PACIFIC JOURNAL OF MATHEMATICS
Vol. 46, No. 2, 1973

ON REALIZING HNN GROUPS IN 3-MANIFOLDS

C. D. FEUSTEL AND R. J. GREGORAC

In this paper we suppose that the fundamental group of a 3-manifold M has a presentation as an HNN group. We then show that under suitable conditions we can realize this presentation by embedding a closed, connected imcompressible surface in M.

In [2], [3], and [4] we show that if $\pi_1(M^3)$ is constructed in certain ways, one can realize this construction by a surface embedded in M^3. In this paper we show that one can realize the HNN construction when certain relationships between $\pi_1(M^3)$ and M^3 are present. The results in this paper are related to Theorem 2.4 in [10].

In this paper all spaces will be simplicial complexes, all maps will be piecewise linear, and all 3-manifolds will be 3-manifolds with boundary. However the boundary may be vacuous. Let X be a connected subspace of a space Y. As usual we shall denote the boundary, closure, and interior of X in Y by $\mathrm{bd}(X)$, $\mathrm{cl}(X)$, and $\mathrm{int}(X)$ respectively. The natural inclusion map from X into Y will be denoted by ρ and the induced homomorphism from $\pi_1(X)$ into $\pi_1(Y)$ by ρ_*. Let S be a closed connected surface other than the 2-sphere of projective plane embedded in a space Y. Then S is *incompressible* in Y if $\rho_*: \pi_1(S) \rightarrow \pi_1(Y)$ is one-to-one. If S is a closed surface embedded in Y, then S is *incompressible* in Y if each component of S is incompressible in Y. Irreducible and P^2-irreducible are defined as in [7]. We denote the unit interval $[0, 1]$ by I throughout.

DEFINITION 1. Let K be a group and A a subgroup of K. Let S be a closed connected surface other than the projective plane or 2-sphere. Let $A_j \cong \pi_1(S)$ and $A_j \subset A$ for $j = 1, 2$. Let k be an element of K not in A such that $A_1 = k^{-1}A_2k$. Then if A and k generate K and all relations of K are consequences of the relations of A together with the relations k induces between the elements of A_1 and A_2, we shall say that *K is an extension of A by k across A_1 and A_2.* The reader will note that the class of groups defined above is a subclass of the Higmann, Neumann, Neumann (H.N.N.) groups [8].

Let M be a 3-manifold, x a point in M, and S an incompressible surface in M such that $M - S$ is connected. Then it is a consequence of Van Kampen's Theorem that $\pi_1(M, x)$ is an extension of $\pi_1(M - S, x)$ by some element of $\pi_1(M, x)$ across appropriate subgroups of $\pi_1(M, x)$. One might then wonder "If $\pi_1(M, x)$ were such an extension, could we embed in M an incompressible surface which realizes this extension." We will show blow that this can, in fact, be done. Let M be a compact 3-manifold and x a point of M. We suppose that $\pi_1(M, x)$ is an extension of A by k across A_1 and A_2 as given in Definition 1 above. We can represent this extension by an ordered sequence $\langle\pi_1(M, x), A, A_1, A_2, k\rangle$. If for each component F of the boundary of M some conjugate $\rho_*\pi_1(F)$ is contained in A, we shall say that the *extension preserves the peripheral structure of* M. Suppose a second representation of $\pi_1(M, x)$ is given by $\langle\pi_1(M, x), B, B_1, B_2, \hat{k}\rangle$ and this extension of B is induced by an incompressible, closed, two-sided surface S embedded in M and a loop l meeting S in the single point x, i.e., B is generated by the elements of $\pi_1(M, x)$ having representative loops which do not cross S, $\hat{k}=[l]$, $B_1=\rho_*\pi_1(S, x)$ and $B_2=[l]B_1[l]^{-1}$. We shall say that S realizes the extension of B if there is an isomorphism

$$\Phi\colon \pi_1(M, x) \longrightarrow \pi_1(M, x)$$

such that

(1) $\Phi(A) = B$

(2) $\Phi(A_j) = B_j \quad j = 1, 2$

(3) $\Phi(k) = \hat{k}$.

