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ENDOMORPHISM RINGS OF FINITELY GENERATED PROJECTIVE MODULES

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Over a ring A let P_A be a finitely generated projective right A-module with A-endomorphism ring B. Anderson has called P_A an injector, perfect injector, projector, perfect projector, if the functor $F = {}_BP \otimes_A (\ )$ preserves injectives, injective hulls, projectives, projective covers, respectively. Call P_A a flatjector if F preserves flat modules. Injectors, flatjectors, and projectors are characterized. The radical of a module over B is studied, and necessary and sufficient conditions are given for the radical of B to be left T-nilpotent. Perfect injectors are characterized. Previous characterizations of perfect projectors have assummed the ring A to be left perfect. Here characterizations are obtained using substantially weaker conditions on P_A.

Over a ring A let P_A be a finitely generated projective right A-module with A-endomorphism ring B. Let T be the trace of P_A and $P^* = \mathrm{Hom}_A(P, A)$ be the A-dual of P. Injectors, flatjectors, and projectors are characterized in §2, the characterizations of injectors and projectors being extensions of results due to Anderson [1]. For example, P_A is a flatjector if and only if ${}_BP$ (equivalently ${}_AP^* \otimes {}_BP$) is flat. P_A is a projector if and only if the natural map $\eta\colon {}_AP^* \otimes P \to {}_AT$ is a projective cover. It is shown that P_A is a flatjector if and only if ${}_AP^*$ is an injector. Furthermore, every projector is a flatjector, the two being equivalent for perfect rings. Examples are given of a flatjector that is not a projector and a nonperfect ring where every flatjector is a projector.

In §3 it is shown that the B-radical of ${}_BP \otimes {}_AX$ is isomorphic to ${}_BP \otimes {}_AJ(TX)$. In particular the B-radical of ${}_BP$ is isomorphic to ${}_BP \otimes {}_AJ(T)$. A definition of U-dominant codimension dual to the definition of V-dominant dimension given in [8] is introduced. The radical N of B being left T-nilpotent is characterized in terms of the full subcategories $\mathscr{C}_1({}_AP^*)$ of ${}_A\mathfrak{M}$ consisting of all left A-modules of ${}_AP^*$-dominant codimension $\geqq 1$, and $\mathfrak{D}_1(Q_A)$ of $\mathfrak{M}_A$ consisting of all right A-modules of Q_A-dominant dimension $\geqq 1$ (where $Q_A = \mathrm{Hom}_B(P, W)$ for ${}_BW$ an injective cogenerator). It is shown that N is left T-nilpotent if and only if JT is left T-nilpotent, or equivalently $J(\bigoplus_I P^*)$ is small in $\bigoplus_I P^*$ for any index set I.

In §4 perfect injectors are characterized in terms of their trace ideal and certain conditions on large submodules. Anderson [1] has studied perfect projectors when the ring A is left perfect. Here characterizations of perfect projectors are obtained using substantially weaker conditions on P_A, e.g. JT being left T-nilpotent. Finally it is shown that for A semiperfect, P_A is a perfect projector if and only if ${}_AP^*$ is a perfect injector.

1. Preliminaries. Throughout this paper we will observe the following definitions and notation. A will be a ring with unit and all modules will be unitary. All maps will be written on the side opposite the scalars. Given an A-module X, its A-dual $\mathrm{Hom}_A(X, A)$ will be denoted by X^*. P_A will always be a finitely generated projective right A-module with A-endomorphism ring B. It is easy to see that ${}_AP^*$ is also finitely generated projective and that $(P^*)^* \cong P$. With P_A we associate the two-side ideal T of A (called the trace of P_A) generated by the images of all A-homomorphisms from P into A, i.e., $T = \sum_{f \in P^*} \mathrm{im}\, f$.

Let ${}_A\mathfrak{M}(\mathfrak{M}_A)$ represent the category of left (right) A-modules. We have functors

$$F = {}_BP \otimes {}_A(\ ) \colon {}_A\mathfrak{M} \to {}_B\mathfrak{M}$$
$$G = {}_AP^* \otimes {}_B(\ ) \colon {}_B\mathfrak{M} \to {}_A\mathfrak{M}$$

and associated natural transformations

$$\eta_X \colon GF({}_AX) \to {}_AX$$
$$\mu_W \colon FG({}_BW) \to {}_BW$$

defined by

$$(f \otimes p \otimes x)\eta_X = f(p)x$$
$$(p \otimes f \otimes w)\mu_W = pf(\ )w$$

for $p \in P$, $f \in P^*$, $x \in X$, and $w \in W$.

The following isomorphisms are well known (see e.g. [4]).

I. For ${}_AX$, ${}_BU_A$, ${}_BW$

$$\mathrm{Hom}_B({}_BU \otimes {}_AX, {}_BW) \cong \mathrm{Hom}_A({}_AX, {}_A\mathrm{Hom}_B({}_BU_A, {}_BW))$$

II. For U_A, ${}_BM_A$, ${}_BW$ where U_A is finitely generated projective

$$U \otimes {}_A\mathrm{Hom}_B({}_BM_A, {}_BW) \cong \mathrm{Hom}_B({}_B\mathrm{Hom}_A(U_A, {}_BM_A), {}_BW)$$

Properties of P, P^*, T, *and* B.

(a) $T^2 = T$, $PT = P$, and $TP^* = P^*$.

(b) $B = \mathrm{End}({}_AP^*)$.

(c) ${}_BP \otimes {}_AT_A \cong {}_BP_A$ and ${}_AT \otimes {}_AP^*_B \cong {}_AP^*_B$.

(d) $\mathrm{im}\, \eta_X = TX$ for all $X \in {}_A\mathfrak{M}$.

(e) FG is naturally equivalent to the identity functor on

${}_B\mathfrak{M}(FG \sim I_{{}_B\mathfrak{M}})$via μ.

(f) ${}_BP \otimes {}_ATX \cong {}_BP \otimes {}_AX$ and $YT \otimes {}_AP_B^* \cong Y \otimes {}_AP_B^*$.

(g) ${}_BP \otimes {}_AX \cong {}_B\mathrm{Hom}_A\,({}_AP_B^*, {}_AX)$ and $Y \otimes {}_AP_B^* \cong \mathrm{Hom}_A\,({}_BP_A, Y_A)_B$.

(h) For $X \in {}_A\mathfrak{M}$, $P \otimes {}_AX = 0$ if and only if $TX = 0$.

(i) For $Y \in \mathfrak{M}_A$, $\mathrm{Hom}_A\,(P, Y) = 0$ if and only if $YT = 0$.

Proof. (*a*), (c), (d), and (e) are essentially in [1].

(b) See [10, Lemma 1.1].

(f) ${}_BP \otimes {}_AX/TX \cong {}_BP \otimes {}_AA/T \otimes X \cong P/PT \otimes {}_AX = 0$ by (a). The second isomorphism follows similarly.

(g) follows easily from II.

(h) follows easily from (d) and (f).

(i) follows easily from (f), (g), and the dual of (d).

Let K be a submodule of X. We say K is small in X (and write $K' \subseteqq X$) if given a submodule H of X such that $K + H = X$, then $H = X$. We say K is large in X (and write $K \subseteqq' X$) if K has nonzero intersection with every nonzero submodule of X.

For an A-module X the radical of X, denoted $J(X)$, is defined to be the intersection of all maximal submodules of X. If X has no maximal submodules, then $J(X) = X$ (e.g. see [7]). We will write J for $J(A)$.

The following properties of the radical are well known.

