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LOCALLY COMPLETE GRAPHS

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If G is the square of a graph H, then each vertex has a closed neighborhood which generates a complete subgraph of G and G is the union of these complete subgraphs. Although the converse fails, it does suggest a classification which yields a theory extensive enough to be of independent interest. This paper develops some basic properties of what will be called locally complete graphs. In § 3 the theory is applied to the problem of square roots, and an existence theorem is proved from which Mukhopadhyay's theorem [3] follows as a corollary. Based on the more general theorem, a technique for square root determination is illustrated in the final section.

1. Introduction. A graph is a finite set of *vertices* together with some of its doubleton subsets, called *edges*. The *closed neighborhood* of a vertex v is the set of all vertices at a distance not greater than 1 from v. If V denotes a subset of the vertex set of a graph G, then the maximal subgraph of G on the vertex set V is said to be *generated by* V. Throughout this paper, G will represent a nontrivial, connected graph. Terminology is essentially that of [2], in which all basic definitions may be found.

DEFINITION 1. Let $\{v_1, v_2, \cdots, v_n\}$ be the vertex set of a graph G, and for each α let N^*_α denote the closed neighborhood of v_α. Let N_α be any subset of N^*_α containing v_α which generates a complete subgraph C_α of G. Then C_α is called a *complete subneighborhood* of v_α, and the indexed family $\mathscr{C} = \{C_1, C_2, \cdots, C_n\}$ is called a *complete family* for G if $G = \bigcup \mathscr{C}$. A graph G is called *locally complete* iff G has at least one complete family.

It is easily seen that complete graphs, trees, and unicyclic graphs are also locally complete. The complete bigraph $K_{3,2}$ is the smallest (nontrivial, connected) graph which fails to be locally complete. The following lemma, suggested by $K_{3,2}$, depends upon the observation that if the given graph contains more than one cycle then the number of edges exceeds the number of vertices.

LEMMA. *A graph G which contains no triangle is locally complete iff it contains at most one cycle.*

No full characterization of local completeness is known. An additional result along this line, however, is given in the theorem below.

THEOREM 1. *Let the graph G contain a clique T of order $r > 2$ and suppose that T has no edge in common with any other clique in G. Then G is locally complete iff for some set E of $\binom{r}{2} - 1$ edges in T each nontrivial component of $G - E$ is locally complete.*

Proof. Let G be locally complete. Some vertex in T has a subneighborhood contained in T, hence there exists a complete family $\mathscr{C}$ in G which contains T. Let v_β denote a vertex in T such that $C_\beta = T$. Let e_β be any edge in T incident with v_β, and let E be the set of all edges in T except e_β. For each α let $C'_\alpha = C_\alpha \cap (G - E)$. If C'_α contains an edge $\neq e_\beta$ then C_α does not contain an edge of T. Therefore $C'_\alpha = C_\alpha$ and we put $C''_\alpha = C'_\alpha = C_\alpha$. If C'_α does not contain any edge and if v_α is not an isolated point in $G - E$, then define C''_α to be any complete subgraph of $G - E$ which contains v_α. We need not be concerned with isolated vertices, and we note that C''_β is defined to be e_β plus its end points.

Now let H be any nontrivial component of $G - E$, and put $K = \bigcup_{v_\alpha \in H} C''_\alpha$. Each C''_α is connected, hence $K \subset H$. If $e_\beta \in H$ then $v_\beta \in H$ and $e_\beta \in K$. For any other edge $e \in H$, there is a γ such that $e \in C_\gamma = C''_\gamma \subset K$. Therefore, $K = H$ and H is locally complete.

Conversely suppose that there is a set E of $\binom{r}{2} - 1$ edges in T such that each nontrivial component of $G - E$ is locally complete. Note that if $v_\alpha \notin T$ then v_α is in a nontrivial component of $G - E$. For v_α is not isolated in G and no edge of G incident with v_α is in T. Let C'_δ be the complete subneighborhood in $G - E$ for each nonisolated vertex v_δ.

