{D}_k^* = D_k^* \bmod (\cup\{S_\alpha|\alpha\in\Gamma_k\})\cup E$. Let U_0^k, U_1^k be the components of $S^3 - X_k$ where $\mathrm{Int}\,B\subset U_0^k$.

Let $\Gamma_{k+1} = \Gamma^k\cup\{\beta\}$ where β is the next element in the lexicographical ordering of the m_α's. That is, if $\alpha = i_1i_2\cdots i_n$ is the last element of Γ_k then $\beta = i_1i_2\cdots i_{n-1}(i_n+1)$ or $\beta = i_1i_2\cdots i_n0$ depending on which of these is the lowest subscript among the symbols labeling the boundary components of the desk-with-holes $\cup\{H_\alpha|\alpha\in\Gamma_k\}$. Let $a_{k+1} = (X_k, U_{|\beta|}^k, q, m_\beta)$ and apply Lemma 5.15 to obtain 5-tuple $(H_\beta, M_\beta, \hat{S}, \hat{X}_{k+1}, \hat{P})$ that satisfies the conclusion of Lemma 5.15 with $\hat{K} = \hat{K}_1\cup\hat{K}_2\cup\hat{K}_3$ the separation complex supported by $\hat{S}$ in M_β. Properties (5.16.1), (5.16.2) as modified above, and (5.16.3) and (5.16.3) are easily established and we leave this to the reader. Label the components (if there are any) of $(\mathrm{Bd}H_\beta) - m_\beta$ with $m_{\beta 0}, m_{\beta 1}, \cdots$.

It follows by (5.14.2), (4.6.1), (4.8.1), (4.7), (5.16.2), and (5.16.4) that $(B\cup(\cup\{G_\alpha|\alpha\in\Gamma_k\}))\cap\hat{P}(M_\beta) = m_\beta$ and $\hat{P}\,|(B\cup(\cup\{G_\alpha|\alpha\in\Gamma_k\})) = 1$ where $G_\alpha = \mathrm{Bd}M_\alpha - \mathrm{Int}\,H_\alpha$. Hence, $\hat{X}_{k+1}$ contains $(T-D)\cup(\cup\{G_\alpha|\alpha\in\Gamma_{k+1}\})$.

For $\sigma\in\hat{K}_3$, $\hat{P}(\sigma)$ may intersect $\cup\{\mathrm{Int}\,M_\alpha|\alpha\in\Gamma_k\}$: however, $\hat{P}(\sigma)$ may intersect at most one of the sets $\mathrm{Int}\,M_\alpha$, $\alpha\in\Gamma_k$. Let $\alpha(\sigma)$ be the subscript such that $\hat{P}(\sigma)\cap\mathrm{Int}\,M_{\alpha(\sigma)}\neq\emptyset$. Proceeding as in Case B, paragraph 4, of the induction step in the proof of Lemma 5.15 we may apply Lemma 3.14 and Lemma 3.10 and assume without loss of generality that $S(\sigma)$ is a compact PL 2-manifold in σ such that if Y is a component of $S(\sigma)$ then $Y\cap\mathrm{Bd}\sigma = Y\cap((\mathrm{Bd}\sigma) - \hat{S}) = \mathrm{Bd}\,Y\neq\emptyset$ and $\hat{P}(Y)$ is a component of $S_{\alpha(\sigma)}\cap\hat{P}(\sigma)$. Let $S_\beta = \hat{S}\cup(\cup\{S(\sigma)|\sigma\in K_3\})$. It is clear that S_β partitions M_β and let $K(\beta) = K_1(\beta)\cup K_2(\beta)\cup K_3(\beta)$ be the separation complex that S_β supports in M_β, thus (5.16.4) is satisfied. It is clear that $\hat{P}$ satisfies (5.16.7) and (5.16.10) for σ, σ_0, $\sigma_1\in K_3(\beta)$ since $\hat{P}$ satisfied (4.6.2) and (4.6.5) for $\sigma, \sigma_0, \sigma_1\in K_3$ and the applications of Lemma 3.14 and Lemma 3.10 haven't altered these properties. Let $K(k+1) = K_1(k+1)\cup K_2(k+1)\cup K_3(k+1) = \cup\{K(\alpha)|\alpha\in\Gamma_{k+1}\} = \hat{K}(k)\cup K(\beta)$, and Let $S_{k+1} = \cup\{S_\alpha|\alpha\in\Gamma_{k+1}\} = S_k\cup S_\beta$.

