PACIFIC JOURNAL OF MATHEMATICS
Vol. 46, No. 2, 1973

# $(KE)$-DOMAINS

SURJEET SINGH

**A commutative ring $R$ is said to have the $(K)$-property if for each of its proper ideals $A$, there exists an ideal $A'$, such that $AA'$ is a nonzero principal ideal of $R$. A domain $D$ with unity $1 \neq 0$ is said to be a $(KE)$-domain, if each of its ideals $A$, considered as a ring, has the $(K)$-property. The concept of a $(KE)$-domain had been studied earlier by the author and R. Kumar. In this paper injective modules and flat modules are studied and characterizations of $(KE)$-domains in terms of these modules are established. Finally the problem of embedding of a $(KE)$-domain in $\hat{Z}_{(p)}$, the $p$-adic completion ($p$ a prime number) of the ring $Z$ of integers, is studied.**

In [11], the concept of a $(KE)$-domain was introduced and a structure theorem for the same was established. The study of $(KE)$-domains was continued in [12], in which, their characterizations in terms of Dedekind domains, Prüfer domains and generalized Krull domains were proved. The present paper is also concerned with the study of $(KE)$-domains and it contains some further characterizations. Let $D$ be a domain with unity $1 \neq 0$. For any proper ideal $A$ of $D$, let $A^*$ denote the subring of $D$ generated by $A \cup \{1\}$. In § 1, we study injective modules and prove that, if a proper ideal $A$ of a domain $D$ is such that $A^*$ is Noetherian and every injective $D$-module is injective as an $A^*$-module, then $D = A^*$(Theorem 2). This theorem yields a characterization of $(KE)$-domains given in Theorem 3. In § 3, we study flat modules and prove that a domain $D$ is a $(KE)$-domain if and only if it is a flat $A^*$-module for each of its proper ideals $A$ (Theorem 6). Theorem 2 in [12] is deduced as a corollary to Theorem 6. The other important result in § 2 is Theorem 5. Example 1 shows that if a domain $D$ is a fiat $A^*$-module for some proper ideal $A$, it need not equal $A^*$. Let $Z$ be the ring of integers and $p$ any prime number; it was shown in [11, Example 4] that $\hat{Z}_{(p)}$, the $p$-adic completion of the quotient ring $Z_{(p)}$ is a $(KE)$-domain. In § 3, we prove that $\hat{Z}_{(p)}$ is a maximal $(KE)$-domain, in the sense that, if $D$ is any $(KE)$-domain, different from its quotient field, such that some prime number $p$ is not invertible in it, then $D$ is embeddable in $\hat{Z}_{(p)}$ (Theorem 8). Other results of interest are Proposition 1, Lemma 13, and Theorem 9. The notations and terminology are essentially the same as in [10, 11], except that, all rings considered here are with unity $1 \neq 0$, all modules are unital, and by a proper prime ideal of a ring $R$ is meant a prime ideal different from both (0) and $R$.

1. **Injective modules.** A ring $R$ (not necessarily with unity) is said to have the $(K)$-property if for each of its proper ideals $A$, there exists an ideal $A'$ of $R$, such that $AA'$ is a nonzero principal ideal of $R$ [11]. A domain $D$ is said to be a $(KE)$-domain if each of its ideals $A$, considered as a ring, has the $(K)$-property [11, Definition 3]. For any domain $D$ (not necessarily with unity) having $F$ as its quotient field, let $D^*$ denote the subring of $F$ generated by $D \cup \{1\}$, where 1 is the unity of $F$. The following lemmas, which we state without proof, were proved in [11, Lemma 1 and Theorem 13].

LEMMA 1. *A domain $D$ (not necessarily with unity) has the $(K)$-property if and only if $D^*$ is a Dedekind domain.*

LEMMA 2. *A proper ideal $A$ of a domain $D$ (with unity) has the $(K)$-property if and only if $D = A^*$ and $D$ is a Dedekind domain.*

The following lemma is an immediate consequence of the above lemmas.

LEMMA 3. *A domain $D$ is a $(KE)$-domain if and only if it is a Dedekind domain and for each of its proper ideals $A$, $A^* = D$.*

For the definitions and fundamental properties of injective modules the reader may refer to Tsai-Chi-Te [13]. A ring $R$ is said to be self-injective ring, if $R_R$ is an injective module. We now establish the following.

PROPOSITION 1. *A domain $D$ is a $(KE)$-domain if and only if $D = A^*$ for each of its proper ideals $A$.*

*Proof.* "Only if" follows from Lemma 3.