THEOREM 1. *Let M be a compact 3-manifold such that $\pi_2(M) = 0$. Let S be a closed connected surface other than the 2-sphere or projective plane. Suppose $\pi_1(M, x)$ has a representation given by*

$$\langle\pi_1(M, x), A, A_1, A_2, k\rangle$$

where $A_1 \cong \pi_1(S)$ and the extension above preserves the peripheral structure of M. Then there is an embedding of S in M which realizes the given extension.

The proof of Theorem 1 above is similar in many respects to the proof of Theorem 1 in [3]. One first constructs a complex X having the same fundamental group as M. One then finds a map $f\colon M \rightarrow X$ inducing an isomorphism from $\pi_1(M)$ to $\pi_1(X)$. The complex X is constructed to contain an embedded surface S realizing the given extension. One shows that there is a map g homotopic to f such that $g^{-1}(S)$ is an incompressible, connected, closed surface in M and that $g^{-1}(S)$ realizes the given extension.

The following three lemmas appear in [4]. We omit the proofs which are not difficult.

LEMMA 1. *Let M be a compact, connected 3-manifold such that $\pi_2(M) = 0$. Let X be a connected complex and S a closed incompressi-*

ble surface embedded in X and having a neighborhood homeomorphic to $S \times I$. We suppose that no component of S is a 2-sphere or projective plane. Let X_k, $k = 1, \cdots, n$ be the components of $X - S$. We suppose that $\pi_i(X) = \pi_i(X_k) = 0$ for $i \geqq 2$ and $k = 1, \cdots, n$. Let $f: M \to X$ be a map such that $f_*: \pi_1(M) \to \pi_1(X)$ is one-to-one $f\,\mathrm{bd}(M)$ does not meet S. Then there is a homotopy, constant on $\mathrm{bd}\,(M)$, of f to a map g such that $g^{-1}(S)$ is an incompressible surface in M.

LEMMA 2. *Let S_1 and S_2 be disjoint, incompressible, connected, two-sided surfaces which are embedded in a P^2-irreducible 3-maifold M. Then if S_1 is homotopic to S_2 in M, $S_1 \cup S_2$ bounds an $S_1 \times I$ embedded in M.*

LEMMA 3. *Let M_1 be a compact, connected, 3-manifold, X a connected complex, and F and S incompressible connected surfaces in M_1 and X respectively. We suppose that S is neither a 2-sphere or projective plane and $\pi_i(X) = 0$ for $i \geqq 2$.*

Let $f: (M_1, F) \to (X, S)$ be a map of pairs such that for some $x \in F$

$$f_*\pi_1(M_1, x) \subset \pi_1(S, f(x)) .$$

Then f is homotopic under a deformation, constant on F, to a map into S.

Proof of Theorem 1. It is a consequence of Remark 1 in [9] that we may assume that M is irreducible.

Let $(M_A, \hat{x}, p)$ be the covering space of (M, x) associated with $A \subset \pi_1(M, x)$. Let f_1, f_2; $(S, y) \to (M, x)$ be maps such that $f_{j*}(\pi_1(S, y)) = A_j$, for $j = 1, 2$. Since $f_{j*}(\pi_1(S, y)) \subset p_*\pi_1(M_A, \hat{x})$, there is a map $\hat{f}_j$: $(S, y) \to (M_A, \hat{x})$ such that $p\hat{f}_j = f_j$ for $j = 1, 2$. Let X be the union of M_A and $S \times I$ with identifications $\hat{f}_1(s) = (s, 0)$ and $\hat{f}_2(s) = (s, 1)$. We note that the arc $\{y\} \times [0, 1] \subset S \times I$ becomes a simple loop $\hat{l}$ after the identification above since $\hat{f}_1(y) = \hat{f}_2(y) = \hat{x}$. Let Φ: $A \cup \{k\} \to \pi_1(X, \hat{x})$ be a function defined by $\Phi(k) = [l]$ and $\Phi(a) = P_*^{-1}(a)$ for $a \in A$. Then Φ can be extended to an isomorphism of $\pi_1(M, x)$ onto $\pi_1(X, \hat{x})$ since X has been constructed so that $\pi_1(X, \hat{x})$ will have a presentation identical to the given presentation of $\pi_1(M, x)$.