Now let $(v_\beta v_\gamma)$ be the edge of T in $G - E$ and suppose it is in a component H of $G - E$. Then $(v_\beta v_\gamma)$ is not in a subgraph of H isomorphic to K_m for any $m > 2$. If $\{C'_\alpha\}$ is the complete family for H, then $(v_\beta v_\gamma) \in C'_\beta$ or C'_γ and is the only edge in that subgraph. For definiteness, assume that $C'_\beta = [\beta, \gamma]$[1]. Then the following is a complete family $\mathscr{C}$ in G:

$$
\begin{aligned}
C_\beta &= T\,, \\
C_\alpha &= T \text{ if } v_\alpha \text{ is isolated in } G - E\,, \\
C_\delta &= C'_\delta \text{ for all other vertices}\,.
\end{aligned}
$$

This theorem and the preceding lemma enable us to characterize all those graphs containing exactly one triangle.

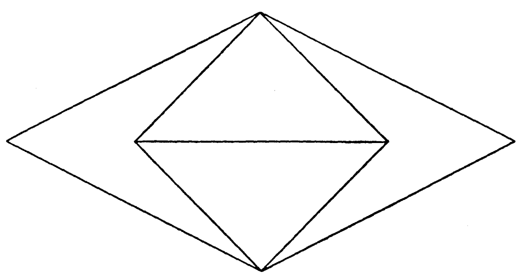
[1] We use $[\alpha, \beta, \cdots, \delta]$ to denote the complete graph on the vertices $v_\alpha, v_\beta, \cdots, v_\delta$.

COROLLARY. *If G contains exactly one triangle T, then G is locally complete iff for some pair, E, of edges in T each nontrivial component of $G - E$ contains at most one cycle.*

2. Derived graphs. The definition of complete family admits the possibility that some C_α may be *trivial* ($C_\alpha = \{v_\alpha\}$) and that nontrivial complete subneighborhoods may have edges in common.

DEFINITION 2. Two complete subneighborhoods are *edge disjoint* iff they have no edge in common. A complete family is *edge disjoint* iff its members are pairwise edge disjoint.

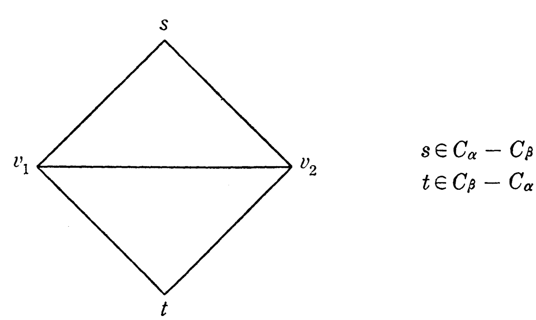
A locally complete graph may have more than one complete family $\mathscr{C}$ associated with it, and some of these may be edge disjoint. But not every locally complete graph has an edge disjoint complete family. The graph G_1 in Figure 1, for example, has no edge disjoint complete family, for if it did then at most one C_α could be a triangle,



G_1

FIGURE 1

leaving five vertices but six edges. That G_1 is the smallest such graph can be seen in the following way. Let $\mathscr{C}$ be a complete family for any graph G. If $C_\alpha \subset C_\beta$ for some $\beta \neq \alpha$ then C_α can be redefined as $C'_\alpha = \{v_\alpha\}$ thus insuring that C'_α and C_β are edge disjoint.



G_2

FIGURE 2

If G is locally complete and has no edge disjoint complete family then there is an edge $(v_1v_2) \in C_\alpha \cap C_\beta$, where neither complete subneighborhood is contained in the other and each contains at least three vertices. Thus G contains a subgraph isomorphic to G_2 in Figure 2. It is easy to see that G_2 has an edge disjoint complete family. Furthermore, each locally complete graph with five vertices having G_2 as a subgraph also has an edge disjoint complete family.

DEFINITION 3. Let G be a locally complete graph and let $\mathscr{C}$ be a complete family for G. Corresponding to $\mathscr{C}$ there is a subgraph of G called the *derived graph* $G(\mathscr{C})$ constructed as follows:

(1) $G(\mathscr{C})$ has the same vertex set as G,

(2) $(v_\alpha v_\beta) \in G(\mathscr{C})$ iff $(v_\alpha v_\beta) \in C_\alpha \cap C_\beta$.

The extreme case in which $G(\mathscr{C})$ is connected is that in which $G(\mathscr{C}) = G$ and, as one might suspect, this case occurs iff G is complete and for each α, $C_\alpha = G$. At the opposite extreme, $G(\mathscr{C})$ is totally disconnected if $\mathscr{C}$ is edge disjoint, but the converse fails as may be observed in G_1. Disconnectedness of $G(\mathscr{C})$ depends upon properties of G as well as upon the complete family $\mathscr{C}$. For example, if G is locally complete and has a cut point, then $G(\mathscr{C})$ is not connected. This observation is generalized in the following theorem.