Suppose that for every ideal $A$ of $D$, we have $D = A^*$. Since $D/A = A^*/A \cong Z/(n)$ for some $n \geqslant 0$ and $Z/(n)$ is Noetherian, we get that $D$ is Noetherian. Consider any proper prime ideal $P$ of $D$. Then $D/P = P^*/P$ is either isomorphic to $Z$ or to $Z/(p)$, for some prime number $p$. In the former case, for every $k(\neq 0) \in Z$, $k1 \notin P$; consequently $k1 \notin P^2$ and $D/P^2 = (P^2)^*/P^2 \cong Z$. This gives that $P^2$ is a prime ideal of $D$: this is not possible in a Noetherian domain. Hence $D/P \cong Z/(p)$, for some prime number $p$ and hence for every proper ideal $A$ of $D$, $D/A \cong Z/(n)$ for some $n \geqslant 2$. Thus every proper homomorphic image of $D$ is self-injective, since every proper homomorphic image of $Z$ is self-injective. Hence by Levy [6], $D$ is a Dedekind domain. Hence by Lemma 3, $D$ is a $(KE)$-domain.

LEMMA 4. *Let $D$ be a domain and $A$ be a proper ideal of $D$. Then $A^*$ is Noetherian if and only if $D$ is Noetherian and a finite $A^*$-module.*

*Proof.* Let $A^*$ be Noetherian. Suppose to the contrary that $D$ is not a finite $A^*$-module. Then there exists a denumerable subset $S = \{b_i: i = 1, 2, \cdots\}$ of $D$ such that the $A^*$-submodule of $D$ generated by $S$ cannot be generated by a finite subset of $S$. Choose $a$ $(\neq 0) \in A$. As $A^*$ is Noetherian and $Sa \subset A^*$, there exists a positive integer $n$ such that the ideal of $A^*$ generated by the elements $b_i a$ $(1 \leqslant i \leqslant n)$ is the same as that generated by $Sa$. This yields that for each $i \geqslant n+1$, $b_i a = \sum_{j=1}^{n} a_{ij} b_j a$ for some $a_{ij} \in A^*$, and hence $b_i = \sum_{j=1}^{n} a_{ij} b_j$. Consequently the finitely many elements $b_i (1 \leqslant i \leqslant n)$ generate the $A^*$-submodule of $D$ generated by $S$; this gives a contradiction. Hence $D$ is a finite $A^*$-module. It is now immediate that $D$ is Noetherian, since $A^*$ is Noetherian. The converse follows by Eakin [5, Theorem 2]. Finally, the second part is an immediate consequence of [14, Chap. V, p. 255].

If $S$ is a subring of a ring $R$ such that it contains the unity element of $R$, then every $R$-module can be regarded as an $S$-module in a natural way. In the following lemmas, $D$ will be a domain having a proper ideal $A$, such that $A^*$ is Noetherian and every injective $D$-module is injective as an $A^*$-module. For any $D$-module $M$ $E(M)$ and $E'(M)$ will denote its $D$-injective hull and $A^*$-injective hull respectively.

LEMMA 5. *Every indecomposable injective $D$-module is an indecomposable injective $A^*$-module.*

*Proof.* Let $M$ be an indecomposable injective $D$-module. By the hypothesis $M$ is also an injective $A^*$-module. Let $M = M_1 \oplus M_2$ for some $A^*$-submodules $M_i (i = 1, 2)$. As $M_1$ is an injective $A^*$-module, it is a divisible $A^*$-module. Consider $b (\neq 0) \in D$. Choose $a (\neq 0) \in A$. As $ab \in A$ and $ab \neq 0$, $M_1 = M_1 ab$. This implies that $M_1 = M_1 b$ and $M_1$ is a $D$-submodule of $M$. Similarly $M_2$ is a $D$-submodule of $M$. Hence $M_1 = (0)$ or $M_2 = (0)$. This proves the lemma.

LEMMA 6. *Let $M$ and $N$ be any two divisible $D$-modules. Then:*
( i ) *Any $A^*$-homomorphism of $M$ into $N$ is a $D$-homomorphism,*

(ii) *M and N are isomorphic as D-modules if and only if they are isomorphic as $A^*$-modules.*

(iii) $\mathrm{Hom}_D(M, M) = \mathrm{Hom}_{A^*}(M, M)$.

*Proof.* Let $\sigma: M \rightarrow N$ be any $A^*$-homomorphism. Let $x \in M$ and $b(\neq 0) \in D$. Choose $a(\neq 0) \in A$. Then $ab \in A^*$. As $M$ is a divisible $D$-module there exists $y_\varepsilon$, $M$ such that $x = ya$. Then $xb = yab$ and $\sigma(xb) = \sigma(yab) = \sigma(y)ab = \sigma(x)b$. Hence $\sigma$ is a $D$-homomorphism (ii) and (iii) are immediate consequences of (i).

We need the following two results due to Matlis [7], which we state without proof.

PROPOSITION 2. *Let $R$ be a commutative Noetherian ring. Then there exists a one-to-one correspondence between the prime ideals $P(\neq R)$ of $R$ and the indecomposable injective $R$-modules, given by $P \leftrightarrow E(R/P)$, where $E(M)$ denotes the injective hull of any $R$-module $M$. If $Q$ is an irreducible $P$-primary ideal, then $E(R/P) = E(R/Q)$.*

THEOREM 1. *With the same notation as in Proposition 2, let $E = E(R/P)$ be an indecomposable injective $R$-module and*

$$H = \mathrm{Hom}_R(E, E) .$$

*Then $H$ is isomorphic to $\hat{R}_P$, the $PR_P$-adic completion of $R_P$. More precisely, $E$ is a faithfull $\hat{R}_P$-module and each $R$-endomorphism of $E$ can be realized by multiplication by an element of $\hat{R}_P$.*

We now prove the following.