Properties of the radical: For ${}_AX$,

(a) If $x \in J(X)$, then $Ax' \subseteqq X$.

(b) If $K' \subseteqq X$, then $K \subseteqq J(X)$.

(c) If $f\colon X \to Y$ is an A-homomorphism, then $(J(X))f \subseteqq J(Y)$.

(d) $JX \subseteqq J(X)$.

(e) If X is projective, then $JX = J(X)$. [3]

(f) If X is projective, then $JX \neq X$. [3]

NOTATION. To minimize confusion in later theorems, the radical of a module W over the ring $B = \mathrm{End}\,(P_A)$ will be denoted by $N(W)$. In particular, $N = N(B)$.

LEMMA 1.1. *Let $f\colon {}_AX \to {}_AY$ be a projective cover. Then* $J(Y) = JY$.

Proof. Since $JY \subseteqq J(Y)$ always, we only need to show that $J(Y) \subseteqq JY$. Let $y \in J(Y)$. Then y is contained in every maximal submodule of Y. Thus $(y)f^{-1}$ is contained in every maximal submodule of X containing $K = \ker f$. But $K' \subseteqq X$ and hence is contained in every maximal submodule of X. Thus $(y)f^{-1} \in J(X) = JX$. So $y \in (JX)f = J(X)f = JY$.

LEMMA 1.2. *If a flat module ${}_AY$ has a projective cover, then ${}_AY$ is projective.*

Proof. Let $f: {}_AX \to {}_AY$ be a projective cover of ${}_AY$. We have the exact sequence

$$0 \longrightarrow K \longrightarrow X \xrightarrow{f} Y \longrightarrow 0$$

where $K = \ker f$. Let $k \in K$. By [11, Lemma 2.2] there exists $\theta_k: X \to K$ such that $(k)\theta_k = k$. But $\operatorname{im} \theta_k \subseteq K' \subseteq X$, hence θ_k is contained in the radical of the endomorphism ring of ${}_AX$ (see [11, Prop. 1.1]). So $1 - \theta_k$ is a unit (see e.g. [7]). Therefore, $(k)(1 - \theta_k) = k - (k)\theta_k = 0$. Hence $k = 0$. Thus ${}_AY$ is projective since f is an isomorphism.

NOTATION. X^I will denote the direct sum of copies of X over the index set I. If $I = \{1, \cdots, n\}$, we will simply write X^n for X^I.

Let ${}_AY \subseteq {}_AX$ and let L be a right ideal of A. Then $(Y: L)_X = \{x \in X \mid Lx \subseteq Y\}$ is a submodule of X containing Y. In particular $r_X(L) = (0: L)_X$ is called the right annihilator of L in X. If $Y_A \subseteq X_A$ and L is a left ideal of A similar definitions apply. We will write $l_X(L)$ for the left annihilator of L in X.

The following lemmas are left to the reader.

LEMMA 1.3. *Let L be a two-sided ideal of A. Then*

(a) *For ${}_AX$, $r_X(L) \cong \operatorname{Hom}_A(A/L, X)$.*

(b) *For X_A, $l_X(L) \cong \operatorname{Hom}_A(A/L, X)$.*

(c) *$r_A(L)$ and $l_A(L)$ are both two-sided ideals of A.*

LEMMA 1.4. *Let ${}_BU_A$ be a bimodule with ${}_BU$ flat. If ${}_AX$ is flat, then ${}_BU \otimes {}_AX$ is flat.*

2. Injectors, flatjectors, and projectors. For a finitely generated projective right A-module P_A with A-endomorphism ring B, Anderson [1] has called P_A an injector (projector) if the functor $F = {}_BP \otimes {}_A(\ )$ preserves injectives (projectives). That is, if ${}_AX$ is injective (projective) in ${}_A\mathfrak{M}$, then ${}_BP \otimes {}_AX$ is injective (projective) in ${}_B\mathfrak{M}$. Similarly, we call P_A a flatjector if the functor $F = {}_BP \otimes {}_A(\ )$ preserves flat modules. Our purpose in this section is to characterize injectors, flatjectors, and projectors.

THEOREM 2.1. *For P_A finitely generated projective the following statements are equivalent.*

(a) *P_A is an injector.*

(b) *P_B^* is flat in $\mathfrak{M}_B$.*

(c) $P^* \otimes_B P_A$ *is flat in* $\mathfrak{M}_A$.

Proof. (a) $\Leftrightarrow$ (b). This is by Anderson [1, Theorem 2.1].
(b) $\Rightarrow$ (c). Since both P_B^* and P_A are flat (c) follows by Lemma 1.4.
(c) $\Rightarrow$ (b). Let $\alpha\colon {}_BU \rightarrow {}_BV$ be a monomorphism and consider the exact sequence

$$0 \longrightarrow K \longrightarrow G(U) \xrightarrow{G(\alpha)} G(V)$$

where $K = \ker G(\alpha)$. Since P_A is flat

$$0 \longrightarrow F(K) \longrightarrow FG(U) \longrightarrow FG(V)$$

is exact. Thus $F(K) = 0$ since $FG \sim I_{B\mathfrak{M}}$. Since $P^* \otimes_B P_A$ is flat

$$0 \longrightarrow GF(K) \longrightarrow GFG(U) \longrightarrow GFG(V)$$

is exact. Thus P_B^* is flat since $GF(K) = 0$ and $FG \sim I_{B\mathfrak{M}}$.

We include the following theorem for completeness. With the obvious changes, the proof is similar to the proof of (c) $\Rightarrow$ (b) in Theorem 2.1.

THEOREM 2.2. (*Anderson* [1, *Theorem* 2.2]). *If the trace ideal* T *of* P_A *is flat in* $\mathfrak{M}_A$, *then* P_A *is an injector.*

The converse to Theorem 2.2 is false as shown by [1, Example 2.3].

For flatjectors we give a characterization similar to the one given for injectors.

THEOREM 2.3. *For* P_A *finitely generated projective the following statements are equivalent.*

(a) P_A *is a flatjector.*
(b) ${}_BP$ *is flat in* ${}_B\mathfrak{M}$.
(c) ${}_AP^* \otimes_B P$ *is flat in* ${}_A\mathfrak{M}$.

Proof. (a) $\Rightarrow$ (b). Since ${}_AA$ is flat and ${}_BP \cong {}_BP \otimes_A A$, we must have ${}_BP$ flat.
(b) $\Rightarrow$ (c). Both ${}_BP$ and ${}_AP^*$ are flat. Thus (c) follows from Lemma 1.4.
(c) $\Rightarrow$ (a). Let $\alpha\colon U_B \rightarrow V_B$ be a monomorphism.

Let F^* and G^* represent the functors $(\ ) \otimes_A P_B^*$ and $(\ ) \otimes_B P_A$ respectively. Consider the exact sequence

$$0 \longrightarrow K \longrightarrow G^*(U) \xrightarrow{G^*(\alpha)} G^*(V)$$

where $K = \ker G^*(\alpha)$. Since ${}_AP^*$ is flat

$$0 \longrightarrow F^*(K) \longrightarrow F^*G^*(U) \longrightarrow F^*G^*(V)$$

is exact. Thus $F^*(K) = 0$ since $F^*G^* \sim I_{\mathfrak{M}_B}$. Since ${}_AP^* \otimes {}_BP$ is flat

$$0 \longrightarrow G^*F^*(K) \longrightarrow G^*F^*G^*(U) \longrightarrow G^*F^*G^*(V)$$

is exact. Thus ${}_BP$ is flat since $G^*F^*(K) = 0$ and $F^*G^* \sim I_{\mathfrak{M}_B}$.