THEOREM 2. *Let G be locally complete, let V denote its vertex set and let $\mathscr{C}$ be a complete family for G. If $U = \{v_1, v_2, \cdots, v_k\}$ is any subset of V let $U(\mathscr{C}) = \{v \in V : v \in (C_1 \cup C_2 \cup \cdots \cup C_k)\}$, and $\bar{U} = U(\mathscr{C}) - U$. Then $G(\mathscr{C})$ is not connected if there is a nonempty proper subset U of V such that*

$$U \cap \bar{U}(\mathscr{C}) = \varnothing \,.$$

Proof. Note that $\bar{U}$ is the set of vertices in the complement of U, each belonging to a complete subneighborhood of some vertex in U. Now consider the complete subneighborboods corresponding to vertices in $\bar{U}$. There can be no edge in $G(\mathscr{C})$ joining U to its complement unless $U \cap \bar{U}(\mathscr{C}) \neq \varnothing$.

In spite of its easy proof, Theorem 2 has a practical value which will appear in the next section: to test whether U and $V - U$ form a disconnection in $G(\mathscr{C})$ it is necessary to look only at vertices in $\bar{U}$, not at all vertices in $V - U$.

COROLLARY. *If $\mathscr{C}$ is a complete family for G, then $G(\mathscr{C})$ is not connected if either*

(a) *G has a vertex v_α such that $v_\alpha \notin \bigcup \{C_\beta : v_\beta \in C_\alpha - v_\alpha\}$, or*

(b) *G has more that 2 vertices and has an edge not contained in a triangle.*

A locally complete graph may contain several derived graphs and to study the relationship between them, it will be convenient to separate the edges of G into the following types with respect to a given complete family:

(I) $(v_\alpha v_\beta) \in C_\alpha \cap C_\beta$,

(II) $(v_\alpha v_\beta) \notin C_\alpha \cup C_\beta$,

(III) $(v_\alpha v_\beta)$ is in exactly one of C_α, C_β.

Let $\mathscr{C}$ and $\mathscr{C}'$ denote complete families for the graph G. It follows immediately that if $C_\alpha \subset C'_\alpha$ for each α then $G(\mathscr{C}) \subset G(\mathscr{C}')$. Simple examples can be constructed to show that the converse fails; however, the next theorem provides a partial converse.

THEOREM 3. *Let $\mathscr{C}$ and $\mathscr{C}'$ denote complete families for the graph G.*

(1) *If G has no edge of Type III with respect to $\mathscr{C}$, then $G(\mathscr{C}) \subset G(\mathscr{C}')$ implies $C_\alpha \subset C'_\alpha$ for each α.*

(2) *If G has no edge of Type III with respect to either $\mathscr{C}$ or $\mathscr{C}'$, then $\mathscr{C} \neq \mathscr{C}'$ implies $G(\mathscr{C}) \neq G(\mathscr{C}')$.*

The proof of this theorem offers no difficulty and is omitted.

If $\mathscr{L}$ is the set of all complete families $\mathscr{C}$ for G, then $\bigcup \{G(\mathscr{C}) \colon \mathscr{C} \in \mathscr{L}\} \subset G$. The inclusion may be proper, as illustrated by the (5,5) graph consisting of a 4-cycle plus an edge with one vertex on the cycle and one off. It would be of interest to determine conditions under which a locally complete graph *is* the union of its derived subgraphs. The (apparently) more complicated problem of reconstructing G from its derived subgraphs has not yet been studied.

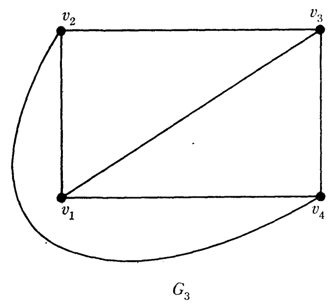
3. Squares and square roots.

We now apply the ideas developed in the preceding sections to the problem of the existence and construction of square roots.

Let H be a nontrivial, connected graph, let N^*_α be the closed neighborhood of v_α in H, and let C^*_α be the complete graph on N^*_α. Although C^*_α need not be a subgraph of H, it is a subgraph of $H^2 = G$, and a simple argument shows that G is the union of these complete subgraphs. Thus G is locally complete, and we shall say that the complete family $\mathscr{C}^* = \{C^*_1, C^*_2, \cdots, C^*_n\}$ for G is *induced* by H.