We now show that $P_k\cdot\hat{P}|M_\beta$ is winding function with respect to $(M_\beta, S_\beta, H_\beta, R)$. Property (4.6.1) is satisfied since by (5.14.2), and (5.16.5), respectively, $\hat{P}$, P_k satisfy (4.6.1). Property (4.6.2) is established by observing that, by the way $K(\beta)$ was constructed, if $\sigma \in K_3(\beta)$ then $(\cup \{M_\alpha | \alpha \in \Gamma_k\}) \cap \operatorname{Int} \hat{P}(\sigma) = \emptyset$ or $\hat{P}(\sigma) = |L|$ where L is a subcomplex of $\bar{\sigma}'_0$ for some $\sigma_0 \in K(k)$ ($\bar{\sigma}'_0$ is the induced subdivision of $\bar{\sigma}_0$ by $K'(k)$). In the former case, (4.6.2) follows by (5.16.5) and in the latter case by (b_k). Properties (4.6.3) and (4.6.4) are straight forward consequences of the corresponding properties for $\hat{P}$, (5.16.8) and (5.16.9) for P_k, and the way S_β was constructed. We now establish property (4.6.5). Suppose $\sigma_0, \sigma_1 \in K_3(\beta)$, $\sigma_0 \neq \sigma_1$, and $P_k\hat{P}(\sigma_0 - S_\beta) \cap P_k\hat{P}(\sigma_1 - S_\beta) \neq \emptyset$ then (a) $M_\alpha \cap \operatorname{Int} \hat{P}(\sigma_i) = \emptyset$ for $i = 0, 1$ and all $\alpha \in \Gamma_k$, or (b) $\hat{P}(\sigma_i) \subset M_\alpha$, $M_\gamma \cap \operatorname{Int} \hat{P}(\sigma_{1-i}) = \emptyset$ for some α, and all $\gamma \in \Gamma_k$, or (c) $\hat{P}(\sigma_i) \subset M_\alpha$, $\hat{P}(\sigma_{1-i}) \subset M_\gamma$ for some $\alpha, \gamma \in \Gamma_k$. In Case (a), (4.6.5) follows from the corresponding properties for $\hat{P}$, the way S_β was constructed, and (5.16.5). In Case (b), $P_k\hat{P}(\sigma_i - S_\beta) \subset \operatorname{Int} P_k\hat{P}(\sigma_{1-i})$ since $P_k\hat{P}(\sigma_{1-i} - S_\beta) \subset X_k - (\cup \{M_\alpha | \alpha \in \Gamma_k\})$, $P_k\hat{P}(\sigma_i - S_\beta)$ is connected, and by (a_k). In Case (c), either (c.1) there exists a $\sigma^* \in K_3(k)$ such that $\hat{P}(\sigma_i) \subset \sigma^*$ for $i = 0, 1$ or (c.2) there exist distinct $\sigma_0^*, \sigma_1^* \in K_3(k)$ such that $\hat{P}(\sigma_i) \subset \sigma_1^*$. In Case (c.1), (4.6.5) follows from the corresponding property for $\hat{P}$ and (b_k). In Case (c.2), (4.6.5) follows from (c_k), sice we must have $P_k(\sigma_i^*) \subset P_k(\sigma_{1-i}^*)$ for $i = 0$, or 1 by (5.16.10).