Let ${}_AX$ be flat. Then ${}_BP \otimes {}_AX$ is flat by Lemma 1.4. So P_A is a flatjector.

Theorems 2.1 and 2.3 show that injector and flatjector are dual properties.

COROLLARY 2.4. *For a finitely generated projective module P_A, P_A is an injector if and only if ${}_AP^*$ is a flatjector.*

A ring A is said to be regular if every (right) A-module is flat. Ware [11] has called a projective module regular if every homomorphic image is flat. In the following corollary we give a new proof to one of Ware's results.

COROLLARY 2.5. (*Ware* [11, *Theorem* 3.6]). *Let P_A be a finitely generated regular module with $B = \operatorname{End}(P_A)$. Then B is a regular ring.*

Proof. We first note that P_A^I is a regular module for any index set I [11, page 239]. Let $U_B \in \mathfrak{M}_B$ and consider $U \otimes_B P_A$. Let $I = U$ and define a map $\varphi \colon P_A^I \to U \otimes {}_BP_A$ by $\varphi[(p_u)_{u \in I}] = \Sigma u \otimes p_u$. Clearly φ is an A-epimorphism. Thus $U \otimes {}_BP_A$ is flat in $\mathfrak{M}_A$.

In particular $P^* \otimes {}_BP_A$ is flat in $\mathfrak{M}_A$. Thus ${}_AP^*$ is a flatjector by Theorem 2.3. Hence $U_B \cong U \otimes {}_BP \otimes {}_AP_B^*$ is flat in $\mathfrak{M}_B$. So B is a regular ring.

THEOREM 2.6. *If the trace ideal T of P_A is flat in ${}_A\mathfrak{M}$, then P_A is a flatjector.*

With the obvious changes, Theorem 2.6 follows in the same manner as (c) $\Rightarrow$ (a) of Theorem 2.3. Hence the proof will be omitted. The converse of Theorem 2.6 is false. In Example 2.3 of [1], P_A is a flatjector, but T is not flat in ${}_A\mathfrak{M}$.

Before giving a characterization for projectors, we need the following lemma.

LEMMA 2.7. *The map $\eta_A \colon P^* \otimes {}_BP \to T$ is both a right and left* *minimal A-epimorphism.*

Proof. We will only prove the left case as the right case follows by symmetry.

Suppose we have $\alpha\colon {}_AX \to {}_AP^* \otimes {}_BP$ such that $\alpha\eta_A$ is onto. Tensoring with P_A we have

$$F(X) \xrightarrow{F(\alpha)} F(P^* \otimes P) \xrightarrow{F(\eta_A)} F(T)$$

where $F(\alpha)F(\eta_A)$ is onto. Since $F(P^* \otimes P) = FG(P) \cong P \cong F(T)$, $F(\eta_A)$ is an isomorphism. Thus $F(\alpha)$ is onto. So

$$\begin{tikzcd} GF(X) \arrow[r, "GF(\alpha)"] \arrow[d, "\eta_X"'] & P^* \otimes P \\ X \arrow[ur, "\alpha"'] & \end{tikzcd}$$

is a commutative diagram where $GF(\alpha)$ is onto. Hence, α is onto.

THEOREM 2.8. *For P_A finitely generated projective the following statements are equivalent.*

(a) *P_A is a projector.*

(b) *${}_BP$ is projective in ${}_B\mathfrak{M}$.*

(c) *${}_AP^* \otimes {}_BP$ is projective ${}_A\mathfrak{M}$.*

(d) *$\eta_A\colon {}_AP^* \otimes P \to {}_AT$ is a projective cover for ${}_AT$.*

Proof. (a) $\Leftrightarrow$ (b). This is by Anderson [1, Theorem 3.1].
(b) $\Rightarrow$ (c). Since ${}_BP$ is projective we have a split exact sequence

$${}_BB^I \rightleftarrows {}_BP \longrightarrow 0$$

where I is some index set. Thus

$$G(B^I) \rightleftarrows G(P) \longrightarrow 0$$

is also split exact. Since $G(B^I) \cong {}_AP^{*I}$, we see that $G(P) = {}_AP^* \otimes {}_BP$ is a direct summand of a projective, and hence is projective.
(c) $\Rightarrow$ (d). This is immediate from Lemma 2.7.
(d) $\Rightarrow$ (b). As in Corollary 2.5, ${}_AP^* \otimes {}_BP$ is the homomorphic image of a direct sum of copies of ${}_AP^*$, say ${}_AP^{*I}$. Since ${}_AP^* \otimes {}_BP$ is projective, we have

$${}_AP^{*I} \rightleftarrows {}_AP^* \otimes {}_BP \longrightarrow 0$$

is split exact. Tensoring with P_A, we see that ${}_BP$ is a direct summand of a direct sum of copies of ${}_BB$. Thus ${}_BP$ is projective.

The following theorem is added for completeness. With the obvious changes, the proof is similar to the proof of (d) $\Rightarrow$ (b) of Theorem 2.8.

THEOREM 2.9. (*Anderson* [1, *Theorem* 3.2]). *If the trace ideal T of P_A is projective in ${}_A\mathfrak{M}$, then P_A is a projector.*

The converse to Theorem 2.9 is false as shown by [1, Example 2.3].

One might ask when the functor $F = {}_BP \otimes {}_A(\ )$ preserves finitely generated projectives? That is, if ${}_AX$ is finitely generated projective, when is ${}_BP \otimes {}_AX$ finitely generated projective? This question is answered by the following theorem.

THEOREM 2.10. *For P_A finitely generated projective the following statements are equivalent.*

(a) *$F = {}_BP \otimes {}_A(\ )$ preserves finitely generated projectives.*

(b) *${}_BP$ is finitely generated projective in ${}_B\mathfrak{M}$.*

(c) *${}_BP^* \otimes {}_BP$ is finitely generated projective in ${}_A\mathfrak{M}$.*

Proof. (a) $\Rightarrow$ (b). Since ${}_AA$ is finitely generated projective, so is ${}_BP \cong {}_BP \otimes A$.
(b) $\Rightarrow$ (c). Since ${}_BP$ is finitely generated, the index set I used in (b) $\Rightarrow$ (c) of Theorem 2.8 can be taken to be finite. Thus ${}_AP^* \otimes {}_BP$ is finited generated projective.
(c) $\Rightarrow$ (b). Again the index set I used in (d) $\Rightarrow$ (b) of Theorem 2.8 can be taken to be finite. Thus ${}_BP$ is finitely generated projective.
(b) $\Rightarrow$ (a). If ${}_AX$ is finitely generated projective, then there is a split exact sequence

$$ {}_AA^n \rightleftarrows {}_AX \longrightarrow 0 . $$

Thus tensoring with P_A we see that ${}_BP \otimes {}_AX$ is a direct summand fo ${}_BP \otimes {}_AA^n \cong ({}_BP \otimes {}_AA)^n \cong {}_BP^n$. Since ${}_BP^n$ is finitely generated projective, so is ${}_BP \otimes {}_AX$.

THEOREM 2.11. *If the trace ideal T of P_A is finitely generated projective in ${}_A\mathfrak{M}$, then $F = {}_BP \otimes {}_A(\ )$ preserves finitely generated projectives.*

Proof. By Theorem 2.9, P_A is a projector, hence ${}_BP$ is projective. By [10, Theorem 2.2, Corollary 4] ${}_BP$ is finitely generated. Hence the theorem follows by Theorem 2.10.