LEMMA 7. *$P \leftrightarrow P \cap A^*$ is a one-to-one correspondence between proper prime ideals $P$ of $D$ and proper prime ideals of $A^*$.*

*Proof.* By Lemma 4, $D$ is Noetherian. Thus by Proposition 2, $P \leftrightarrow E(R/P)$ is a one-to-one correspondence between the prime ideals $P$ of $D$ and the indecomposable injective $D$-modules. By Lemma 5 $E(D/P) = E'(A^*/A^* \cap P)$, the $A^*$-injective hull of $A^*/A^* \cap P$. From Proposition 2 and Lemma 6 we get that $P \rightarrow A^* \cap P$ is a one-to-one mapping of the set of all prime ideals $P$ of $D$ into the set of all prime ideals of $A^*$. By Lemma 4, $D$ is integral over $A^*$. Therefore given a prime ideal $P'$ of $A^*$, there exists a prime ideal $P$ of $D$ such that $P \cap A^* = P'$ [14, p. 223, Theorem 3]. This completes the proof.

LEMMA 8. *Let $P$ be a proper prime ideal of $D$. There exists a one-to-one inclusion preserving correspondence between the $P$-primary ideals of $D$ and the $P \cap A^*$-primary ideals of $A^*$. Further for any irreducible $P$-primary ideal $Q$ of $D$, the corresponding primary ideal* *of* $A^*$ *is* $A^* \cap Q$.

*Proof*. Consider $E = E(D/P) = E'(A^*/A^* \cap P)$. By Lemma 6, $\mathrm{Hom}_A^*(E, E) = \mathrm{Hom}_D(E, E)$. It follows from Theorem 1 that there exists an isomorphism $\sigma$ of $\hat{D}_P$ onto $\hat{A}^*_{P'}$, where $P' = P \cap A^*$, such that for any $d \in \hat{D}_P$ and $x \in E$, $xd = x\sigma(d)$. By Cohen [3, Theorem 2], for any local ring $(R, M)$, if $\hat{R}$ is the completion of $R$, then $\hat{M} = M\hat{R}$ is the unique maximal ideal of $R$ and $Q \leftrightarrow Q\hat{R}$ is a one-to-one correspondence between the $M$-primary ideals $Q$ of $R$ and the $\hat{M}$-primary ideals of $\hat{R}$. Thus $Q \leftrightarrow Q\hat{D}_P$ is a one-to-one correspondence between the $P$-primary ideals $Q$ of $D$ and $P\hat{D}_P$-primary ideals of $\hat{D}_P$. For any $P$-primary ideal $Q$ of $D$, $\sigma(Q\hat{D}_Q) \cap A^*$ is a $P'$-primary ideal of $A^*$, and $Q \leftrightarrow \sigma(Q\hat{D}_P) \cap A^*$ is a one-to-one correspondence between the $P$-primary ideals $Q$ of $D$, and $P'$-primary ideals of $A^*$. Let $Q$ be an irreducible $P$-primary ideal of $D$. By Matlis [7, Lemma 32], there exists $x \in E$ for which $\mathrm{ann}_D(x) = Q$. Then $\mathrm{ann}_{\hat{D}_P}(x) = Q\hat{D}_P$ and $\mathrm{ann}_{\hat{A}^*_P}(x) = \sigma(Q\hat{D}_P)$, so that $\mathrm{ann}_{A^*}(x) = \sigma(Q\hat{D}_P) \cap A^*$. At the same time $\mathrm{ann}_{A^*}(x) = \mathrm{ann}_D(x) \cap A^* = Q \cap A^*$. This shows that

$$Q \cap A^* = \sigma(Q\hat{D}_P) \cap A^* .$$

Hence the lemma follows.

THEOREM 2. *If* $A$ *is any proper ideal of a domain* $D$ *such that* $A^*$ *is Noetherian and every injective* $D$*-module is an injective* $A^*$*-module then* $D = A^*$.

*Proof*. Let $A = P$ be a prime ideal. Then either $P^*/P \cong Z/(p)$, for some prime number $p$ or $P^*/P \cong Z$. Now $E(D/P) = E(P^*/P)$ implies that $\hat{D}_P = \hat{P}^*_P$. From this we obtain that the quotient field of $D/P$ is isomorphic to the quotient field of $P^*/P$. If $P^*/P \cong Z/(p)$, then $D/P \cong Z/(p) \cong P^*/P$ and $D = P^*$. If $P^*/P \cong Z$, then the quotient field of $D/P$ is isomorphic to the field $R$ of rational numbers. Since every overring of $Z$, contained in $R$, is of the type $Z_S$, we get that $D/P \cong Z_S$ for some multiplicative subset $S$ of $Z$. It follows from Lemma 4, that $D/P$ is integral over $P^*/P$. However $Z$ is integrally closed in $R$. Consequently $D/P \cong P^*/P \cong Z$. Since $Z$ has no proper subring containing 1, we get that $D = P^* = A^*$.