Let $\hat{P}_{k+1} = P_k\hat{P}$. We showed in the previous paragraph that $\hat{P}_{k+1} | M_\beta$ is a winding function with respect to $(M_\beta, S_\beta, H_\beta, R)$. This, combined with (5.16.5), (5.16.7), (5.16.8) and (5.16.9) for P_k establishes (5.16.5), (5.16.7), (5.16.8), and (5.16.9) for $\hat{P}_{k+1}$. Property, (5.16.6) is valid for $\hat{P}_{k+1}$ since it is valid for P_k, $\hat{P} | \cup \{G_\alpha | \alpha \in \Gamma_k\} = 1$, and $\hat{P} | G_\beta$ is a separation isomorphism onto $J_{a_{k+1}}$ of Lemma 5.15.

Property (5.16.10) follows by the corresponding property for P_k or $\hat{P}$ if either $\sigma_0, \sigma_1 \in K_3(k)$ or $\sigma_0, \sigma_1 \in K_3(\beta)$. The other cases are (a)$\sigma_0 \in K_3(k)$, $\sigma_1 \in K_3(\beta)$, and $\hat{P}(\sigma_1) \cap \operatorname{Int} M_\alpha = \emptyset$ for all $\alpha \in \Gamma_k$ or (b) $\sigma_0 \in K_3(k)$, $\sigma_1 \in K_3(\beta)$, and $\hat{P}(\sigma_1) \subset M_\alpha$ for some $\alpha \in \Gamma_k$. In Case (a), $\hat{P}_{k+1}(\sigma_0 - S_{k+1}) \subset \operatorname{Int} \hat{P}_{k+1}(\sigma_1)$, and $\hat{P}_{k+1}(\sigma_0 \cap (S_{k+1} \cup E)) \subset \operatorname{Int} \hat{P}_{k+1}(\sigma_1 \cap (S_{k+1} \cup E))$ since $\hat{P}_{k+1}(\sigma_1 - S_{k+1}) \subset X_k - (\cup \{M_\alpha | \alpha \in \Gamma_k\})$, $\hat{P}_{k+1}(\sigma_0 - S_{k+1})$ is connected and by (a_k). In Case (b) either (b.1) $\hat{P}(\sigma_1) \subset \sigma_0$ or (b.2) there exists a $\sigma^* \in K_3(k)$ such that $\hat{P}(\sigma_1) \subset \sigma^*$. In Case (b.1) (5.16.10) follows by b_k since $\hat{P}(\sigma_1)$ must be the underlying set of a subcomplex of $K'(k)$ in σ^*. In Case (b.2), (5.16.10) follows from c_k, since we must have either $P_k(\sigma^*) \subset P_k(\sigma_0)$, or $P_k(\sigma_0) \subset P_k(\sigma^*)$ by (5.16.10) for P_k.

The properties remaining to be varified are (a_{k+1}), (b_{k+1}), and (c_{k+1}). We establish these by using the lifting property for winding functions in Lemma 4.9. For $\sigma \in K_3(k+1)$ let $N(\sigma) = \{\tau \in K_3(k+1) | \hat{P}_{k+1}(\tau) \subset \hat{P}_{k+1}(\sigma)$ and $\tau \neq \sigma\}$ and let $W(\sigma) = \hat{P}_{k+1}(\hat{X}_{k+1} - (\cup \{G_\alpha | \alpha \in \Gamma_{k+1}\})) \cap (\hat{P}_{k+1}(\sigma) - \cup \{\hat{P}_{k+1}(\tau) | \tau \in N(\sigma)\})$. It is clear that $(\hat{P}_{k+1}(\cup \{M_\alpha | \alpha \in \Gamma_{k+1}\}) - \hat{P}_{k+1}(\cup \{G_\alpha | \alpha \in \Gamma_{k+1}\})) \cap \hat{P}_{k+1}(\hat{X}_{k+1}) = \cup \{W(\sigma) | \sigma \in K_3(k+1)\}$ and if $\sigma_0 \neq \sigma_1$ then $\operatorname{Int} W(\sigma_0) \cap \operatorname{Int} W(\sigma_1) = \emptyset$. Also, by (5.16.8), (5.16.9), and (5.16.10), if $\sigma_0 \cap \sigma_1$ is a component of S_{k+1} then $W(\sigma_0) \cap W(\sigma_1) = \hat{P}_{k+1}(\hat{X}_{k+1} - \cup \{G_\alpha | \alpha \in$