REMARK. By Theorems 2.3 and 2.8 we see that every projector is a flatjector. On page 328 of [1] Anderson gives an example of a projector (hence flatjector) that is not an injector. Also, Anderson constructs an injector that is not flat over its endomorphism ring, and thus is neither a flatjector nor a projector.

In [3] Bass defines a ring A to be left (right) perfect if every left (right) A-module has a projective cover, and semiperfect if every cyclic (left) A-module has a projective cover. Bass shows that A is left perfect if and only if evey flat left A-module is projective. This is also easily seen from Lemma 1.2.

By Theorems 2.3 and 2.8, for a left perfect ring A, P_A is a flatjector if and only if P_A is a projector. The question arises as to whether a left perfect ring can be characterized in terms of every flatjector being a projector. We exhibit an example to show that the answer to this question is no.

EXAMPLE 2.12. Let A be a simple ring that is not left perfect. Let P_A be a finitely generated projective right A-module. Since A is simple, P_A is a generator, and hence $_BP$ is projective (see [2]). Thus over A, every flatjector is a projector, but A is not left perfect.

We end this section with an example of a flatjector that is not a projector.

EXAMPLE 2.13. Let B be a ring that is not left perfect and let $_BQ_C$ be a $B-C$ bimodule such that $_BQ$ is flat, but not projective. Let A be the ring

$$A = \begin{pmatrix} B & _BQ_C \\ 0 & C \end{pmatrix}$$

and let

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $P_A = eA$ is finitely generated projective and $\mathrm{End}\,(P_A) = eAe \cong B$. As a B-module, $_BP \cong {_BB} \oplus {_BQ}$. Thus $_BP$ is flat, but $_BP$ is not projective. Hence P_A is a flatjector but not a projector.

3. **Radicals.** Our purpose in this section is to answer the following questions. For P_A finitely generated projective with $B = \mathrm{End}\,(P_A)$

(1) given $_AX$, what is the radical of $_BP \otimes {_AX}$ (in particular $_BP \cong {_BP} \otimes {_AA}$), and

(2) when is the radical of B left T-nilpotent?

Letting T be the trace ideal of P_A, Sandomierski [10] has defined an A-module ${}_AX(X_A)$ to be T-accessible if $TX = X(XT = X)$. Clearly, both P_A and ${}_AP^*$ are T-accessible.

LEMMA 3.1. *${}_AX(X_A)$ is T-accessible if and only if ${}_AX(X_A)$ is the homomorphic image of a direct sum of copies of ${}_AP^*(P_A)$.*

Proof. We will do the left case only as the right case follows in a similar manner.

Suppose that $X = TX$. Let $I = F(X)$ and define $\varphi\colon P^{*I} \to GF(X)$ by $[(z_i)_{i \in I}]\varphi = \sum z_i \otimes i$. Clearly φ is an A-epimorphism. But via η_X, TX is the homomorphic image of ${}_AP^{*I}$.

Conversely suppose that we have an A-epimorphism $\varphi\colon {}_AP^{*I} \to {}_AX$ for some index set I. Letting $Y = P^{*I}$, $X = (Y)\varphi = (TY)\varphi = T(Y)\varphi = TX$.

If ${}_BL \subseteqq {}_B\mathrm{Hom}_A(P^*, X)$, then ${}_AP^*L = \{\sum zg \mid z \in P^*, g \in L\}$ is a T-accessible submodule of ${}_AX$. In fact, $P^* \,\mathrm{Hom}_A(P^*, X) = TX$.

THEOREM 3.2. (*Sandomierski* [10, *Theorem* 2.2]). *The correspondence ${}_BL \to {}_AP^*L$ is a one-to-one inclusion preserving correspondence between the submodules ${}_BL$ of ${}_B\mathrm{Hom}_A(P^*, X)$ and the T-accessible submodules of ${}_AX$. The inverse correspondence is given by ${}_AY \to {}_B\mathrm{Hom}_A(P^*, Y)$.*

By Theorem 3.2 we see that if ${}_AS$ is a simple A-module, then ${}_B\mathrm{Hom}_A(P^*, S) \cong {}_BP \otimes {}_AS$ is either zero or a simple B-module.

We say that a proper submodule ${}_AY$ is a maximal T-accessible submodule of ${}_AX$ if ${}_AY$ is T-accessible and there are no T-accessible submodules strictly between ${}_AY$ and ${}_AX$. By Theorem 3.2, ${}_AY$ is a maximal T-accessible submodule of TX if and only if ${}_BP \otimes {}_AY \cong {}_B\mathrm{Hom}_A(P^*, Y)$ is a maximal submodule of ${}_BP \otimes {}_AX \cong {}_B\mathrm{Hom}_A(P^*, X)$.

THEOREM 3.3. *The correspondence $Y \to (Y\colon T)_X$ is a one-to-one inclusion preserving correspondence between the T-accessible submodules ${}_AY$ of ${}_AX$ and the submodules ${}_AU$ of ${}_AX$ such that $(TU\colon T)_X = U$. The inverse correspondence is given by $U \to TU$.*

Proof. We show that both composites yield the identity.

(i) Consider $W = (Y\colon T)_X$. Since $Y \subseteqq W$ and $TY = Y$, it is easy to see that $TW = Y$. Hence $(TW\colon T)_X = (Y\colon T)_X = W$. Thus

$$Y \longrightarrow W \longrightarrow TW = Y\,.$$

(ii) Consider U such that $(TU\colon T)_X = U$. Then

$$U \longrightarrow TU \longrightarrow (TU\colon T)_X = U\,.$$

One easily checks that the correspondence is inclusion preserving.

THEOREM 3.4. *For $_AX$ T-accessible the correspondence defined in Theorem 3.3 yields a one-to-one correspondence between the maximal T-accessible submodules of X and the maximal submodules of X.*

Proof. (i) Let Y be a maximal T-accesible submodule of $_AX$ and suppose that $(Y\colon T)_X \subseteqq V \subsetneqq X$. Then $Y = T(Y\colon T)_X \subseteqq TV \subsetneqq TX = X$. Thus $TV = Y$ so that $V \subseteqq (Y\colon T)_X$. Hence $(Y\colon T)_X$ is maximal in X.
(ii) Let U be a maximal submodule of $_AX$ and note that $U \subseteqq (TU\colon T)_X \subseteqq X$. If $(TU\colon T)_X = X$, then $X = TX = T(TU\colon T)_X \subseteqq TU$. That is, $U = X$, a contradiction to the maximality of U. Thus $U = (TU\colon T)_X$. Now the fact that TU is a maximal T-accessible submodule of X follows easily from the correspondence and the maximality of U.

COROLLARY 3.5. *Let $_AX$ be T-accessible. Then $_AX$ has a maximal submodule if and only if $_AX$ has a maximal T-accessible submodule. Furthermore, if U is a maximal submodule of X, then U contains a unique maximal T-accessible submodule TU of X; and if Y is a maximal T-accessible submodule of X, then Y is contained in a unique maximal submodule $(Y\colon T)_X$ of X.*

The next two lemmas follow from Theorems 3.4 and 3.2 respectively and are needed to answer our first question.

LEMMA 3.6. *For $_AX$, let $_AM$ be the intersection of all maximal T-accessible submodules of TX. Then $TM = TJ(TX)$.*

REMARK. If there are no maximal T-accessible submodules of TX, we set $M = TX$.

Proof. By Corollary 3.5 there are no maximal T-accessible submodules of TX if and only if there are no maximal submodules of TX. In this case the lemma becomes trivial.

