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ON SOME MEAN VALUES ASSOCIATED WITH A RANDOMLY SELECTED SIMPLEX IN A CONVEX SET

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For any convex body K in euclidean n-space denote by $m(K)$ the mean value of the volume of a simplex with vertices at $n+1$ randomly selected points from K. It is shown that among all convex bodies of given volume the mean value $m(K)$ is minimal if and only if K is an ellipsoid. Actually, a more general result is obtained which shows that the higher order moments of the volume of a randomly selected simplex in a convex set have similar minimal properties.

Throughout this paper R^n denotes euclidean n-space, where n is a given fixed positive integer. A compact convex subset of R^n which has interior points will be called a convex body. The volume of a convex body X will be denoted by $v(X)$. If $p_1, p_2, \cdots, p_{n+1}$ are $n+1$ points of R^n we write $C(p_1, p_2, \cdots, p_{n+1})$ to denote the convex hull of the points $p_1, p_2, \cdots, p_{n+1}$. Including various forms of degeneracy, $C(p_1, p_2, \cdots, p_{n+1})$ will be called a simplex with vertices at $p_1, p_2, \cdots, p_{n+1}$.

Let K be a given convex body. If $x_1, x_2, \cdots, x_{n+1}$ are $n+1$ points from K the volume of the simplex with vertices at $x_1, x_2, \cdots, x_{n+1}$ is given by $v(C(x_1, x_2, \cdots, x_{n+1}))$ and, assuming that the points $x_1, x_2, \cdots, x_{n+1}$ are variable, the mean value of this volume is defined by

$$(1)\qquad m(K) = (1/v(K))^{n+1}\int_{x_1\in K}\cdots\int_{x_{n+1}\in K} v(C(x_1, \cdots, x_{n+1}))\, dx_1 \cdots dx_{n+1}\,.$$

Since $v(C(x_1, x_2, \cdots, x_{n+1})$ is a continuous function in the space $R^{n(n+1)}$ and since the set defined by the $n+1$ conditions $x_i \in K$ $(i = 1, 2, \cdots, n+1)$ is a compact convex set in $R^{n(n+1)}$ it is obvious that $m(K)$ exists for every convex body K.

Blaschke [1], [2] has proved that for convex bodies in R^2 of given volume (i.e., area) the mean value $m(K)$ is minimal if and only if K is an ellipse. See also Klee [11] for the history of this problem. Kingman [10] has conjectured that for any dimension n and fixed volume $v(K)$ the minimum of $m(K)$ is reached if K is a (solid) sphere in R^n. In addition, he pointed out that the higher order moments of the expected volume, i.e., the expressions

$$(2)\qquad m_r(K) = (1/v(K))^{n+1}\int_{x_1\in K}\cdots\int_{x_{n+1}\in K}(v(C(x_1, \cdots, x_{n+1})))^r dx_1 \cdots dx_{n+1}$$

are of interest. The definitions (1) and (2) show that $m_1(K) = m(K)$.

Just as before, it is seen that $m_r(K)$ exists for every convex body K and every $r \geqq 0$. It is also clear that $m_r(K)$ is invariant under volume preserving affine transformation.

The main purpose of this paper is to provide a proof of Kingman's conjecture and of a similar but more general statement for the higher order moments. The following theorem contains the precise formulation of our result.

THEOREM. *For any convex body K in R^n and any real number r with $r \geqq 1$ the moments $m_r(K)$ satisfy the inequality*

$$m_r(S) \leqq m_r(K)$$

where S is a solid sphere in R^n which has the same volume as K. Equality holds if and only if K is an ellipsoid.

Because of $m_1(K) = m(K)$ this theorem has the following corollary as an obvious consequence.