It can be shown as in the proof of the theorem in [2] that $\pi_i(X) = \pi_i(X - S) = 0$ for $i \geqq 2$.

We denotes $S \times \{1/2\} \subset X$ by S.

Let the boundary of M be expressed as $\bigcup_{m=1}^n F_m$ where F_m is a closed connected 2-manifold. Then some conjugate of $\rho_*\pi_1(F_m)$ is contained in A for $m = 1, \cdots, n$. Thus we can find a collection $\{\alpha_m \mid m = 1, \cdots, n\}$ of simple arcs embedded in M such that intersection of each pair of these arcs is x, α_m meets F_m in a single point, and there is a map $\hat{\rho}: \bigcup_{m=1}^{n} (F_m \cup \alpha_m) \to M_A$ such that $p\hat{\rho} = \rho$. Note that for each loop l_0 in $\bigcup_{m=1}^{n} (F_m \cup \alpha_m)$ based at x, $[\hat{\rho} l_0] = \Phi[l_0]$. Since $\hat{\rho}_* \rho_* = \Phi \rho_*: \pi_1(\bigcup_{m=1}^{n} (F_m \cup \alpha_m), x) \to \pi_1(X, \hat{x})$, we can extend $\hat{\rho}$ to a map $f: M \to X$ such that $\Phi = f_*: \pi_1(M, x) \to \pi_1(X, \hat{x})$ by using standard techniques from obstruction theory. (See [2] or [3] for the details of this construction.) It is a consequence of Lemma 1 that there is a map g_1 homotopic to f such that $g_1^{-1}(S)$ is an incompressible surface in M and $g_1 = f$ on the boundary of M.

Since $g_1^{-1}(S)$ and S are incompressible in M and X respectively, if S_0 is any component of $g_1^{-1}(S)$, the homomorphism $(g_1 | S_0)_*: \pi_1(S_0) \to \pi_1(S)$ is one-to-one. Thus by Theorem 1 in [6] $g_1 | S_0$ is homotopic to a covering map. Thus after a deformation, constant outside of a small neighborhood of S_0, we may assume that $g_1 | S_0$ is a local homeomorphism. Thus we may assume that g_1 is a local homeomorphism on $g_1^{-1}(S)$.

Let z be a point on S_0. Suppose that the isomorphism $\Phi_0 = g_{1*}: \pi_1(M, z) \to \pi_1(X, g_1(z))$ does not carry $\pi_1(S_0, z)$ onto $\pi_1(S, g_1(z))$. It is a consequence of the result in [1] that M is P^2-irreducible. Since $\Phi_0^{-1}\pi_1(S, g_1(z))$ would properly contain $\pi_1(S_0, x)$, we would have by Theorem 6 in [7] that S_0 bounds a twisted line bundle $N \subset M$. One can easily show using the techniques of [7], as has been done in [5], that $\rho_*\pi_1(N, z)$ may be taken to be $\Phi_0^{-1}(\rho_*\pi_1(S, g_1(z)))$. It follows from Lemma 3 that there is a deformation of g_1 to a map g_2 which pushes $g_1(N)$ first onto S and then to one side of S so that $g_2^{-1}(S) = g_1^{-1}(S) - S_0$. Thus we can assume that $(g_1 | S_0)_*: \pi_1(S_0) \to \pi_1(S)$ is an epimorphism for each component S_0 of $g_1^{-1}(S)$.

Since $\pi_1(M) \not\subset A$, $g_1^{-1}(S)$ is not empty.