We note first that G has no edge of Type III with respect to $\mathscr{C}^*$ and that $H = G(\mathscr{C}^*)$. Thus from Theorem 3 in the preceding section, if $\mathscr{C}$ is any complete family for G, then $H \subset G(\mathscr{C})$ iff

$C_\alpha^* \subset C_\alpha$ for each α. Although each square root is a derived graph, a derived graph, in general, need not be a square root. The graph G_3 in Figure 3, for example, is the complete graph K_4 and has a square root (indeed, its derived graph with respect to $\mathscr{C}$ is connected), yet $(G(\mathscr{C}))^2 \neq G$. On the other hand, it is always true that $(G(\mathscr{C}))^2 \subset G$.



$\mathscr{C}$

1 : [1, 2, 4]
2 : [1, 2, 3]
3 : [2, 3, 4]
4 : [2, 3, 4]

FIGURE 3

Summarizing these remarks, if G has a square root, then it is locally complete and each of its square roots is a derived graph. But a locally complete graph G may have derived graphs which are not square roots; in fact, each of its derived graphs may fail to be a square root. The next theorem characterizes those graphs which do have square roots.

THEOREM 4. *Let G be locally complete, with complete family $\mathscr{C}$. Then $(G(\mathscr{C}))^2 = G$ iff for each $(v_\alpha v_\beta) \in G$ there is an index γ such that* (1) $v_\gamma \in C_\alpha \cap C_\beta$ *and* (2) $(v_\alpha v_\beta) \in C_\gamma$.

Proof. Since $(G(\mathscr{C}))^2 \subset G$ we need to show that the given condition is equivalent to $G \subset (G(\mathscr{C}))^2$.

First suppose that $G \subset (G(\mathscr{C}))^2$ and that $(v_\alpha v_\beta) \in G$. Then either $(v_\alpha v_\beta) \in G(\mathscr{C})$ or for some γ $(v_\gamma v_\alpha) \in G(\mathscr{C})$ and $(v_\gamma v_\beta) \in G(\mathscr{C})$. In the first case, conditions (1) and (2) hold with $\gamma = \alpha$ or $\gamma = \beta$. In the second case, $(v_\gamma v_\alpha) \in C_\gamma \cap C_\alpha$ and $(v_\gamma v_\beta) \in C_\gamma \cap C_\beta$, so that (1) and (2) hold.

Next suppose that conditions (1) and (2) hold and that $(v_\alpha v_\beta) \in G$. Then for some γ, $(v_\gamma v_\alpha) \in C_\alpha \cap C_\gamma$ and $(v_\gamma v_\beta) \in C_\beta \cap C_\gamma$. Hence $(v_\gamma v_\alpha) \in G(\mathscr{C})$ and $(v_\gamma v_\beta) \in G(\mathscr{C})$, so $(v_\alpha v_\beta) \in (G(\mathscr{C}))^2$.

COROLLARY. (*Mukhopadhyay*) *A connected graph G has a square root iff G is locally complete and for some complete family $\mathscr{C}$ G contains no edge of Type III with respect to $\mathscr{C}$.*

Proof. Necessity is evident from the construction of the induced family. Now let the given conditions hold and let $(v_\alpha v_\beta) \in G$. If $(v_\alpha v_\beta) \in C_\alpha \cap C_\beta$ then (1) and (2) of Theorem 4 hold with $\gamma = \alpha$ or β. If $(v_\alpha v_\beta) \notin (C_\alpha \cup C_\beta)$, then for some $\gamma(\neq \alpha, \beta)$ $(v_\alpha v_\beta) \in C_\gamma$. Thus $(v_\alpha v_\gamma) \in C_\gamma$ and $(v_\beta v_\gamma) \in C_\gamma$. Since G has no edge of Type III with respect to $\mathscr{C}$, $(v_\alpha v_\gamma) \in C_\alpha$ and $(v_\beta v_\gamma) \in C_\beta$. Therefore (1) and (2) of Theorem 4 hold, and $(G(\mathscr{C}))^2 = G$.

To be used as a test for the existence of a square root, Mukhopadhyay's theorem requires that a complete family $\mathscr{C}$ be found corresponding to which G has no edge of Type III. This is unnecessarily restrictive, as Theorem 4 suggests. In Figure 4, for example, G_4 has several edges of Type III with respect to $\mathscr{C}$.