COROLLARY 1. *Among all convex bodies of given volume the mean value $m(K)$ of the volume of a simplex with vertices at $n + 1$ randomly selected points from the convex body K is minimal if and only if K is an ellipsoid.*

Kingman [10] has been able to find an explicit formula for $m(K)$ in the case when K is an ellipsoid of R^n, namely

$$m(K) = 2^n \binom{n+1}{\frac{1}{2}(n+1)}^{n+1} \binom{(n+1)^2}{\frac{1}{2}(n+1)^2}^{-1} v(K) .$$

Corollary 1 is related to a problem which, in two dimensional space, is frequently referred to as Sylvester's problem (cf. Kendall and Moran [9]). If $n + 2$ points of R^n are selected at random from a convex body K the problem consists of finding the probability, say $P(K)$, that none of these $n + 2$ points is in the interior of their convex hull. A simple calculation shows that (see Kingman [10])

$$P(K) = 1 - \frac{(n+2)m(K)}{v(K)} .$$

It follows that Corollary 1 is equivalent with the following statement.

COROLLARY 2. *For any convex body K of R^n the probability $P(K)$ that the convex hull of $n + 2$ randomly selected points from K con-*

tains none of these points in its interior is maximal if and only if K is an ellipsoid.

Similarly as the proof given by Blaschke for $n = 2, r = 1$ our proof of the above theorem depends on a property of the Steiner symmetrization of a convex body and on a certain characterization of ellipsoids. Since this characterization, which is of independent interest, appears to have been investigated only in the cases $n = 2$ and $n = 3$ (see Bonnesen and Fenchel [4], p. 143) we supply a new proof which imposes no restriction on the dimension or regularity of the convex body (Lemma 2).

First, we prove a lemma which shows that there exist convex bodies which have the desired minimal property with respect to $m_r(K)$.

LEMMA 1. *If r is a given positive number there exists a convex body K_0 in R^n such that $v(K_0) = 1$ and*

$$m_r(K_0) \leqq m_r(K) \tag{3}$$

for every convex body K with $v(K) = 1$.

Proof. For every convex body K there exist, according to a theorem of John [8], two ellipsoids E, E' such that $E' \subset K \subset E$ and $v(E) \leqq n^n v(E')$. Because of $v(E') \leqq v(K)$ this implies $v(E) \leqq n^n v(K)$. It follows that to any K with $v(K) = 1$ there is a volume preserving affine transformation σ such that $\sigma K \subset B$, where B is a sphere of volume n^n and center at the origin of the coordinate system. Because of this fact and because of the invariance of m_r under volume preserving affine transformations it is evident that it suffices to prove (3) under the additional assumptions that $v(K) = 1$ and $K \subset B$. Let us denote by $\mathscr{K}$ the class of all convex bodies for which these two conditions are satisfied. If a number μ is defined by

$$\mu = \inf m_r(K) \qquad (K \in \mathscr{K})$$

then μ has obviously the property that for every $K \in \mathscr{K}$

$$\mu \leqq m_r(K) \tag{4}$$

and that there exists a sequence $K_1, K_2, \cdots$ of convex bodies in $\mathscr{K}$ such that

$$\lim_{i \to \infty} m_r(K_i) = \mu \, . \tag{5}$$

Because of $K_i \subset B$ the selection theorem of Blaschke can be applied to the class of convex bodies K_i. This justifies the assumption that the sequence $K_1, K_2, \cdots$ converges (in the Hausdorff-Blaschke metric) to some convex set K_0. Note that $K_i \in \mathscr{K}$ implies $K_0 \in \mathscr{K}$.

The functional m_r is obviously translation invariant, monotone and homogeneous in the sense that $m_r(sK) = s^{n-r}m_r(K)$ for any $s \geqq 0$. It is known that such a functional is also continuous (cf. Hadwiger [7], p. 204 and the proof of the continuity of the volume in Blaschke [3], p. 61 or Eggleston [6], p. 72). Therefore, the convergence of K_i to K_0 implies

$$\lim_{i\to\infty} m_r(K_i) = m_r(K_0) \ . \tag{6}$$

Since (3) is an immediate consequence of (4), (5), and (6) the proof of the Lemma is finished.

K_0 will be referred to as a *minimum body* for m_r. Actually, K_0 does not depend on r if $r \geqq 1$; but this cannot be concluded from our proof of Lemma 1.