Suppose that $A$ is not a prime ideal. Then $A = \bigcap_{i=1}^t Q_i$ for some irreducible ideals $Q_i$ of $D$ such that $\bigcap_{j \neq i} Q_j \not\subset Q_i$ for every $i$. Now

$$A = A \cap A^* = \bigcup_{i=1}^{t} (Q_i \cap A^*) . \tag{1}$$

Suppose that $A$ is a prime ideal of $A^*$. Then (1) yields that

$A = Q_i \cap A^*$ for some $i$ and $Q_i \cap A^* \subset Q_j \cap A^*$ for every $j$. In view of Lemmas 6(i), 7 and 8, $t = 1$, $A = Q_1 \cap A^*$ and $Q_1$ is a prime ideal of $D$, since $A$ is a prime ideal of $A^*$. Thus $A = Q_1$ is a prime ideal of $D$. This is a contradiction. Hence $A$ is not a prime ideal of $A^*$. Consequently $A^*/A \cong Z/(n)$, for some composite integer $n > 2$. Since in $Z/(n)$ every prime ideal different from $Z/(n)$ is a maximal ideal of $Z/(n)$, the prime radical of $Q_i \cap A^*$ in $A^*$ is a maximal ideal of $A$. Then by Lemma 7, the prime radical of $Q_i$ in $D$ is a maximal ideal of $D$. Further, since in $Z/(n)$ any family of primary ideals, which have common radical, is totally ordered and by Lemmas 6(i), 7 and 8, $Q_i \cap A^* \not\subset Q_j \cap A^*$ for $i \neq j$, we get that the prime radical of these $Q_i$ are all distinct and maximal. Thus $A = \bigcap_{i=1}^{t} Q_i$ is an irredundant decomposition of $A$ into primary ideals. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_u^{\alpha_u}$ be the factorization of $n$ into distinct prime powers. It is immediate that $t = u$, and we can arrange the $Q_i'^s$ in such a way that $(Q_i \cap A^*)/A \cong (p_i^{\alpha_i})/(n)$. Now by Zariski and Samuel [14, p. 178. Theorem 32], $D/A \cong \bigoplus \sum_{i=1}^{t} D/Q_i$. Further

$$D/Q_i \cong D_{M_i}/Q_i D_{M_i} \cong \hat{D}_{M_i}/Q_i \hat{D}_{M_i} \cong A^*_{M'}/Q'_i A^*_{M_i} = A^*/Q_i \ ,$$

where $M'_i = M_i \cap A^*$ and $Q'_i = Q_i \cap A^*$: as $A^*/Q_i = Z/(p_i^{\alpha_i})$, it follows that $D/A = \bigoplus \sum_{i=1}^{t} Z/(p_i^{\alpha_i}) = Z/(n)$. Thus the additive group of $D/A$ is cyclic and is generated by its unity. Hence $A^* = D$. This proves the theorem.

REMARK. In the above theorem, it can be easily seen from the proof that it is enough to assume that every indecomposable injective $D$-module is $A^*$-injective. However in that case a simple application of a theorem due to Matlis [7] yields that every injective $D$-module is an injective $A^*$-module. Proposition 1 and the above theorem immediately yield the following characterization of a $(KE)$-domain.

THEOREM 3. *A domain $D$ is a $(KE)$-domain if and only if for each of its proper ideals $A$, $A^*$ is a Noetherian ring and every injective $D$-module is an injective $A^*$-module.*

2. **Flat modules.** For definitions and some well known results on flat modules the reader may see Bourbaki [2]. Let $D$ be a domain having $K$ as its quotient field. By an overring of $D$, we mean any domain $D'$ such that $D \subset D' \subset K$. In [8], Richman studied those overdomains of a domain $D$ which are flat as $D$-modules. The following theorem which we state without proof was proved by Richman.

THEOREM 4. *Let $D'$ be an over domain of a domain $D$. Then*

*$D'$ is a flat $D$-module if and only if $D'_M = D_{(M \cap D)}$ for all maximal ideals $M$ of $D'$.*

Let us recall from [11] that a ring $R$ is said to have dimension $n$, if it contains a chain $P_0 < P_1 < P_2 < \cdots < P_n \, (\neq R)$ of prime ideals, but it contains no such chain of greater length.