A graph G may have more than one square root, and a given square root may be the derived graph corresponding to complete families different from the induced family (again Figure 4). Each square root H of G contains a minimal square root (a concept discussed briefly in [3]), and is contained in a *maximal* square root H_m (i.e., $H \subset H_m$, $H_m^2 = G$, no proper supergraph of H_m is a square root). By concentrating on maximal square roots, the theory developed earlier leads to a more efficient square root test (Theorem 5, below).

If H is a nonmaximal square root of G, let it be a proper subgraph of the square root K. Let $\mathscr{C}^*$ be induced by H and $\mathscr{C}'$ induced by K. Since G has no edge of Type III with respect to either C^* or $\mathscr{C}'$, by Theorem 3 $C_\alpha^* \subset C_\alpha'$ for each α and $C_\beta^* \neq C_\beta'$ for some β. This suggests the concept defined next.

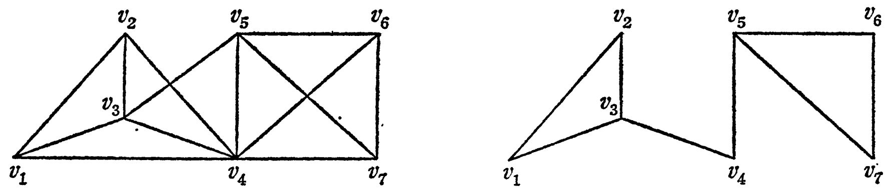
DEFINITION 4. The complete family $\mathscr{C} = \{C_\alpha\}$ is called *maximal* iff for each α C_α is a clique.

It is clear that each graph which has a complete family has a maximal complete family and that a given vertex may belong to more than one clique.

THEOREM 5. *Let G be locally complete and let $\mathscr{M}$ denote the set of all maximal complete families for G. If G has a square root, then some member of the set $\{G(\mathscr{C}): \mathscr{C} \in \mathscr{M}\}$ is a square root.*

Proof. Let $H^2 = G$ and let $\mathscr{C}^*$ be its induced family. For each α, let C_α be a clique containing C_α^*. Then $\mathscr{C} \in \mathscr{M}$, $H \subset G(\mathscr{C})$ and $G(\mathscr{C})$ is a square root.

EXAMPLE. Consider the graph shown in Figure 4. First, tabulate the cliques containing each vertex, and note that G_4 is



G_4 $\qquad$ $G_4(\mathscr{C}) = H$

				$\mathscr{C}$	$\mathscr{C}^*$
1	[1, 2, 3, 4]			[1, 2, 3, 4]	[1, 2, 3]
2	[1, 2, 3, 4]			[1, 2, 3, 4]	[1, 2, 3]
3	[1, 2, 3, 4]	[3, 4, 5]		[1, 2, 3, 4]	[1, 2, 3, 4]
4	[1, 2, 3, 4]	[3, 4, 5]	[4, 5, 6, 7]	[3, 4, 5]	[3, 4, 5]
5	[3, 4, 5]	[4, 5, 6, 7]		[4, 5, 6, 7]	[4, 5, 6, 7]
6	[4, 5, 6, 7]			[4, 5, 6, 7]	[5, 6, 7]
7	[4, 5, 6, 7]			[4, 5, 6, 7]	[5, 6, 7]

FIGURE 4

locally complete. Among its maximal complete families, some may be eliminated from consideration fairly quickly because the derived graphs are not connected (see Theorem 2 and its corollary). For example, the choice $C_1 = C_2 = C_3 = C_4 = [1, 2, 3, 4]$ is not admissible. By Theorem 4, the following complete family $\mathscr{C}$ is admissible:

$$C_1 = C_2 = C_3 = [1, 2, 3, 4]\,,\ C_4 = [2, 4, 5]\,,\ C_5 = C_6 = C_7 = [4, 5, 6, 7]\,.$$

The derived graph $G_4(\mathscr{C})$ is shown in Figure 4 and it is easily seen to be a square root of G_4. The induced family $\mathscr{C}^*$ is also shown, and it clearly is not maximal even though $G_4(\mathscr{C})$ is maximal.

It would be of interest to modify the concepts introduced in this paper for application to directed graphs and to the square roots of digraphs [1].

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