For the formulation of our next lemma it is convenient to call a subset of R^n *flat* if it is contained in some plane. It should be noted that in this paper a plane is always understood to be a hyperplane. As a further notational simplification the following concept will be used. If K is a convex body and if G is a line in R^n we denote by $\mathscr{P}(K, G)$ the set of midpoints of all line segments of the form $X \cap K$ where X ranges over all lines that are parallel to G and meet K. $\mathscr{P}(K, G)$ will be called a *midpoint set* of K.

LEMMA 2. *A convex body K is an ellipsoid if and only if the midpoint set $\mathscr{P}(K, G)$ is flat for every line G of R^n.*

Proof. If K is a sphere the midpoint set $\mathscr{P}(K, G)$ is obviously flat for every line G. Applying an affine transformation the same result is seen to be true for ellipsoids.

Assume now that for a given convex body K the midpoint set $\mathscr{P}(K, G)$ is flat for every line G. Let H be any plane, and choose a coordinate system in R^n which has the property that H is given by $H = \{(x^1, x^2, \cdots, x^n) \mid x^n = 0\}$. Then, if G is a line that is orthogonal to H, the equation of the plane which contains $\mathscr{P}(K, G)$ can be written in the form

$$x^n = a_0 + a_1x^1 + \cdots + a_{n-1}x^{n-1} \ .$$

The symmetrization of K with respect to the plane H is achieved by mapping each point $(p^1, p^2, \cdots, p^n)$ of K onto the point

$$(p^1, p^2, \cdots, p^{n-1}, p^n - (a_0 + a_1p^1 + \cdots + a_{n-1}p^{n-1})) \ .$$

This mapping is obviously an affine transformation. Hence, one can conclude that every symmetrization is a volume preserving affine transformation, provided that the midpoint set $\mathscr{P}(K, G)$ is flat for every line G of R^n.

The convex body obtained from K by symmetrization with respect to a plane H will be denoted by $\tilde{K}(H)$.

It is known (see Danzer, Laugwitz, and Lenz [5]) that there is an ellipsoid, say L, which contains K and has smallest possible volume. It is also known (see Hadwiger [7], p. 170) that there is a sequence of planes, say $H_1, H_2, \cdots$, in R^n such that the sequence of convex bodies which is defined by $K_1 = K$, $K_{i+1} = \tilde{K}_i(H_i)$ $(i = 1, 2, \cdots)$ contains a subsequence that converges to a sphere S. It follows that there are volume preserving affine transformations $\sigma_1, \sigma_2, \cdots$ such that the sequence $\sigma_1 K, \sigma_2 K, \cdots$ converges to S. If $K = L$ the proof of the lemma is obviously finished. Let us assume that $K \neq L$. In this case we have

$$v(K) = v(S) < v(L) \; . \tag{7}$$

Since the sequence $\sigma_1 K, \sigma_2 K, \cdots$ converges to S there exists for any positive ε an index h such that

$$\sigma_h K \subset S^\varepsilon \; . \tag{8}$$

Here, S^ε denotes the parallel domain of S, which, in this case is a sphere of radius $r + \varepsilon$ if S has radius r. Because of (7) ε can be taken so small that

$$v(S^\varepsilon) < v(L) \; . \tag{9}$$

(8) implies that the ellipsoid $\sigma_h^{-1} S^\varepsilon$ contains K, and (9) shows that $v(\sigma_h^{-1} S^\varepsilon) < v(L)$. However, according to the definition of L it is impossible that an ellipsoid which contains K has smaller volume than L. It follows that the trivial case $K = L$ is the only possiblility.