LEMMA 9. *Let $P$ be a proper prime ideal of a domain $D$ such that for every nonzero primary ideal $Q$ of $D$ contained in $P$ (not necessarily a $P$-primary ideal), $D$ is a flat $Q^*$-module. Then:*

(i) *Height $P \leqslant 2$.*

(ii) *If $P$ is not a minimal proper prime ideal, then $P$ is a maximal ideal.*

(iii) *There exists a $P$-primary ideal $Q \neq P$.*

*Proof.* Suppose that $P$ is not a minimal prime ideal. Then there exists a proper prime ideal $P' < P$. Let $M$ be a maximal ideal of $D$ containing $P$. Since by the hypothesis, $D$ is a flat $P'^*$-module, Theorem 4 yields that $D_M = (P')^*_{(P' \cap M)}$. Since $(P')^*/P' \cong Z/(n)$ for some $n$ and $\dim Z/(n) \leqslant 1$, we have $\dim (P')^*/P' \leqslant 1$: thus

$$\dim D/P' \leqslant 1 .$$

It follows that there exists no prime ideal of $D$ properly between $P'$ and $M$. Consequently $M = P$. By considering $P'$ instead of $P$, we also get that $P'$ is a minimal prime. Hence height $P \leqslant 2$. This proves (i) and (ii).

Let $P$ be a minimal prime ideal of $D$. The contraction in $D$ of any proper ideal of $D_P$, not equal to $PD_P$ is a $P$-primary ideal of $D$ different from $P$. Now let $P$ be not a minimal prime ideal. Then there exists a proper prime ideal $P' < P$. By (i) $D_P/P'D_P$ is a one dimensional domain. Choose any proper ideal $T/P'D_P$ of $D_P/PD_P$, not equal to its maximal ideal, then the contraction of $T$ in $D$ is a $P$-primary ideal of $D$, not equal to $P$. This proves (iii).

LEMMA 10. *Let $P$ be a proper prime ideal of $D$, satisfying the hypothesis of Lemma 9. Then $P^* = D$, $P^*/P \cong Z/(p)$, for some prime number $p$, and $P$ is a maximal ideal of $D$.*

*Proof.* By Lemma 9, there exists a $P$-primary ideal $Q \neq P$. Let $M$ be a maximal ideal of $D$ containing $P$. Theorem 4 yields that,

$$D_M = P^*_{(P^* \cap M)} = Q^*_{(Q^* \cap M)} . \tag{2}$$

Now $P^*/P \cong Z$ or $P^*/P = Z/(p)$, for some prime number $p$. Let

$P^*/P = Z$. Then for every $n(\neq 0) \in Z$, $n1 \notin P$: consequently $n1 \notin Q$. This yields that $Q^*/Q \cong Z$ and that $Q$ is a prime ideal of $Q^*$. Then from (2) it follows that $Q$ is a prime ideal of $D$. This is a contradiction. Hence $P^*/P \cong Z/(p)$ and that $P$ is a maximal ideal of $P^*$. Consequently (2) yields that $M \cap P^* = P$ and $D_M = P^*_P$. So that $P = M$ and $D/P \cong P^*/P \cong Z/(p)$. Thus $P^*/P$ is a subring of $D/P$ such that both of them have $p$ elements. Hence $P^* = D$ and the lemma follows.

COROLLARY 1. *If $P$ is a proper prime ideal of a domain $D$, satisfying the hypothesis of Lemma 9, then height $P = 1$.*

*Proof.* If $P'$ is any proper prime ideal of $D$ contained in $P$, then $P'$ also satisfies the hypothesis of Lemma 9. By Lemma 10, $P'$ is a maximal ideal of $D$. Hence $P' = P$ and height $P = 1$.

THEOREM 5. *Let $P$ be a proper prime ideal of domain $D$ such that for every nonzero primary ideal $Q$ of $D$ contained in $P$, $D$ is a flat $Q^*$-module. Then every nonzero primary ideal $Q$ of $D$ contained in $P$ is $P$-primary, $D/Q \cong Z/(p^\alpha)$ for some power $p^\alpha$ of a prime number $p$ and $Q^* = D$.*

*Proof.* By Corollary 1, height $P = 1$. So that $\sqrt{Q} = P$. In case $P = Q$, the result follows from Lemma 10. Let $Q \neq P$. Since $D$ is a flat $Q^*$-module, by Theorem 4,

$$D_P = Q^*_{(Q^* \cap P)} \,. \tag{3}$$

This equation along with Lemma 10, yields that there exists a prime number $p$ such that $Z/(p) \cong D/P \cong Q^*/Q^* \cap P$. However $Q^*/Q \cong Z/(n)$, for some $n$, and $Q$ is a $(Q^* \cap P)$-primary ideal of $Q^*$. Therefore $n = p^\alpha$, for some $\alpha > 2$. Then from (3) $D/Q \cong Q^*/Q \cong Z/(p^\alpha)$: as a consequence we get that $D = Q^*$. This proves the theorem.

Henceforth the domain $D$ will always be assumed to be different from its quotient field. The following corollary is an immediate consequence of the above theorem.

COROLLARY 2. *If $D$ is a flat $A^*$-module for each of its proper ideals $A$, then* $\dim D = 1$.