Let S_0 and S_1 be components of $g_1^{-1}(S)$. We claim that $S_0 \cup S_1$ bounds a copy of $S_0 \times [0, 1]$ embedded in M. Since M is P^2-irreducible, this will follow from Lemma 2 after we show that S_0 and S_1 are homotopic. Let z_0 be a point on S_0. Since $g_1 | S_0$ and $g_1 | S_1$ are assumed to be homeomorphisms, there is a unique point z_1 on S_1 such that $g_1(z_0) = g_1(z_1)$. Let α be an arc running from z_0 to z_1. Since g_{1*} is an isomorphism, we can find a loop l_1 based at z_0 such that the loops $g_1(l_1)$ and $g_1(\alpha)$ represent the same element in $\pi_1(X, g_1(z_0))$. Thus we may assume that $[g_1(\alpha)] = 1 \in \pi_1(X)$. Let λ_0 be a loop on S_0 based at z_0 and λ_1 a loop on S_1 such that $g_1(\lambda_0) = g_1(\lambda_1)$. Since the loop $g_1(\lambda_0)g_1(\alpha)(g_1(\lambda_1))^{-1}(g_1(\alpha))^{-1}$ is nullhomotopic and $\pi_2(X) = 0$, we can show as in the proof of Theorem 1 in [3] that S_0 and S_1 are homotopic. Our claim follows.

We wish to show that we may assume $g_1^{-1}(S)$ contains exactly one component.

Suppose there is more than one component in $g_1^{-1}(S)$ and that the number of components of $g_1^{-1}(S)$ cannot be decreased by a small deformation of g_1. Let $l: S^1 \to M$ be a loop in M such that $g_{1*}[l] = [\hat{l}]$. We may assume that

(i) $g_1(l)$ meets S since the intersection number of $[\hat{l}]$ and S is one. Thus we can take our basepoint to lie on one of the surfaces in $g_1^{-1}(S)$.

(ii) l crosses $g_1^{-1}(S)$ at each point in $l \cap g^{-1}(S)$ and thus $(g_1 l)^{-1}(S)$ is a finite set whose cardinality cannnot be reduced.

(iii) $g_1(l \cap g_1^{-1}(S))$ is a single point.

Let D be a disk and β_1 and β_2 arcs in the boundary of D such that $\beta_1 \cap \beta_2 = \mathrm{bd}\,(\beta_1)$. Then we can define a map $\gamma: D \to X$ such that $\gamma(\beta_1)$ is the loop $g_1 l(S^1)$ and $\gamma(\beta_2)$ is the loop $\hat{l}$.

We wish to show that $g_1^{-1}(S)$ may be taken to be homeomorphic to S (connected). Assume that $g_1^{-1}(S)$ is not connected; then it has been shown that each pair of distinct surfaces in $g_1^{-1}(S)$ bounds a copy of $S \times I$ embedded in M. If this is the case, it is clear that $l^{-1}g_1^{-1}(S)$ contains more than one point. Let $\delta_1, \cdots, \delta_v$ be the closures of the components of $S^1 - l^{-1}g_1^{-1}(S)$. After a general position argument we may assume $\gamma^{-1}(S)$ contains an arc β_3 which cuts off an arc $\beta_4 \subset \beta_1$ and that $g_1 l(\delta_1) = \gamma(\beta_4)$. Now l carries $\mathrm{bd}\,(\delta_1)$ to one or two components of $g_1^{-1}(S)$.

If $l(\mathrm{bd}\,(\delta_1))$ is a single point, the loop $l(\delta_1)$ is homotopic to a loop $l_1 \subset g_1^{-1}(S)$ such that $g_1(l_1) = \gamma(\beta_3)$ since the restriction of g_1 to each component of $g_1^{-1}(S)$ is a homeomorphism and g_{1*} is an isomorphism. It would follow that the number of points in $l^{-1}g_1^{-1}(S)$ could have been reduced by a different choice of l. Thus we conclude that l carries the points of $\mathrm{bd}\,(\delta_1)$ to distinct components of $g_1^{-1}(S)$.