LEMMA 3. *Let $G_1, G_2, \cdots, G_{n+1}$ be $n + 1$ distinct lines in R^n which are of the form $G_k = \{(c_k^1, c_k^2, \cdots, c_k^{n-1}, z_k) \mid -\infty < z_k < \infty\}$. Assume that to each G_k there corresponds an interval I_k of the form $I_k = \{(c_k^1, c_k^2, \cdots, c_k^{n-1}, z_k) \mid |z_k - p_k| \leqq l_k\}$ where $l_k > 0$. Write $z = (z_1, z_2 \cdots, z_{n+1})$, $p = (p_1, p_2, \cdots, p_{n+1})$, $e = (1, 1, \cdots, 1)$, $c^j = (c_1^j, c_2^j, \cdots, c_{n+1}^j)$ and*

$$D(z) = \frac{1}{n} \det (e, c^1, c^2, \cdots, c^{n-1}, z) \; .$$

Finally, if r is a given real number with $r \geqq 1$ write

$$M(p) = \int_{|z_k - p_k| \leqq l_k} |D(z)|^r \, dz \; . \tag{10}$$

Then, if the numbers c_k^i *and the interval lengths* l_k *are fixed,* $M(p)$ *attains its absolute minimum value exactly for those vectors* p *for which all the midpoints* $(c_k^1, c_k^2, \cdots, c_k^{n-1}, p_k)$ *of the intervals* I_k $(k = 1, 2, \cdots, n+1)$ *are contained in some plane of* R^n.

Proof. Since $D(z)$ is a linear function of z (10) can be written in the form

$$M(p) = \int_{|u_k| \leqq l_k} |D(u) + D(p)|^r du \tag{11}$$

where $u = (u_1, u_2, \cdots, u_{n+1})$ and $u = z - p$. If p varies over the total R^{n+1} the linear function $D(p)$ takes on any value between $-\infty$ and ∞. Therefore, a comparison of (11) with the function

$$F(y) = \int_{|u_k| \leqq l_k} |D(u) + y|^r du \tag{12}$$

shows that $M(p)$ and $F(y)$ have the same greatest lower bound. If all y-values for which $F(y)$ is (absolutely) minimal are known, the set of all vectors p for which $M(p)$ is minimal are found by solving the linear equation

$$y = D(p) \tag{13}$$

for each such known y-value.

Now, to investigate the minimum value of $F(y)$ we note that $D(u) = -D(-u)$ implies

$$\int_{|u_k| \leqq l_k} |D(u) + y|^r du = \int_{|u_k| \leqq l_k} |D(u) - y|^r du .$$

This, together with the definition (12), shows that

$$\begin{aligned} &F(y) - F(0) \\ &= \frac{1}{2} \int_{|u_k| \leqq l_k} (|D(u) + y|^r + |D(u) - y|^r - 2|D(u)|^r) du . \end{aligned} \tag{14}$$

Since for a fixed value of r $(r \geqq 1)$ the function $|\zeta|^r$ is convex it follows that the integrand in (14), say $T(u, y)$, has the property that for all values of u and y

$$T(u, y) \geqq 0 . \tag{15}$$

(The convexity of the function $|\zeta|^r$, i.e., the relation $|(\zeta_1 + \zeta_2)/2|^r \leqq (|\zeta_1|^r + |\zeta_2|^r)/2$, is a special case of Hölder's inequality $|\alpha a + \beta b| \leqq (|\alpha|^p + |\beta|^p)^{1/p}(|a|^q + |b|^q)^{1/q}$, namely the case $\alpha = \beta = 1/2$, $a = \zeta_1$, $b = \zeta_2$, $p = r/r-1$, $q = r$). In addition to (15) it is clear that for $y \neq 0$

(16) $$T(0, y) = 2\,|\,y\,|^r > 0 \,.$$

Because of the continuity of $T(u, y)$ as a function in u (16) implies that for a given value of y with $y \neq 0$ the inequality

(17) $$T(u, y) > 0$$

holds not only for $u = 0$ but for a whole interval with center at $u = 0$. From (14), (15), and (17) it follows that for any $y \neq 0$

$$F(y) > F(0) \,.$$

Hence, $F(y)$ attains an absolute minimum value at $y = 0$ and nowhere else. This result in conjunction with (13) shows that $M(p)$ is minimal if and only if $D(p) = 0$. Since $D(p)$ is the volume of a simplex with vertices at the points $(c_k^1, c_k^2, \cdots, c_k^{n-1}, p_k)$ we find finally that these points are contained in a plane if and only if $M(p)$ is minimal.