$\Gamma_{k+1}\}) \cap (\hat{P}_{k+1}(\sigma_0 \cap \sigma_1) - \cup \{\hat{P}_{k+1}(\tau) \mid \tau \in N(\sigma_i)\})$ for $i = 0$ or 1. It follows that for each $\alpha \in \Gamma_{k+1}$, $W_\alpha = \cup \{W(\sigma) \mid \sigma \in K_3(\alpha)\}$ is a compact 2-manifold that satisfies the hypothesis of Lemma 4.9 along with winding function $\hat{P}_{k+1} \mid M_\alpha$ with respect to $(M_\alpha, S_\alpha, H_\alpha, R)$. Apply Lemma 4.9 for each $\alpha \in \Gamma_{k+1}$ and let $\bar{W}_\alpha$ be the resulting compact 2-manifold, $K'(\alpha)$ the resulting subdivision of $K(\alpha)$ satisfying 4.9.3 and $P_{k+1} \mid M_\alpha$ the resulting function that satisfies (4.9.1), $\cdots$, (4.9.6).

Let $X_{k+1} = (\cup \{\bar{W}_\alpha \mid \alpha \in \Gamma_{k+1}\}) \cup (\cup \{G_\alpha \mid \alpha \in \Gamma_{k+1}\}) \cup (\hat{X}_{k+1} - \hat{P}_{k+1}(\cup \{M_\alpha \mid \alpha \in \Gamma_{k+1}\}))$, let $P_{k+1} \mid \cup \{M_\alpha \mid \alpha \in \Gamma_{k+1}\} = \cup \{(P_{k+1} \mid M_\alpha) \mid \alpha \in \Gamma_{k+1})\}$ and let $K'(k+1) = K_1'(k+1) \cup K_2'(k+1) \cup K_3'(k+1) = \cup \{K'(\alpha) \mid \alpha \in \Gamma_{k+1}\}$. By (4.9.1), P_{k+1} satisfies (5.16.5), $\cdots$, (5.16.10), thus it is only necessary to varify (a_{k+1}), (b_{k+1}), and (c_{k+1}). Property (a_{k+1}) follows since $X_{k+1} - \cup \{M_\alpha \mid \alpha \in \Gamma_{k+1}\} = \hat{X}_{k+1} - \hat{P}_{k+1}(\cup \{M_\alpha \mid \alpha \in \Gamma_{k+1}\})$. Property (b_{k+1}) follows by (4.9.4). Property (c_k) is established by observing that if $P_{k+1}(\sigma_0) \subset P_{k+1}(\sigma)$ then $P_{k+1}(\sigma_0)$ is in a component Z of $P_{k+1}(\sigma_1) - W(\sigma_1)$ since $P_{k+1}(\sigma_0)$ is connected and $W(\sigma_1) \cap P_{k+1}(\sigma_0) = \emptyset$ by definition of $W(\sigma_1)$. The lift of $\mathrm{Cl}\, Z$ is the required σ_2, that is, $\sigma_2 = (P_{k+1} \mid \sigma_1)^{-1}(\mathrm{Cl}\, Z)$.

The induction step is now complete. The above process terminates when $\cup\{H_\alpha \mid \alpha \in \Gamma_k\}$ is the disk E, thus completing property (5.16.2). The reason for not including properties (a_k), (b_k), (c_k) in the conclusion to Theorem 5.16 is that they are "empty" in the sense that $(\bar{D} \cup (T - D)) \cap (\cup \{\mathrm{Int}\, M_\alpha \mid \alpha \in \Gamma\}) = \emptyset$.