Thus let $\{H_i\}_{i\in I}$ be the set of all maximal submodules of TX. Then by Theorem 3.4 $\{TH_i\}_{i\in I}$ is the set of all maximal T-accessible submodules of TX. Clearly $M \subseteqq J(TX)$ so that $TM \subseteqq TJ(TX)$. But $TJ(TX) = T(\bigcap_{i\in I} H_i) \subseteqq \bigcap_{i\in I} TH_i = M$. Thus $TJ(TX) \subseteqq TM$.

LEMMA 3.7. *For $_AX$, let $_AM$ be defined as in Lemma 3.6 and let $\{L_i\}_{i\in I}$ be the collection of maximal submodules of $_B\mathrm{Hom}_A(P^*, X)$. Then $P^*(\bigcap_{i\in I} L_i) = TM$.*

Proof. By Theorem 3.2 there are no maximal T-accessible modules of TX if and only if there are no maximal submodules of $_B\mathrm{Hom}_A(P^*, X)$.

In this case the lemma becomes trivial.

By the remarks following Theorem 3.2 $\{P^*L_i\}_{i\in I}$ is the set of all maximal T-accessible submodules of TX. Thus $P^*(\bigcap_{i\in I} L_i) \subseteqq \bigcap_{i\in I} P^*L_i = M$, and so $P^*(\bigcap_{i\in I} L_i) \subseteqq TM$.

Conversely, $TM \subseteqq P^*L_i$ for all $i \in I$. By Theorem 3.2 $\mathrm{Hom}_A(P^*, P^*L_i) = L_i$, hence $\mathrm{Hom}_A(P^*, TM) \subseteqq L_i$ for all $i \in I$. That is, $\mathrm{Hom}_A(P^*, TM) \subseteqq \bigcap_{i\in I} L_i$. Again by Theorem 3.2 $TM = P^* \mathrm{Hom}_A(P^*, TM) \subseteqq P^*(\bigcap_{i\in I} L_i)$.

THEOREM 3.8. *Let P_A be finitely generated projective with trace ideal T and $B = \mathrm{End}(P_A)$. For ${}_AX$, we have*

(a) $P^*N(\mathrm{Hom}_A(P^*, X)) = TJ(TX)$.

(b) $N(\mathrm{Hom}_A(P^*, X)) = \mathrm{Hom}_A(P^*, J(TX))$.

Proof. (a) By Lemmas 3.7 and 3.6, $P^*N(\mathrm{Hom}_A(P^*, X)) = P^*(\bigcap_{i\in I} L_i) = TM = TJ(TX)$.

(b) By Theorem 3.2 and (a) we have that $N(\mathrm{Hom}_A(P^*, X)) = \mathrm{Hom}_A(P^*, P^*N(\mathrm{Hom}_A(P^*, X)) = \mathrm{Hom}_A(P^*, TJ(TX))$. But it is easy to see that $\mathrm{Hom}_A(P^*, TJ(TX)) = \mathrm{Hom}_A(P^*, J(TX))$.

Thus we have the following description of the radical of ${}_BP \otimes {}_AX$.

COROLLARY 3.9. (a) *For ${}_AX$, $N({}_BP \otimes {}_AX) \cong P \otimes {}_AJ(TX)$.*

(b) $N({}_BP) \cong P \otimes {}_AJ(T)$.

Proof. (a) follows from Theorem 3.8 since $P \otimes_A X \cong \mathrm{Hom}_A(P^*, X)$ for all ${}_AX$.

(b) follows in the same manner as (a) using ${}_BP \cong {}_BP \otimes {}_AA$.

The following corollary is well known. See for example [11, Prop. 1.1].

COROLLARY 3.10. $N = N(B) = \mathrm{Hom}_A(P^*, JP^*) = \mathrm{Hom}_A(P, PJ)$.

Proof. That $N = \mathrm{Hom}_A(P^*, JP^*)$ follows from the fact that $B = \mathrm{End}({}_AP^*)$ and $TP^* = P^*$. That $N = \mathrm{Hom}_A(P, PJ)$ follows by a dual argument.

COROLLARY 3.11. *Letting $N = N(B)$, we have*

(a) $P^*N = TJP^*$.

(b) $NP = PJT$.

Proof. (a) $P^*N = P^* \mathrm{Hom}_A(P^*, JP^*) = TJP^*$.

(b) follows by symmetry.

We now move on to our second question. For a right A-module V_A, Morita [8] has defined a right A-module Y to be of V-dominant dimension $\geqq n$ (written V-dom. dim. $Y \geqq n$) if there is an exact sequence

$$0 \longrightarrow Y \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_n$$

such that each Y_i is a direct product of copies of V. Dually, for a left A-module ${}_AU$, we say that a left A-module X is of U-dominant codimension $\geqq n$ (written U-dom. codim. $X \geqq n$) if there is an exact sequence

$$X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X \longrightarrow 0$$

such that each X_i is a direct sum of copies of U.

Let $\mathfrak{D}_n(V_A)$ be the full subcategory of $\mathfrak{M}_A$ consisting of all A-modules of V-dominant dimension $\geqq n$. Similarly, let $\mathscr{C}_n({}_AU)$ be the full subcategory of ${}_A\mathfrak{M}$ consisting of all A-modules of U-dominant codimension $\geqq n$.

In particular $\mathscr{C}_1({}_AP^*)$ consists of all modules ${}_AX$ that are homomorphic images of a direct sum of copies of ${}_AP^*$. By Lemma 3.1 we see that $\mathscr{C}_1({}_AP^*)$ is the full subcategory of T-accessible left A-modules.

Let W_B be an injective cogenerator in $\mathfrak{M}_B$. That is W_B is injective and $\mathfrak{D}_1(W_B) = \mathfrak{M}_B$. Let $Q_A = \mathrm{Hom}_B({}_AP_B^*, W_B)_A$. Then Q_A is injective by [8, Lemma 1.3]. By the natural isomorphism (see I)

$$\mathrm{Hom}_A(Y_A, \mathrm{Hom}_B({}_AP_B^*, W_B)_A) \cong \mathrm{Hom}_B(Y \otimes {}_AP_B^*, W_B)$$

and the fact that W_B is a cogenerator, we have

$$Y \otimes {}_AP^* = 0 \quad \text{if and only if} \quad \mathrm{Hom}_A(Y_A, Q_A) = 0\,.$$

LEMMA 3.12. (a) *${}_AX \in \mathscr{C}_1({}_AP^*)$ if and only if $A/T \otimes {}_AX = 0$.*
(b) *$Y_A \in \mathfrak{D}_1(Q_A)$ if and only if* $\mathrm{Hom}_A(A/T, Y) = 0$.

Proof. (a) Since $A/T \otimes {}_AX \cong X/TX$, we have that $A/T \otimes X = 0$ if and only if X is T-accessible.
(b) Let $K = \bigcap \{\ker f \mid f \in \mathrm{Hom}_A(Y, Q)\}$. Since Q_A is injective $\mathrm{Hom}_A(K, Q) = 0$, which implies that $K \otimes {}_AP^* = 0$. Thus $KT = 0$. That is, $K \subseteqq l_Y(T)$. On the other hand, $l_Y(T) \otimes_A P^* = 0$, so that $\mathrm{Hom}_A(l_Y(T), Q) = 0$. Thus $l_Y(T) \subseteqq K$. Therefore, $K = l_Y(T) \cong \mathrm{Hom}_A(A/T, Y)$.

If $Y \in \mathfrak{D}_1(Q_A)$, there is a monomorphism $\varphi\colon Y \longrightarrow \prod_{i \in I} Q^{(i)}$ where $Q^{(i)} \cong Q$ and I is some index set. Let ρ_i be the ith projection map and let $f_i = \rho_i\varphi$. Then $K \subseteqq \bigcap_{i \in I} \ker f_i\} = 0$.

