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THE PERIPHERALITY OF IRREDUCIBLE ELEMENTS OF A LATTICE

J. W. LEA, JR.

Irreducible elements, which are not cutpoints, and meet complemented elements are peripheral in certain compact lattices and semilattices.

The purpose of this note is to extend the results of A. D. Wallace [12]; i.e., to show that meet irreducible elements and meet complemented elements are peripheral in certain topological lattices and semilattices. Meet irreducible elements have played a key role in embedding theorems for topological lattices obtained by K. A. Baker and A. R. Stralka [2] and by the author [10].

1. Preliminaries. If S is a semilattice and $x \in S$, then $M(x) = \{y \in S: x \leqslant y\}$; $L(x)$ is defined dually; if $x \leqslant y$, then

$$[x, y] = M(x) \cap L(y) .$$

An element x in a semilattice S is *meet irreducible* if $a, b \in S$ and $x = a \wedge b$ imply $x = a$ or $x = b$. We denote the set of all meet irreducible elements of S by $MI(S)$. If S has a 0, an element $x \in S$ is *meet complemented* if there is a $y \in S$ such that $y \neq 0$ and $x \wedge y = 0$. The *width* $w(X)$ of a partially ordered set X is the maximum number of elements in a set of incomparable elements.

A topological semilattice S is said to have *small semilattices at* x if x has a basis of neighborhoods which are subsemilattices of S; S is a *Lawson semilattice* if it has small semilattices at every point.

Basic definitions, notations, and properties of Alexander cohomology and codimension may be found in [11] and [4]. The point x is *marginal* in a regular space X if and only if for any open set U containing x, there exists an open set V containing x and contained in U such that the natural homomorphism $H^*(X) \to H^*(X\backslash V)$ is an isomorphism [8, Th. 1.3]. A point $x \in X$, a topological space, is *peripheral* if for any open set U containing x, there exists an open set V containing x and contained in U such that the homomorphism $i^*: H^*(X, X\backslash V) \to (X, X\backslash U)$ induced by the inclusion mapping i is the trivial or zero homomorphism. A point is *inner* if it is not peripheral.

2. Peripheral elements. J. D. Lawson and B. Madison [9, Th. 3.2] have proved that cutpoints of compact, connected spaces are inner points. This result and Theorem 2.1 locate all the meet irreducible elements of a semilattice.

THEOREM 2.1. *Let S be a compact, connected, locally connected Lawson semilattice. If $p \in S$ is meet irreducible and is not a cutpoint of S, then p is marginal in S.*

Proof. Let $p \in U$, an open subset of S. Since p is not a cutpoint of S, it is known [14, III 4.15] that there exists an open set V containing p such that $V \subset U$ and $S \backslash V$ is connected. If

$$x, y \in S \backslash \{p\},$$

then $x \wedge y \in S \backslash \{p\}$ because p is meet irreducible. Thus $S \backslash \{p\}$ is a locally compact Lawson semilattice. Hence there exists a compact subsemilattice $W \subset S \backslash \{p\}$ such that $S \backslash V \subset W$ [6, Lemma 5.2]. Since $S \backslash V$ is connected, the closed semilattice B generated by $S \backslash V$ is a compact, connected subsemilattice of W; B is acyclic [13]; thus $H^*(S) \rightarrow H^*(B)$ is an isomorphism. Since $S \backslash V \subset B \subset W \subset S \backslash \{p\}$, we have $p \in S \backslash B \subset V \subset U$. Therefore p is marginal in S and hence peripheral in S [8, Th. 1.8].

COROLLARY 2.2. *Let L be a compact, connected topological lattice of finite codimension. If $p \in L$ is not a cutpoint of L and p is either meet irreducible or join irreducible, then p is marginal in L.*

Proof. Since L is compact and connected, its breadth is equal to its codimension [7, Cor. 2.4]. Hence L is a Lawson semilattice with respect to either $\vee$ or $\wedge$ [7, Th. 1.1]. Finally, L is locally connected [1, Th. 2]; therefore the conclusion follows from Theorem 2.1.

Peripheral elements need not belong to $(MI(L) \cup JI(L))^*$. Examples may be found in I^3, the unit cube.