6. Disjoint singular disks.

We now prove the following corollary to §§3, 4, and 5.

COROLLARY 6.1. *Suppose T is a 2-sphere is S^3, D is a PL subdisk of T, and E is a PL disk in S^3 in general position with respect to T such that $E \cap T \subset D$ and $\mathrm{Bd}E = \mathrm{Bd}D$. Let U_0, U_1 be the components of $S^3 - T$ and suppose that $E_i (i = 0, 1)$ is a PL disk in general position with respect to D such that $\mathrm{Bd}E_i \subset U_i$, E_i fails to intersect the 2-sphere $E \cup (T - D)$, and $T - D$, $\mathrm{Bd}E_0$, $\mathrm{Bd}E_1$ are in the same component of $S^3 - (D \cup E)$. Let F_i $(i = 0, 1)$ be the component of $E_i \cap U_i$ that contains $\mathrm{Bd}E_i$. Then for $i = 0, 1$ there exist collections $\mathscr{D}_i$, $\mathscr{E}_i$ of components of $D - E_i$ and of $E_i \cap U_i$ respectively such that*

(6.1.1) *if $A \in \mathscr{D}_0$, $B \in \mathscr{D}_1$ then $\mathrm{Cl}\, A \cap \mathrm{Cl}\, B = \emptyset$,*

(6.1.2) *if $A \in \mathscr{E}_i$ and $B \in \mathscr{D}_{1-i}$ then E separates A from B in $\mathrm{Cl}\, U_i$, and*

(6.1.3) *the set $D_i = \cup\{\mathrm{Cl}\, A \mid A \in \mathscr{D}_i \cup \mathscr{E}_i\}$ is a singular disk with $\mathrm{Bd}D_i = \mathrm{Bd}E_i$ obtained by pasting in the sense that there exists a map $f_i: E_i \to D_i$ such that $f_i \mid F_i = 1$ and if K is any component of $f_i^{-1}(\mathrm{Cl}\, A)$ for $A \in \mathscr{D}_i \cup \mathscr{E}_i$ then $f_i \mid K$ is a PL homeomorphism onto $\mathrm{Cl}\, A$.*

Proof. It follows from the polyhedral approximation theorem of Bing [1] that we may assume that the 2-sphere T is piecewise linear. In order to use Theorem 5.16, we assume that the component of $S^3 - T$ have been labeled so that $\mathrm{Cl}\,(U_1)$ contains a collar of $\mathrm{Bd}E$ in E and B is a collar of $\mathrm{Cl}\,(T - D)$ in $\mathrm{Cl}\,U_0$ such that $B \cap E = \mathrm{Bd}E$, and for $i = 0$, or 1, $B \cap E_i = \varnothing$. Let R be the 2-sphere defined by $R = E \cup ((\mathrm{Bd}B) - (T - D))$ and let the components of $S^3 - R$ be V_0, V_1 where $\mathrm{Int}\,B \subset V_0$. It follows that $B - m$, $\mathrm{Bd}E_0$, $\mathrm{Bd}E_1$ are in the same component of $S^3 - (D \cup E)$. We use the notation of Theorem 5.16, in particular let Γ, m_α, H_α, M_α, S_α, $K(\alpha)$, K and P be as in the conclusion. The map f_i is found by cutting E_i off on $\cup\{P((\mathrm{Bd}M_\alpha) - H_\alpha)\,|\,\alpha \in \Gamma,\ |\alpha| = 1 - i\}$, but first we observe some useful facts.

It follows from (5.16.2), (5.16.3) that

(6.1.4) M_α contains a corner in $\mathrm{Cl}\,(U_{|\alpha|} \cap V_{|\alpha|})$.

Also, by (5.16.5), (4.8.2) we have that

(6.1.5) $P(M_\alpha)$ contains a collar of $P(\mathrm{Bd}M_\alpha - \mathrm{Int}\,H_\alpha)$ in $\mathrm{Cl}\,U_{|\alpha|}$.