Conversely, if $K = 0$, let $I = \mathrm{Hom}_A(Y, Q)$ and define $\varphi\colon Y \longrightarrow \prod_{i \in I} Q^{(i)}$ by $\varphi(y) = (f(y))_{f \in I}$. Since $\ker \varphi = K = 0$, φ is monomorphism.

LEMMA 3.13.
(a) *For $X \in \mathscr{C}_1({}_AP^*)$, $P/PJ \otimes {}_AX \cong P/NP \otimes {}_AX$.*
(b) *For $Y \in \mathfrak{D}_1(Q_A)$,* $\mathrm{Hom}_A(P/PJ, Y) \cong \mathrm{Hom}_A(P/NP, Y)$.

Proof. By Corollary 3.11, $NP = PJT$. Thus we have the following exact sequence

$$0 \longrightarrow \frac{PJ}{PJT} \longrightarrow \frac{P}{NP} \longrightarrow \frac{P}{PJ} \longrightarrow 0 .$$

(a) For $X \in \mathscr{C}_1({}_AP^*)$

$$PJ/PJT \otimes {}_AX \longrightarrow P/NP \otimes {}_AX \longrightarrow P/PJ \otimes_A X \longrightarrow 0$$

is exact. But $PJ/PJT \otimes {}_AX \cong PJ \otimes A/T \otimes X = 0$ by Lemma 3.12.
(b) For $Y \in \mathfrak{D}_1(Q_A)$

$$0 \longrightarrow \mathrm{Hom}_A (P/PJ, Y) \longrightarrow \mathrm{Hom}_A (P/NP, Y) \longrightarrow \mathrm{Hom}_A (PJ/PJT, Y)$$

is exact. But $\mathrm{Hom}_A (PJ/PJT, Y) \cong \mathrm{Hom}_A (PJ \otimes A/T, Y) \cong \mathrm{Hom}_A(PJ, \mathrm{Hom}_A (A/T, Y)) = 0$ by Lemma 3.12.

A two-sided ideal H of a ring A is said to be left T-nilpotent if given any sequence $\{h_i\}_{i=1}^{\infty} \subseteq H$ there is a finite index n such that $h_1h_2 \cdots h_n = 0$. We need the following necessary and sufficient conditions for a two-sided ideal H to be left T-nilpotent.

THEOREM 3.14. *Let H be a two-sided ideal of A. Then the following statements are equivalent.*

(a) *H is left T-nilpotent.*
(b) $\mathrm{Hom}_A (A/H, Y_A) = 0$ *implies* $Y_A = 0$.
(c) $A/H \otimes {}_AX = 0$ *implies* ${}_AX = 0$.
(d) *For* ${}_AX \neq 0$, $HX' \subseteq Y$.

REMARK. Condition (b) says that if $Y_A \neq 0$, then $l_Y(H) \neq 0$. Condition (c) says that if ${}_AX \neq 0$, then $HX \neq X$. The proof of (d) $\Rightarrow$ (a) is essentially in Bass [3] but will be included here for completeness.

Proof. (a) $\Rightarrow$ (b). Let $0 \neq y \in Y_A$. If $l_Y(H) = 0$, then there exists $a_1 \in H$ such that $ya_1 \neq 0$. Again, there exists $a_2 \in H$ such that $ya_1a_2 \neq 0$. Clearly this will lead to a contradiction of (a) if $l_Y(H) = 0$. Thus $l_Y(H) \neq 0$.
(b) $\Rightarrow$ (c). Suppose that $A/H \otimes {}_AX = 0$. Let $D = \mathrm{End}\,({}_AX)$ and let W_D be a cogenerator in $\mathfrak{M}_D$. Then

$$0 = \mathrm{Hom}_D (A/H \otimes {}_AX_D, W_D) \cong \mathrm{Hom}_A (A/H, \mathrm{Hom}_D ({}_AX_D, W_D)) .$$

Thus by (b) $\mathrm{Hom}_D (X_D, W_D) = 0$. Since W_D is a cogenerator, $X_D = 0$. Hence ${}_AX = 0$.
(c) $\Rightarrow$ (d). Let ${}_AX \neq 0$ and suppose HX is not small in X. Then there exists $K \subsetneqq X$ such that $HX + K = X$. However, for $\bar{X} = X/K$ we have $H\bar{X} = \bar{X}$. Thus by (c) $\bar{X} = 0$, a contradiction. So $HX' \subseteq X$.

(d) $\Rightarrow$ (a). Let $\{h_i\}_{i=1}^{\infty} \subseteqq H$ and let ${}_AN$ be a free left A-module with basis $\{n_i\}_{i=1}^{\infty}$. Now $N = N_1 + N_2$ where N_1 is the submodule of N generated by $\{h_i n_{i+1}\}_{i=1}^{\infty}$ and N_2 is the submodule of N generated by $\{n_i - h_i n_{i+1}\}_{i=1}^{\infty}$. Since $N_1 \subseteqq HN$ we have that $N = N_2$. So

$$\begin{aligned} n_1 &= \sum_{i=1}^{t} a_i(n_i - h_i n_{i+1}) \\ &= a_1 n_1 + \sum_{i=2}^{t} (a_i - a_{i-1}h_{i-1})n_i - a_t h_t n_{t+1} . \end{aligned}$$

Thus, by uniqueness of representation, $a_1 = 1$ and $0 = a_t h_t = a_{t-1}h_{t-1}h_t = \cdots = h_1 h_2 \cdots h_t$. Hence H is left T-nilpotent.

For P_A finitely generated projective we now give necessary and sufficient conditions for the radical N of the endomorphism ring B of P_A to be left T-nilpotent in terms of the subcategories $\mathscr{C}_1({}_AP^*)$ and $\mathfrak{D}_1(Q_A)$.

THEOREM 3.15. *Let P_A be finitely generated projective with trace ideal T and $B = \mathrm{End}(P_A)$. Let N be the radical of B. Then the following statements are equivalent.*

(a) *N is left T-nilpotent.*

(b) *For $Y \in \mathfrak{D}_1(Q_A)$, $\mathrm{Hom}_A(A/J, Y_A) = 0$ implies $Y = 0$.*

(c) *For $X \in \mathscr{C}_1({}_AP^*)$, $A/J \otimes {}_AX = 0$ implies $X = 0$.*

(d) *JT is left T-nilpotent.*

Proof. (*a*) $\Rightarrow$ (b). Let $Y \in \mathfrak{D}_1(Q_A)$. If $\mathrm{Hom}_A(A/J, Y) = 0$, then by Lemma 3.13 $\mathrm{Hom}_A(P, \mathrm{Hom}_A(A/J, Y)) \cong \mathrm{Hom}_A(P \otimes {}_AA/J, Y) \cong \mathrm{Hom}_A(P/PJ, Y) \cong \mathrm{Hom}_A(P/NP, Y) \cong \mathrm{Hom}_A(B/N \otimes_B P, Y) \cong \mathrm{Hom}_B(B/N, \mathrm{Hom}_A(P, Y)) = 0$. Thus $\mathrm{Hom}_A(P, Y) = 0$ by Theorem 3.14. Hence $\mathrm{Hom}_A(Y, Q) = 0$ which implies that $Y = 0$ since $Y \in \mathfrak{D}_1(Q_A)$.