Let N be closure of the component of $M - g_1^{-1}(S)$ which meets $l(\delta_1)$. Let S_0 be a component of $\mathrm{bd}\,(N)$. Since $g_1 | S_0$ is a homeomorphism and the loop $g_1 l(\delta_1)$ is homotopic to a loop in S, we may assume that the loop $g_1 l(\delta_1)$ is homotopic to a point. (One alters the image of l in a neighborhood of S_0.)

Since the loop $g_1 l(\delta_1)$ is nullhomotopic in X, it can be shown that the map $g_1 | N$ is homotopic mod $\mathrm{bd}\,(N)$ to a map into S; full details of a similar argument appear in [3]. It follows after an argument by induction that there exists a map $g: M \to X$ homotopic to g_1 mod $\mathrm{bd}\,(M)$ such that $g^{-1}(S)$ contains exactly one component S_0 and $g | S_0$ is a homeomorphism. After an argument similar to the one given above, we can find a loop l meeting S_0 in a single point and based at $x \in M$ such that $g_*[l] = [\hat{l}]$.

We observe that S_0 and l induce an expression of $\pi_1(M, x)$ as an extension of a subgroup B of $\pi_1(M, x)$. Let B_1 and B_2 be the associated subgroups of $\pi_1(M, x)$. Then we see that our map g induces an isomorphism $g_*: \pi_1(M, x) \to \pi_1(M, x)$ such that

(1) $g_*(B) \subset A$

(2) $g_*(B_1) = A_1$

(3) $g_* B_2 = A_2$.

Thus Theorem 1 is an immediate consequence of the remark preceding Lemma 2 on page 238 in [8] which shows that g_* sends B onto A.

REMARK 1. The remark mentioned above allows us to strengthen the statement of the theorem in [2] so that the splitting and the cutting are both actually realized.

REMARK 2. We can also realize geometrically more general presentations of $\pi_1(M)$ as an HNN group. In particular one might have that $\pi_1(M)$ has a presentation as in the first definition in §4 in [8] where each of the subgroups L_i of K is isomorphic to the fundamental group of a closed connected surface other than S^2 or the projective plane and there are only finitely many of the t_i. The proof of this result varies only slightly from the one given above.

REMARK 3. Theorem 1 in this paper together with Theorem 1 in [3] or [4] give us a sort of converse to Van Kampen's theorem as applied to a closed, connected, incompressible surface, other than S^2 or the projective plane, embedded in the interior of a compact 3-manifold.

REMARK 4. This paper is in some sense a generalization of Stalling's work in [11].

REFERENCES

1. C. D. Feustel, *M^3 admitting a certain embedding of P^2 is a pseudo P^3*, Proc. Amer. Math. Soc., **26** (1970), 215-216.

2. ———, *On replacing proper Dehn maps with proper embeddings*, Trans. Amer. Math. Soc., **166** (1972), 261-267.

3. ———, *A splitting theorem for closed orientable 3-manifolds*, Topology, **11** (1972), 151-158.

4. ———, *A generalization of Kneser's conjecture*, Pacific J. Math., to appear.

5. ———, *On S-maximal subgroups of $\pi_1(M^2)$*, Canad. J. Math., **24** (1972), 439-449.

6. W. Heil, *On P^3-irreducible 3-manifolds*, Bull. Amer. Math. Soc., **75** (1969), 772-775.

7. W. Jaco, *Finitely presented subgroups of three-manifold groups*, Inventions Math., **13** (1971), 335-346.

8. A. Karrass and D. Solitar, *The subgroups of a free product of two groups with an amalgamated subgroups*, Trans. Amer. Math. Soc., **150** (1970), 227-255.

9. J. Milnor, *A unique decomposition theorem for 3-manifolds*, Amer. J. Math., **84** (1962), 1-7.

10. G. P. Scott, *On sufficiently large 3-manifolds*, Quart. J. Math. Oxford, **23** (1972), 159-172.

11. J. Stallings, *On Fibering Certain 3-Manifolds*, Topology of 3-manifolds, Prentice-Hall, Englewood Cliffs, N. J., 1962, 95-100.

Received April 28, 1972 and in revised form October 3, 1972.

VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY
AND
IOWA STATE UNIVERSITY