By (5.16.5), (5.16.7), $\cdots$, (5.16.10), respectively, we have, respectively, (4.6.1), (4.6.2), $\cdots$, (4.6.5). Thus $P|M_\alpha$ is a winding function with respect to $(M_\alpha, S_\alpha, H_\alpha, R)$; consequently, by (4.8.1) we have

(6.1.6) $((B - m) \cup \mathrm{Bd}E_0 \cup \mathrm{Bd}E_1) \cap P(M_\alpha) = \varnothing$ for all $\alpha \in \Gamma$.

We wish to establish that

(6.1.7) $E_i \cap U_{1-i} \subset E_i \cap P(\cup\{M_\alpha|\alpha \in \Gamma\}) \subset E_i \cap P(\cup\{M_\alpha|\alpha \in \Gamma,\ |\alpha| = 1 - i\})$ for $i = 0$ or 1.

Let $A = \{Y|\,Y$ is a component of $E_i \cap U_{1-i}$ such that there does not exist a $\sigma \in K_3$ such that $Y \subset P(\sigma)\}$ and let $B = \{Z|Z$ is a component of $E_i \cap \mathrm{Int}\,P(\sigma)$ where $\sigma \in \cup\{K_3(\alpha)|\alpha \in \Gamma,\ |\alpha| = 1\}$ and there does not exist a $\tau \in \cup\{K_3(\alpha)|\alpha \in \Gamma,\ |\alpha| = 1 - i\}$ such that $P(\sigma) \subset P(\tau)\}$. Suppose $A \cup B \neq \varnothing$, then there exists an outermost element of $A \cup B$ in E_i. That is, there exists an element $W \in A \cup B$ such that W' fails to separate W from $\mathrm{Bd}E_i$ in E_i for all $W' \in A \cup B$. Let w be the outermost component of $\mathrm{Bd}\,W$ and let X be the component of $E_i - T$ such that $w \subset \mathrm{Bd}X$ but $X \cap W = \varnothing$. If $W \in A$ then, since $w \subset P(\mathrm{Bd}\sigma - (S \cup E))$ for some $\sigma \in K_3$ and because of (6.1.5), it follows that there exists a $Z \in B$ such that $X \subset Z$ contradicting that W is an outermost member of $A \cup B$ in E_i. If $W \in B$, then, by (6.1.5) since W is also an outermost member of B, it follows that $X \in A$ contradicting that W is an outermost member of $A \cup B$ in E_i. Hence we must have $A \cup B = \varnothing$ and (6.1.7) is established.

We wish now to establish that

(6.1.8) E separates $E_i - P(\cup\{M_\alpha|\alpha \in \Gamma,\ |\alpha| = 1 - i\})$ from $P(\cup\{(\mathrm{Bd}M_\alpha) - H_\alpha|\alpha \in \Gamma,\ |\alpha| = i\})$ in $\mathrm{Cl}\ U_i$.

Suppose (6.1.8) is false, then by (6.1.7) there exists an arc ab from a to b such that $a \in E_i - P(\cup\{M_\alpha|\alpha \in \Gamma\})$, $b \in P((\mathrm{Bd}M_\alpha) - H_\alpha)$ for some $\alpha \in \Gamma$ with $|\alpha| = i$, and $\mathrm{Int}\,ab \subset U_i - E$. However, by (6.1.5) there exists a point c in ab near b such that $c \in (P(M_\alpha) - P(\mathrm{Bd}M_\alpha)) \cap U_i$. But this contradicts (4.8.1), thus (6.1.8) is established.