(b) $\Rightarrow$ (c). Let $X \in \mathscr{C}_1({}_AP^*)$ and suppose $A/J \otimes {}_AX = 0$. Let $D = \mathrm{End}({}_AX)$ and let W_D be a cogenerator in $\mathfrak{M}_D$. We have $\mathrm{Hom}_D(A/J \otimes {}_AX, W) \cong \mathrm{Hom}_A(A/J, \mathrm{Hom}_D(X, W)) = 0$. By Lemma 3.12 $\mathrm{Hom}_D(X, W)_A \in \mathfrak{D}_1(Q_A)$ since $\mathrm{Hom}_A(A/T, \mathrm{Hom}_D(X, W)) \cong \mathrm{Hom}_D(A/T \otimes {}_AX, W) = 0$. Thus $\mathrm{Hom}_D(X, W) = 0$ by (b). So $X_D = 0$ since W_D is a cogenerator. Thus ${}_AX = 0$.

(c) $\Rightarrow$ (d). For ${}_AK$, suppose that $A/JT \otimes {}_AK = 0$; that is, $JTK = K$. Since $JTK \subseteqq TK \subseteqq K$, we see that $JTK = TK$, i.e., $A/J \otimes TK = 0$. Thus $K = TK = 0$ by (c). So JT is left T-nilpotent by Theorem 3.14.

(d) $\Rightarrow$ (c). Let $X \in \mathscr{C}_1({}_AP^*)$ and suppose that $A/J \otimes {}_AX = 0$, i.e., $JX = X$. Since $TX = X$, $JTX = X$. Hence $X = 0$ by (d).

(c) $\Rightarrow$ (a). For ${}_BU$, if $B/N \otimes {}_BU = 0$, then $P^* \otimes_B B/N \otimes {}_BU \cong P^*/P^*N \otimes {}_BU = 0$. By Corollary 3.11

$$0 \longrightarrow \frac{JP^*}{TJP^*} \longrightarrow \frac{P^*}{P^*N} \longrightarrow \frac{P^*}{JP^*} \longrightarrow 0$$

is exact. Hence $P^*/JP^* \otimes_B U \cong A/J \otimes_A P^* \otimes_B U = 0$. Thus $P^* \otimes_B U = 0$ since $P^* \otimes_B U \in \mathscr{C}_1({}_AP^*)$. But then ${}_BU \cong P \otimes_A P^* \otimes_B U = 0$.

If the radical J of A is left T-nilpotent, then (b) of Theorem 3.15 holds. Hence if J is left T-nilpotent we have that N is left T-nilpotent. This gives a functorial proof of a well-known result.

COROLLARY 3.16. *For P_A finitely generated projective the following statements are equivalent.*

(a) *N is left T-nilpotent.*

(b) *For $X \in \mathscr{C}_1({}_AP^*)$, $JX' \subseteqq X$.*

(c) *$J({}_AP^{*I})' \subseteqq {}_AP^{*I}$ for any index set I.*

REMARK. R. Ware [11, Lemma 5.3] has shown the equivalence of (a) and (c) in the case that P_A is a projective module which is a finite direct sum of cyclic modules.

Proof. (a) $\Rightarrow$ (b). Let ${}_AX$ be T-accessible. If JX is not small in X, there exists $H \subsetneqq X$ such that $JX + H = X$. However $\bar{X} = X/H$ is also T-accessible, and $J\bar{X} = \bar{X}$. Thus by Theorem 3.15, $\bar{X} = 0$, a contradiction.

(b) $\Rightarrow$ (c). This is trivial since ${}_AP^{*I} \in \mathscr{C}_1({}_AP^*)$ for every index set I.

(c) $\Rightarrow$ (b). Let $\alpha: {}_AU \to {}_AX$ be an A-epimorphism where $JU' \subseteqq U$. Then $X \cong U/K$ where $K = \ker \alpha$. Since $J[U/K] = (JU+K)/K$, if $(JU+K)/K + H/K = U/K$, then $JU + K + H = U$. Thus $K + H = U$, which implies that $H = U$. Hence $JX' \subseteqq X$. (b) now follows from Lemma 3.1.

(b) $\Rightarrow$ (a). Let $X \in \mathscr{C}_1({}_AP^*)$ and suppose $JX = X$. Then $X = 0$ since $JX' \subseteqq X$. Hence N is left T-nilpotent by Theorem 3.15.

4. Perfect injectors and perfect projectors.

For P_A finitely generated projective Anderson [1] has called P_A a perfect injector (perfect projector) if the functor $F = {}_BP \otimes_A (\ )$ preserves injective hulls (projective covers). Clearly a perfect injector (perfect projector) is also an injector (projector). Perfect injectors are characterized in terms of their trace ideal and certain conditions on large submodules.

LEMMA 4.1. *Let I be a right ideal of A. Then A/I is flat in $\mathfrak{M}_A$ if and only if $x \in I$ implies $x \in Ix$.*

Proof. Let $x \in I$ and consider the exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0\,.$$

By [5, Proposition 2.2], assuming $(A/I)_A$ is flat, there exists a map $\theta: A_A \to I_A$ such that $\theta(x) = x$. Thus $x = \theta(x) = \theta(1x) = \theta(1)x \in Ix$.

Conversely if $x \in Ix$, then $x = ix$ for some $i \in I$. Define $\theta_x: A \to I$

by $\theta_x(a) = ia$ for $a \in A$. Then $\theta_x(x) = ix = x$. Again by [5, Proposition 2.2] we have that $(A/I)_A$ is flat.

THEOREM 4.2. *For P_A finitely generated projective the following statements are equivalent.*

(a) *P_A is a perfect injector.*

(b) *$F = {}_BP \otimes {}_A(\ )$ preserves essential monomorphisms.*

(c) *A/T is flat in $\mathfrak{M}_A$.*

(d) *The functor ${}_AT \otimes {}_A(\ )$ preserves essential monomorphisms.*

(e) *${}_AX \subseteq' {}_AE$ implies $TX \subseteq' TE$.*

REMARK. The equivalence of (a), (b), and (c) is essentially (using Lemma 4.1) Anderson's result [1, Theorem 2.4].

Proof. (c) $\Leftrightarrow$ (e). Let ${}_AX \subseteq' {}_AE$ and suppose $(A/T)_A$ is flat. Tensoring with $(A/T)_A$ we get the exact sequence

$$0 \longrightarrow X/TX \longrightarrow E/TE\ .$$

Therefore, $X \cap TE = TX$. Let $0 \neq K \subseteq TE \subseteq E$. Then $K \cap TX = K \cap TE \cap X = K \cap X \neq 0$ since $X \subseteq' E$. Thus $TX \subseteq' TE$.

Conversely, by Lemma 4.1, to show $(A/T)_A$ is flat, it is sufficient to show $x \in T$ implies $x \in Tx$. If $x \notin Tx$, by Zorn's Lemma choose ${}_AI \subseteq {}_AA$ maximal with respect to the property that $Tx \subseteq I$, but $x \notin I$. Then $(Ax + I)/I \subseteq' A/I$, hence by assumption $T((Ax + I)/I) \subseteq' T(A/I)$. Since $T((Ax + I)/I) = 0$, we must have $T(A/I) = (T + I)/I = 0$. That is, $T \subseteq I$. This is a contradiction since $x \in T$.

(c) $\Rightarrow$ (d). Let $\alpha\colon {}_AX \to {}_AE$ be an essential monomorphism. Clearly we may assume α is the inclusion map. Since $(A/T)_A$ is flat, so is T_A, and thus $T \otimes {}_AX \xrightarrow{1 \otimes \alpha} T \otimes {}_AE$ is one-to-one. Again since $(A/T)_A$ is flat, $T \otimes X \cong TX$ and $T \otimes E \cong TE$. Hence (d) follows by (e).