LEMMA 11. *Let $D$ be a domain such that $D$ is a flat $A^*$-module for each of its proper ideals $A$. If $P_1$ and $P_2$ are two distinct proper prime ideals of $D$, such that $D/P_1 \cong Z/(p_1)$ and $D/P_2 \cong Z/(p_2)$, then $p_1 \neq p_2$.*

*Proof*. Suppose that $p_1 = p_2 = p$. Then $p1 \in P_1 \cap P_2 = P_1P_2$. Hence $(P_1P_2)^*/P_1P_2 \cong Z/(p)$ and $N = P_1P_2$ is a maximal ideal of $(P_1P_2)^*$. Consequently $P_1 \cap (P_1P_2)^* = N = P_2 \cap (P_1P_2)^*$. By Theorem 4,

$$D_{P_1} = (P_1P_2)^*_N = D_{P_2} .$$

This yields that $P_1 = P_2$. Hence the lemma follows.

THEOREM 6. *A domain $D$ is a* (*KE*)*-domain if and only if it is a flat $A^*$-module for each of its proper ideals $A$.*

*Proof*. Let $D$ be a (*KE*)-domain. By Proposition 1, given any proper ideal $A$ of, $D = A^*$. Then obviously $D$ is a flat $A^*$-module for each of its proper ideals $A$.

Conversely let $D$ be a flat $A^*$-module for each of its proper ideals $A$. Consider any proper prime ideal $P$ of $D$. By Theorem 5, $P$ is a maximal ideal and there exists a prime number $p$ such that for any nonzero primary ideal $Q$ of $D$ contained in $P$, $D/Q \cong Z/(p^\alpha)$ for some $\alpha \geqslant 1$. Consequently $D_P/QD_P \cong Z/(p^\alpha)$, a *PIR* with d.c.c. So that $D_P$ is a discrete valuation ring of rank one. As an immediate consequence we get that every nonzero primary ideal of $D$ contained in $P$ is a power of $P$ and $D/P^\alpha \cong Z/(p^\alpha)$ for every $\alpha$. Thus $p1 \in P\backslash P^2$. Now for any given proper prime ideal $P' \neq P$, $D/P' \cong Z/(p')$, for some prime number $p'$, which, because of Lemma 11, is not equal to $p$. So that $p1 \notin P'$. Then using the fact that for any ideal $A$ of $D$, $A = \bigcap AD_T$, where $T$ runs over all the maximal ideals of $D$, we get that $P = (p1)$, a principal ideal of $D$. By Cohen [4, Theorem 2], $D$ is Noetherian. Let $A$ be a proper ideal of $D$ and $A = \bigcap_{i=1}^t Q_i$ be an irredundant decomposition of $A$ into primary ideals. For each $i$, since $D/Q_i \cong Z/(p_i^{\alpha_i})$, for some prime power $p_i^{\alpha_i}$ and the prime number $p_i$ are all distinct, we get that, $D/A \cong \bigoplus \sum_{i=1}^t D/Q_i \cong \bigoplus \sum_{i=1}^t Z/(p_i^{\alpha_i}) \cong Z/(n)$, where $n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_t^{\alpha_t}$. Since the ring $Z/(n)$ is generated by its unity element, it follows that $D = A^*$. Hence by Proposition 1, $D$ is a (*KE*)-domain.

We now obtain Theorem 2 of [12] as a corollary to the above theorem.

COROLLARY 3. *A domain $D$ is a* (*KE*)*-domain if and only if for each proper ideal $A$ of $D$, one of the following holds:*

(i) *$A^*$ is a Dedekind domain.*

(ii) *$A^*$ is a Prüfer domain.*

(iii) *$A^*$ is a generalized Krull domain.*

(iv) *$A^*$ is an almost Krull domain.*

*Proof.* If $D$ is a $(KE)$-domain, then by Lemma 3, $D$ satisfies the given conditions.

Let $D$ satisfy the given conditions. Let $A$ be a proper ideal of $D$. If $A^*$ satisfies any of the conditions: (i), (iii), and (iv) then for each of its minimal prime ideals $P'$, $A^*_{P'}$ is a rank one valuation ring and $A^*$ is an intersection of these rings. Now $AP'$ is a nonzero ideal of $D$ contained in $P'$. For $S = A^* \backslash P$, $A^*_{P'} \subset D_S$. Since

$$S \cap AP' = \varnothing ,$$

$D$ is not a field. However $A^*_{P'}$ is a maximal subring of its quotient field. Consequently $D_S = A^*_{P'}$ and $D \subset A^*_{P'}$. Hence $D = A^*$. In this case $D$ is trivially an $A^*$-flat module. If $A^*$ is a Prüfer domain, then again by Richman [8], $D$ is a flat $A^*$-module. Hence, by Theorem 6, $D$ is a $(KE)$-domain.

The following theorem is also an immediate consequence of Theorem 6. It also follows from Lemma 13 given below, and which is analogous to Theorem 2.