THEOREM 2.3. *Let S be a compact, connected, locally connected topological semilattice with 1. If $p \in S$ and $M(p)^\circ = \varnothing$, then $M(p)$ is contained in the set of peripheral elements of L.*

Proof. We define $F: S \times S \rightarrow S$ by $F(x, y) = x \wedge y$ for all $x, y \in S$. Then F is continuous and $F(1, x) = x$ for all $x \in S$. If $s \in M(p)$ and s is an inner point of S, then there exists an open set U containing 1 such that for each $u \in U$ there is a $v \in S$ with $u \wedge v = s$ [8, Th. 3.4]. This implies $U \subset M(s) \subset M(p)$ so that $M(p)^\circ \neq \varnothing$ contrary to hypothesis. Thus s is peripheral in S; since s was an arbitrary element of $M(p)$, $M(p)$ consists entirely of peripheral elements of S.

As noted above a compact, connected topological lattice L is locally connected and has a 1; thus if $M(p)^\circ = \emptyset$ in such a lattice, then $M(p)$ consists of peripheral elements of L.

The set of peripheral elements of a topological space need not be closed [8, p. 261]. However, we have the following.

COROLLARY 2.4. *Let L be a compact, connected topological lattice of finite codimension. If $A = \{x \in L: M(x)^\circ = \emptyset\}$, then each element of A^* is peripheral in L.*

Proof. Let $x \in A^*$ and suppose that x is an inner point of L. Let $\{x_\alpha\}$ be a net in A which converges to x. Then $\{x_\alpha \vee x\}$ also converges to x, and since $M(x_\alpha \vee x) \subset M(x_\alpha)$, we must have

$$M(x_\alpha \vee x)^\circ = \emptyset .$$

Thus $\{x_\alpha \vee x\} \subset A$. By Theorem 2.3 and our assumption that x is inner, $M(x)^\circ \neq \emptyset$. Since the codimension of L is finite we may choose an inner point y of L such that $y \in M(x)^\circ$ [8, Th. 2.6]; thus y is also an inner point of $M(x)$ [8, Th. 1.4]. Since x is the zero of $M(x)$ and y is inner in $M(x)$, it follows from the proof of Thorem 2.3 that there must be an open subset U of $M(x)$ which contains x and such that $u \in U$ implies $u \leqslant y$. The net $\{x_\alpha \vee x\}$ converges to x and $\{x_\alpha \vee x\} \subset M(x)$; therefore there exists an α such that $x_\alpha \vee x \in U$. Hence $x_\alpha \vee x \leqslant y$; therefore $y \in M(x_\alpha \vee x)$ which implies

$$M(y) \subset M(x_\alpha \vee x) .$$

But $M(y)^\circ \subset M(y) \subset M(x_\alpha \vee x)$ and $M(y)^\circ \neq \emptyset$ since y is an inner point of L. Thus $M(x_\alpha \vee x)^\circ \neq \emptyset$ contrary to $x_\alpha \vee x \in A$. This contradiction completes the proof.

The set A^* of Corollary 2.4 has some interesting properties not necessarily held by either the set of all peripheral elements or by the set of all meet irreducible elements of a lattice.

PROPOSITION 2.5. *Let L and A^* be as in Corollary 2.4.*

(1) *$x \leqslant y$ and $x \in A^*$ imply $y \in A^*$.*

(2) *A^* is connected.*

(3) *If the breadth of L, $b(L)$, is two then A^* consists of meet irreducible elements of L.*

(4) *If $b(L) = 2$ and $w(MI(L)) = n$, then A^* is the union of at most n compact, connected chains.*

(5) *If A^* is a sublattice of L, then $A^* = M(\bigwedge A^*)$ and $b(A^*) = cd(A^*) < b(L) = cd(L)$.*

Proof. (1) Clearly A is an increasing set; hence A^* is also.

(2) For each $x \in A^*$, $M(x)$ is a connected subset of A^* which contains 1. Thus A^* is connected.

(3) Let $x \in A^*$. Since $M(x) \subset A^*$ and $b(L) = 2$, $b(M(x)) = cd(M(x)) = 1$ [8, Th. 3.2]. Thus $x = a \wedge b$ implies $x = a$ or $x = b$.