Proof of the Theorem. Since it has already been pointed out that $m_r(K)$ is a homogeneous function of K it suffices to prove the Theorem under the assumption $v(K) = 1$.

As before, let H be the plane $\{(x^1, x^2, \cdots, x^n) \mid x^n = 0\}$. Assume that $G_1, G_2, \cdots, G_{n+1}$ are $n + 1$ given lines which are orthogonal to H and have the property that each G_k intersects K in a line segment I_k of positive length l_k. The midpoint of I_k will again be denoted by $(c_k^1, c_k^2, \cdots, c_k^{n-1}, p_k)$. Under these assumptions the number $M(p)$ can be defined by (11). However, since in this case the vector p is completely determined if K and $G_1, G_2, \cdots, G_{n+1}$ are given we write now $M(K; G_1, G_2, \cdots, G_{n+1})$ instead of $M(p)$. Let $\widetilde{K} = \widetilde{K}(H)$ be the convex body which is obtained from K by symmetrization with respect to the plane H. Since all the segments $\widetilde{K} \cap G_k$ have midpoints that are contained in a plane, namely H, Lemma (3) shows that

(18) $$M(\widetilde{K}; G_1, G_2, \cdots, G_{n+1}) \leqq M(K; G_1, G_2, \cdots, G_{n+1})$$

where equality holds if and only if the midpoints of the segments $K \cap G_k$ are already contained in some plane. Assume now that K is a minimum body for m_r and that K is not an ellipsoid. Then Lemma 2 shows that there is a line G such that the midpoint set $\mathscr{P}(K, G)$ is not flat. This implies obviously that $\mathscr{P}(K, G)$ contains $n + 1$ points which are not contained in a plane of R^n. A simple continuity argument shows further that one may assume that the line segments corresponding to these $n + 1$ midpoints have positive lengths. A suitable selection of the coordinate system permits us to assume that the plane $H = \{(x^1, x^2, \cdots, x^n) \mid x^n = 0\}$ is orthogonal to G. Hence, if $G_1, G_2, \cdots, G_{n+1}$ is any system of $n + 1$ lines that are parallel to G

and meet K in intervals of positive lengths one obtains (18) and the additional information that strict inequality holds for at least one such system of $n+1$ lines.

Denote now by K_H the projection of K onto the plane H. Further, if w_k is a point of K_H denote by $G(w_k)$ the line which is orthogonal to H and contains w_k. Using the definitions (2) and (10) an obvious rearrangement of the order of integration shows that

$$\begin{aligned} &m_r(K) \\ (19)\qquad &= \int_{w_1 \in K_H} \cdots \int_{w_{n+1} \in K_H} M(K; G(w_1), G(w_2), \cdots, G(w_{n+1})) dw_1 \cdots dw_{n+1} \,. \end{aligned}$$

(Since the integrand has been defined only if the intervals $K \cap G(w_k)$ have positive lengths and if the points w_k are distinct, a set of measure 0 has been neglected.) Because of (18) with strict inequality for at least one system $w_1, w_2, \cdots, w_{n+1}$ and because of the continuity of the integrand in (19) (considered as a function of $w_1, w_2, \cdots, w_{n+1}$) the equation (19) implies that

$$m_r(\tilde{K}(H)) < m_r(K) \,.$$

This contradicts the assumption that K be a minimum body for m_r. Therefore, only ellipsoids can be minimal bodies. Because of Lemma 1 and since m_r is invariant under volume preserving affine transformations it follows that any sphere S of unit volume is a minimal body, that

$$m_r(S) < m_r(K)$$

if K is not an ellipsoid, and that

$$m_r(S) = m_r(K)$$

if K is an ellipsoid. Hence, the Theorem is proved.

It might be worth noting that essentially the same method of proof can be used to establish a similar theorem with the higher order moments replaced by more general types of functions.

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