THEOREM 7. *A domain $D$ is a $(KE)$-domain if and only if it is a projective $A^*$-module for each of its proper ideals $A$.*

LEMMA 13. *If for a proper ideal $A$ of a domain $D$, $D$ is a projective $A^*$-module, then $D = A^*$.*

*Proof.* As $D$ is a projective $A^*$-module, by the dual basis theorem for projective modules, there exists a family $\{\sigma_\alpha\}_{\alpha \in \Lambda}$ of elements of $\mathrm{Hom}_{A^*}(D, A^*)$ and a corresponding family $\{d_\alpha\}_{\alpha \in \Lambda}$ of elements of $D$ such that for each $d \in D$, $\sigma_\alpha(d) = 0$, for all but a finite number of values of $\alpha$, and $d = \sum_\alpha \sigma_\alpha(d) d_\alpha$.

Let $\sigma \in \mathrm{Hom}_{A^*}(D, A^*)$. Consider $b$, $c \in D$. Choose a $(\neq 0) \in A$. Then $\sigma(bc)a = \sigma(bca) = \sigma(b)ca$, since $ca \in A^*$: consequently $\sigma(bc) = \sigma(b)c$. Thus $\sigma$ is a $D$-homomorphism. Hence for any

$$d \in D, d = \sum_\alpha \sigma_\alpha(d) d_\alpha = \sum_\alpha \sigma_\alpha(d d_\alpha) \in A^* .$$

This proves that $D = A^*$.

The above lemma does not hold for flat modules, as is evident from the following example.

EXAMPLE 1. Consider the formal power series ring $D = R[[X]]$, over the field $R$ of rational numbers. Its maximal ideal is $M = (X)$. Now $M^* = Z + M \neq D$ and $D = M^*_S$, where $S$ is the set of all nonzero integers. Hence $D$ is a flat $M^*$-module, but $D \neq M$.

3. The ring $\hat{Z}_{(p)}$. In [11, Example 4], it was shown that for any prime number $p$, $\hat{Z}_{(p)}$, the $p$-adic completion of $Z_{(p)}$, is a $(KE)$-domain. In this section we prove that $\hat{Z}_{(p)}$ is a maximal $(KE)$-domain, in the sense that if in a $(KE)$-domain $D$, which is not a field, some prime number $p$ is not invertible, then $D$ is embeddable in $\hat{Z}_{(p)}$. Some other results on $(KE)$-domains are also established. The following structure theorem on $(KE)$-domains was proved in [11, Theorem 14].

THEOREM 8. *Any domain $D$, which is not a field, is a $(KE)$-domain if and only if it satisfies the following*:

(i) *There exists a multiplicative subset $S$ of the ring of integers $Z$, such that $Z_S$ is embeddable in $D$.*

(ii) *The correspondence $A \leftrightarrow A \cap Z_S$ is one-to-one between the ideals $A$ of $D$ and those of $Z_S$.*

(iii) *For every proper prime ideal $P$ of $D$, $D/P \cong Z_S/P \cap Z_S$.*

If a $(KE)$-domain $D$ satisfies conditions (i) to (iii) of Theorem 8 we say that $D$ is a $(KE)$-domain associated with $Z_S$: in that case it is immediate that a prime number $p$ is invertible in $D$ if and only if it is invertible in $Z_S$.

DEFINITION 1. A $(KE)$-domain $D$ associated with $Z_S$ is said to be a maximal $(KE)$-domain associated with $Z_S$, if there exists no $(KE)$-domain $D'$ associated with $Z_S$ such that it contains $D$ properly.

THEOREM 9. *Let $D$ be a $(KE)$-domain, which is not a field and in which some prime number $p$ is not invertible, then $D$ is embeddable in $\hat{Z}_{(p)}$.*

*Proof.* Let $D$ be associated with $Z_S$. Since $Z_S$ is a *PID* of characteristic zero, Theorem 8 yields that $D$ is a *PID* of characteristic zero. Further as $pZ_S$ is a maximal ideal of $Z_S$, Theorem 8 also yields that $P = pD$ is a maximal ideal of $D$ such that $D/P \cong Z/(p)$. By Theorem 5, for each $n \geqslant 1$, $D/P^n = Z/(p^n)$ and hence every element of $D$ is of the form $k1 + p^n a$; $k \in Z$, $a \in D$. Consequently there exists a natural homomorphism $\sigma_n$: $D \rightarrow Z/(p^n)$ such that

$$\sigma_n(k1 + p^n a) = k + (p^n) .$$

For $m \leqslant n$, we have the natural homomorphism $\pi_n^m$: $Z/(p^n) \rightarrow Z/(p^m)$. Then $\{Z/(p^n), \pi_n^m\}$ form a projective system and $\varprojlim Z/(p^n) = \hat{Z}_{(p)}$ [9, Chap. 1, p. 55]. For each $n$, let $\pi_n$: $\hat{Z}_{(p)} \rightarrow Z/(p^n)$ be the canonical mapping. It can be easily seen that $\sigma_m = \pi_n^m \sigma_n$ whenever $m \leqslant n$.

Thus there exists a homomorphism $\sigma$ of $D$ into $\hat{Z}_{(p)}$ such that $\sigma_n = \pi_n\sigma$ for every $n$. Since $\bigcap_n \ker \sigma_n = (0)$, $\sigma$ is a monomorphism. Hence the theorem follows.