The collections $\mathscr{E}_i$, $\mathscr{D}_i$ $(i = 0, 1)$ of the conclusion will be chosen such that $\cup\{A \mid A \in \mathscr{E}_i\} \subset E_i - P(\cup\{M_\alpha \mid \alpha \in \Gamma, |\alpha| = 1-i\})$ and $\cup\{\mathrm{Cl}\,A \mid A \in \mathscr{D}_i\} \subset P(\cup\{\mathrm{Bd}M_\alpha) - H \mid \alpha \in \Gamma, |\alpha| = 1-i\})$. Properties (6.1.2) and (6.1.1) will then follow by (6.1.8) and the fact that $P(\cup\{(\mathrm{Bd}M_\alpha) - H_\alpha \mid \alpha \in \Gamma, |\alpha| = i\}) \cap P(\cup\{(\mathrm{Bd}M_\alpha) - H_\alpha \mid \alpha \in \Gamma, |\alpha| = 1 - i\}) = \varnothing$.

If m is a component of $D \cap E_i$ then we let $E_i(m)$ be the disk m bounds is E_i. Let $\mathscr{E}_i' = \{X \mid X$ is a component of $E_i - P(\cup\{M_\alpha \mid \alpha \in \Gamma, |\alpha| = 1-i\})$. Each member of $\mathscr{E}_i'$ is an open disk-with-holes whose boundary consists of one "outer" simple closed curve and perhaps several "inner" simple closed curves. We accordingly let $\mathscr{O}_i = \{m \mid m$ is a component of $\mathrm{Bd}X$ for some $X \in \mathscr{E}_i'$, and $X \subset E_i(m)\}$ and $\mathscr{I}_i = \{m \mid m$ is a component of $\mathrm{Bd}X$ for some $X \in \mathscr{E}_i'$, and $X \cap E_i(m) = \varnothing\}$. For each $m \in \mathscr{I}_i$ we find a disk-with-holes $D(m) \subset P(\cup\{(\mathrm{Bd}M_\alpha) - H_\alpha \mid \alpha \in \Gamma, |\alpha| = 1 - i\})$ such that $m \subset \mathrm{Bd}D(m)$ and if u is a component of $(\mathrm{Bd}D(m)) - m$ then $u \in \mathscr{O}_i$ and $u \subset E_i(m)$. The $D(m)$'s bridge the gaps between elements of $\mathscr{E}_i'$ and enable us to define the map f_i of the conclusion (see Figure 5).