(4) Since $A^* \subset MI(L)$, $w(A^*) \leqslant n$. Hence by Dilworth's theorem [5, Th. 1.1], A^* is the union of n or fewer chains. These chains may be chosen to be compact and connected.

(5) If A^* is a sublattice of L, then $z = \bigwedge A^*$ belongs to A^* and $A^* = M(z)$. As noted above $b(A^*) = cd(A^*) < cd(L) = b(L)$.

If $b(L) > 2$, then $M(x) \subset A^*$ need not imply $M(x)$ is a chain; examples may be found in I^3, the unit cube.

Let $L = I^2 \backslash \{(x, y)\colon 0 \leqslant x < 1/4, 3/4 < y \leqslant 1\}$. Then L is a compact, connected, distributive topological lattice of breadth two and A^* is a proper subset of $MI(L)$.

EXAMPLE 2.6. Let

$$L = \{(x_i)\colon 0 \leqslant x_i \leqslant 1\} \cup \{(x_i)\colon -1 \leqslant x_i \leqslant 0\} \subset \prod_{i=1}^{\infty} R_i ,$$

R_i the set of real numbers for $i = 1, 2, \cdots$. With the order and topology inherited from $\prod_{i=1}^{\infty} R_i$, i.e., $(x_i) \leqslant (y_i)$ if and only if $x_i \leqslant y_i$ for $i = 1, 2, \cdots$, L is a compact, connected topological lattice. Since $p = (p_i)$ with $p_i = 0$ for $i = 1, 2, \cdots$ is a cutpoint of L, p is an inner point of L. Any $(x_i) \in L$ with $0 < x_i \leqslant 1$ for infinitely many i has the property that $M((x_i))^\circ$ is empty. Thus

$$p \in \{(x_i)\colon M((x_i))^\circ = \varnothing\}^* .$$

THEOREM 2.7. *Let L be a compact, connected topological lattice. If $a, b \in L$ and a is a meet complement for b, then $[0, a]$ and $[0, b]$ are contained in the set of peripheral elements of L.*

Proof. We define $F\colon L \times L \to L$ by $F(x, y) = x \vee y$ for all $x, y \in L$. Then F is continuous and $F(0, y) = y$ for all $y \in L$. Let $x \in (0, a] = [0, a] \backslash \{0\}$; then $x \wedge b \leqslant a \wedge b = 0$ which implies $x \wedge b = 0$. If x is not peripheral in L, then there exists an open set U containing 0 such that for each $s \in U$ there is a $t \in L$ for which $s \vee t = x$ [8, Th. 3.4]. Since $[0, b] = b \wedge L$, it is connected; thus $U \cap (0, b] \neq \varnothing$. Let $s \in U \cap (0, b]$ and let $t \in L$ be such that $s \vee t = x$. Then $s \leqslant x$ and $s \leqslant b$; thus $s \leqslant x \wedge b = 0$ which implies $s = 0$ contrary to

$s \in (0, b]$. Hence x is peripheral in L. That 0 is peripheral is a consequence of Theorem 2.3. Thus each element of $[0, a]$ is peripheral in L. The proof for $[0, b]$ is similar.

The following corollaries are immediate.

COROLLARY 2.8. *Let L be a compact, connected topological lattice. If $a, b \in L$ are not related, then $[a, a \vee b]$, $[b, a \vee b]$, $[a \wedge b, a]$, and $[a \wedge b, b]$ are contained in the set of peripheral elements of*

$$[a \wedge b,\ a \vee b]\ .$$

COROLLARY 2.9. *Let L be a compact, connected topological lattice. If for $p \in L$ there is a $q \in L$ such that q is not related to p and either $p \in M(p \wedge q)^\circ$, or $p \in L(p \vee q)^\circ$, then p is peripheral in L.*

3. **Questions.** 3.1. It is known [8, Ex. 1.9] that peripheral elements of topological spaces need not be marginal. Is this true for compact, connected lattices?

3.2. If B is the set of all peripheral (marginal) elements of a compact lattice L, is B closed in L?

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MIDDLE TENNESSEE STATE UNIVERSITY