THEOREM 10. *Let $\{D_\alpha, \pi_\alpha^\beta\}_{\alpha,\beta\in\Lambda}$ be an injective system of $(KE)$-domains associated with the same $Z_S$ ($\neq$ the field of rational numbers). Then the injective limit $D = \varinjlim D_\alpha$ is a $(KE)$-domain associated with $Z_S$. (It is assumed that each of $\pi_\alpha^\beta$ is a nonzero mapping.)*

*Proof.* For each $\alpha \in \Lambda$, there exists a homomorphism $\pi_\alpha: D_\alpha \rightarrow D$ satisfying the following:

(i) $\pi_\alpha = \pi_\beta\pi_\alpha^\beta$ for $\alpha$, $\beta \in \Lambda$ such that $\alpha \leqslant \beta$.

(ii) $D = \bigcup \pi_\alpha(D_\alpha)$

(iii) If for some $\alpha$, there exists $x_\alpha \in D_\alpha$ such that $\pi_\alpha(x_\alpha) = 0$, then there exists $\beta \geqslant \alpha$ such that $\pi_\alpha^\beta(x_\alpha) = 0$.

Using the above properties, it follows that $D$ is an integral domain. As $\pi_\alpha^\beta \neq 0$, $\pi_\alpha^\beta(1) = 1$. We get that $\pi_\alpha^\beta$ is an identity map on $Z_S$. Consequently each $\pi_\alpha$ is also identity map on $Z_S$. Consider any $x_\alpha(\neq 0) \in D_\alpha$. As seen in the proof of Corollary 3 in [11], $x_\alpha = n_\alpha u_\alpha$ for some $n_\alpha \in Z$ and a unit $u_\alpha$ in $D_\alpha$: thus $\pi_\alpha(x_\alpha) = n_\alpha\pi_\alpha(u_\alpha)$. Clearly $\pi_\alpha(u_\alpha)$ is a unit in $D$. It follows that every element of $D$ is of the type $nu$; $n \in Z$ and $u$ a unit in $D$. Consider any proper ideal $A$ of $D$. Now for every $\alpha$, $A_\alpha = \pi_\alpha^{-1}(A)$ is a proper ideal of $D$ and $A = \bigcup \pi_\alpha(A_\alpha)$. Thus $A^* = \bigcup \pi_\alpha(A_\alpha^*) = \bigcup \pi_\alpha(D_\alpha) = D$. Hence by Proposition 1, $D$ is a $(KE)$-domain. Since every prime number invertible in $Z_S$ is invertible in every $D_\alpha$, we get it is also invertible in $D$. Conversely if any prime number $p$ is invertible in $D$, then the above properties of $D$ imply that $p$ is invertible in some $D_\alpha$ and hence $p$ is invertible in $Z_S$. This shows that $D$ is associated with $Z_S$.

We end this paper with a few remarks.

1. Some of the lemmas, for example Lemmas 4 to 8, and 12 can be proved by replacing $A^*$ by any Noetherian subring of $D$, containing a nonzero ideal of $D$ and keeping the other hypotheses unchanged. It is not clear whether in that case, we obtain $B = D$, as in Theorem 2.

2. Theorems 9 and 10 can be proved in more general settings. To explain the point, let $T$ be a fixed Noetherian domain, which is not a field. Let us call a domain $D$ containing $T$ lattice equivalent to $T$ if it has the following properties:

(i) $A \leftrightarrow A \cap T$, is a one-to-one correspondence between the ideals $A$ of $D$ and those of $T$.

(ii) For any proper ideal $A$ of $D$, $D = A + T$.

Take any proper prime ideal $P$ of $T$. Then as in Theorem 8, it can be shown that $D$ is embeddable in $\hat{T}_P$, the $PT_P$-adic completion of $T_P$. In Theorem 9, we had $T = Z_S$. In Theorem 10, if we replace each $D_\alpha$ by a domain lattice equivalent to a fixed Noetherian domain $T$ and let each $\pi_\alpha^\beta$ be identity on $T$, then their injective limit is also lattice equivalent to $T$. The only reason for not proving Theorems 9 and 10 in this more general setting is that the paper is essentially concerned with (*KE*)-domains.

3. By Theorem 9, given a $Z_S$ (not equal to the field of rational numbers), all (*KE*)-domains associated with $Z_S$ can be regarded as subrings of a fixed $\hat{Z}_{(p)}$. It can be easily seen that the family of all (*KE*)-domains associated with the same $Z_S$ is inductive. Hence by Zorn's lemma it has maximal members. It remains open whether any two maximal (*KE*)-domains associated with a $Z_S$ are isomorphic or not.

ACKNOWLEDGEMENTS. The author is extremely grateful to the referee for his various suggestions and to Dr. W. Stephenson of Bedford College, London, for helping him to improve the exposition.

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Received June 30, 1971 and in revised form November 30, 1972. This research was supported by a grant from the University grants Commission, India.

ALIGARH MUSLIM UNIVERSITY,
ALIGARH, U. P. INDIA