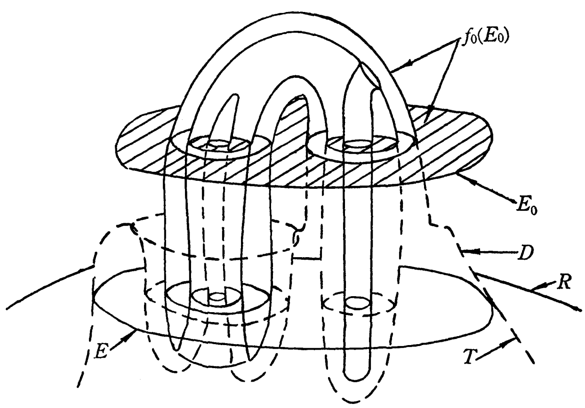


FIGURE 5

Let $m \in \mathscr{I}_i$; then $m \subset P(\mathrm{Bd}\sigma - (S \cup E))$ for a unique $\sigma \in K_3(\alpha)$ with $|\alpha| = 1 - i$. Let Y be the component of $P(\sigma) \cap E_i$ that contains m. Clearly $Y \subset E_i(m)$, and since there does not exist a $\tau \in K_3$ such that $P(\sigma) \subset \mathrm{Int}\,P(\tau)$ it follows by (6.1.5) that each component of $(\mathrm{Bd}\,Y) - m$ is in $\mathscr{O}_i$. Since each component t of $\mathrm{Bd}\,Y$ separates T, we have that t separates the 2-face τ of $\{P(\tau) \mid \tau \in \bar{\sigma}\}$ that contains t. By Lemma 3.14, Y separates $P(\sigma)$, and by Lemma 3.10 we may assume without loss of generality that there exists a compact, connected PL 2-manifold $\sigma_0 \subset \sigma$ such that $\sigma_0 \cap \mathrm{Bd}\sigma = \sigma_0 \cap ((\mathrm{Bd}\sigma) - (S \cup E)) = \mathrm{Bd}\sigma_0$ and $P \mid \sigma_0$ is a separation isomorphism from $\bar{\sigma}_0$ to $\{t \mid t$ is a component of $\mathrm{Bd}\,Y\} \cup \{Y\}$. Let m' be the component of $\mathrm{Bd}\sigma$ such that $P(m') = m$. The 2-manifold σ_0 separates $M(\alpha)$ so let $D'(m)$ be the component of $(\mathrm{Bd}M(\alpha)) - \sigma_0$ that contains m' in its boundary but does not contain $H(\alpha)$. It follows that $\mathrm{Bd}D'(m) \subset \mathrm{Bd}\sigma_0$. Let $D(m) = P(D'(m))$.

For $m_\alpha \in \mathscr{O}_i$, let $X(m_\alpha)$ be the element of $\mathscr{E}_1'$ that contains m_α in its boundary and let $m_{\alpha 0}, m_{\alpha 1}, \cdots$ be the remaining components of $\mathrm{Bd}X(m_\alpha)$. Also, let $m_{\beta 0}, m_{\beta 1}, \cdots$ be the components of $(\mathrm{Bd}D(m_\beta)) - m_\beta$. Let $m_0^i = \mathrm{Bd}E_i$, then for $i = 0, 1, f_i, \mathscr{E}_i, \mathscr{D}_i$ of conclusion are given respectively by

$$\begin{aligned} f_i(E_i) &= X(m_0^i) \cup \Big(\bigcup_{i_1} D(m_{0i_1})\Big) \cup \Big(\bigcup_{i_1, i_2} X(m_{0i_1i_2})\Big) \cup \Big(\bigcup_{i_1, i_2, i_3} D(m_{0i_1, i_2, i_3})\Big) \cup \cdots \\ \mathscr{E}_i &= \{X(m_0^i)\} \cup \{X(m_{0i_1i_2})\} \cup \{X(m_{0i_1i_2i_3i_4})\} \cup \cdots \\ \mathscr{D}_i &= \{D(m_{0i_1})\} \cup \{D(m_{0i_1i_2i_3})\} \cup \{D(m_{0i_1i_2i_3i_4i_5})\} \cup \cdots . \end{aligned}$$

That there exists no circularity in $f_i(E_i)$ follows from the fact that $\mathrm{Bd}D(m_\alpha) \subset E_i(m_\alpha)$ or, in other words, we proceed always from the "outside" to the "inside" along E_i.

REFERENCES

1. R. H. Bing, *Approximating surfaces with polyhedral ones*, Ann. of Math., (2) **65** (1957), 456-483.
2. ———, *Approximating surfaces from the side*, Ann. of Math., (2) **77** (1963), 145-192.
3. ———, *Improving the side approximation theorem*, Trans. Amer. Math. Soc., **116** (1965), 511-525.
4. ———, *Each disk in E^3 contains a tame arc*, Amer. J. Math., **84** (1962), 583-590.
5. ———, *Pushing a 2-sphere into its complement*, Mich. Math. J., **11** (1964), 33-45.
6. ———, *Improving the intersection of lines and surfaces*, Michigan Math. J., **14** (1967), 155-159.
7. W. T. Eaton, *The sum of solid spheres*, Michigan Math. J., **19** (1972), 193-207.
8. ———, *Applications of a mis-match theorem to decomposition spaces*, to appear.
9. F. M. Lister, *Simplifying intersections of disks in Bing's side approximation theorem*, Pacific J. Math., **22** (1967), 281-295.
10. C. D. Papakyriakopoulos, *On Dehn's lemma and the asphericity of knots*, Ann. of Math., (2) **66** (1957), 1-26.

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