(d) $\Rightarrow$ (c). The proof of (d) $\Rightarrow$ (c) is almost identical (with the obvious changes) to the proof of (e) $\Rightarrow$ (c), and hence will be omitted.

The following corollary indicates when T is a direct summand of A.

COROLLARY 4.3. *For P_A finitely generated projective the following statements are equivalent.*

(a) *P_A is a perfect injector and T_A is finitely generated.*

(b) *$T_A = eA$ for some idempotent $e \in A$.*

REMARK. Corollary 4.3 generalizes a result of Anderson's [1, Corollary 2.7] that over a right perfect ring A, P_A is a perfect injector if and only if T_A is a direct summand of A_A.

Proof. (a) $\Rightarrow$ (b). By Theorem 4.2 we have that $(A/T)_A$ is flat. Since T_A is finitely generated, $(A/T)_A$ is projective by [5, Corollary to Proposition 2.2]. Therefore T_A is a direct summand of A_A.
(b) $\Rightarrow$ (a). Clearly $(A/T)_A$ is flat and T_A is finitely generated. Thus (a) follows by Theorem 4.2.

We now give an example of a perfect injector whose trace is not finitely generated.

EXAMPLE 4.4. Let $A = \prod_{i \in I} K_i$ where $K_i = K$ a field and the index set I is infinite. Let $T = K_i^I$. We may write $T = (e_iA)^I$ where $e_i^2 = e_i \in A$ and $K_i \cong e_iA$. Let $P_A = e_1A$. Then P_A is finitely generated projective and the trace ideal of P_A is T. $(A/T)_A$ is flat since A is a regular ring, hence P_A is a perfect injector. However, T_A is not finitely generated.

Ideally we would like to give a characterization of perfect projectors dual to that of Theorem 4.2. In general we do not know enough pertinent information about small submodules to obtain an exact dualization. Anderson circumvented this problem by assuming that the ring A is left perfect. He gives the following theorem.

THEOREM 4.5. (*Anderson* [1, *Theorem* 3.3]). *Let A be left perfect and let P_A be finitely generated projective. Then the following statements are equivalent.*

(a) *P_A is a perfect projector.*
(b) *$F = {}_BP \otimes {}_A(\ )$ preserves minimal epimorphisms.*
(c) *$JT = J \cap T$.*
(d) *${}_A(A/T)$ is projective in ${}_A\mathfrak{M}$.*

Our purpose in this discussion is to extend Anderson's result by replacing the condition that A be left perfect with substantially weaker conditions on P_A.

LEMMA 4.6. *For ${}_AK \subseteqq {}_AX$, the following statements are equivalent.*
(a) ${}_BP \otimes {}_AK' \subseteqq {}_BP \otimes {}_AX$.
(b) $TK' \subseteqq TX$.

Proof. (a) $\Rightarrow$ (b). Suppose that $TK + H = TX$. Since F is an exact additive functor, we may write $F(TK + H) = F(TK) + F(H) = F(K) + F(H) = F(TX) = F(X)$. Thus $F(H) = F(X)$ which implies that $TH = TX$. Hence $H = TX$ since $TH \subseteqq H \subseteqq TX$.
(b) $\Rightarrow$ (a). Suppose that $F(K) + L = F(X)$. Then $P^*F(K) + P^*L = TK + P^*L = P^*F(X) = TX$. Thus $P^*L = TX$ which implies that $L = F(P^*L) = F(TX) = F(X)$.

COROLLARY 4.7. *The functor* $F = {}_BP \otimes {}_A(\ )$ *always preserves projective covers of T-accessible left A-modules.*

Proof. Let ${}_AY$ be T-accessible and let $\alpha\colon {}_AX \to {}_AY$ be a projective cover of ${}_AY$. We have the following commutative diagram

$$\begin{tikzcd} & & & {}_AP^{*I} \arrow[dl, dashed, "f"'] \arrow[d, "g"] & \\ 0 \arrow[r] & {}_AK \arrow[r] & {}_AX \arrow[r] & {}_AY \arrow[r] & 0 \text{ (exact)} \end{tikzcd}$$

where $K = \ker \alpha$, g follows by Lemma 3.1, and f follows by the projectivity of ${}_AP^{*I}$. Since $K' \subseteqq X$, f is onto. Thus ${}_AX$ is T-accessible, so $TK' \subseteqq X = TX$. Therefore, $F(\alpha)\colon F(X) \to F(Y)$ is a minimal epimorphism by Lemma 4.6.

Since ${}_AX$ is projective

$${}_AP^{*I} \longrightarrow {}_AX \longrightarrow 0$$

is split exact. Thus

$$F(P^{*I}) \longrightarrow F(X) \longrightarrow 0$$

is also split exact. Hence $F(X)$ is projective since it is a direct summand of $F(P^{*I}) \cong {}_BB^I$.

The following result follows easily from Theorem 2.8 and Lemma 4.6, hence the proof will be omitted.

THEOREM 4.8. *For* P_A *finitely generated projective the following statements are equivalent.*

(a) P_A *is a perfect projector.*

(b) ${}_BP$ *is projective and for* ${}_AX$ *projective,* ${}_AK' \subseteqq {}_AX$ *implies* $TK' \subseteqq TX$.

THEOREM 4.9. *Let* P_A *be finitely generated projective with trace ideal* T *and* $B = \mathrm{End}\,(P_A)$. *Let the radical* N *of* B *(equivalently* JT*) be left T-nilpotent. Then the following statements are equivalent.*

(a) P_A *is a perfect projector.*

(b) ${}_BP$ *is projective and* $JT = J \cap T$.

(c) ${}_BP$ *is projective and* $TJ \subseteqq JT$.

(d) ${}_BP$ *is projective and* $PJ = NP$.

REMARK. The proof that $JT = J \cap T$ in (a) $\Rightarrow$ (b) is due to Anderson [1].

Proof. (a) $\Rightarrow$ (b). By Theorem 2.8, ${}_BP$ is projective and $\eta_A\colon {}_AP^* \otimes {}_BP \to {}_AT$ is a projective cover. Hence $J(T) = JT$ by Lemma 1.1.

Clearly $JT \subseteqq J \cap T$. Let $x \in J \cap T$ and suppose that $x \notin JT = J(T)$. Then there is a maximal submodule H of T such that $x \notin H$. Since $x \in J$, the natural map $A \longrightarrow A/Ax$ is a projective cover. By (a) $P \to P/Px$ is a projective cover, hence ${}_BPx' \subseteqq {}_BP$. But $P = PT = PH + Px$, since $T = H + Ax$. Thus $P = PH$, and so $P^* \otimes P = P^* \otimes PH$. Hence $T = TH \subseteqq H$, a contradiction. Thus $JT = J \cap T$.
(b) $\Rightarrow$ (c). Clearly $TJ \subseteqq J \cap T$. By assumption $J \cap T = JT$. Thus $TJ \subseteqq JT$.
(c) $\Rightarrow$ (d). Clearly $PJT \subseteqq PJ$. Since $TJ \subseteqq JT$ we have that $PJ = PTJ \subseteqq PJT$. Thus $PJ = PJT$. Hence $PJ = NP$ by Corollary 3.11.
(d) $\Rightarrow$ (a). Let $\alpha: {}_AX \to {}_AY$ be a projective cover. ${}_BP \otimes {}_AX$ is projective since P_A is a projector. Let $K = \ker\, \alpha$. Since $K' \subseteqq X$, $K \subseteqq J(X) = JX$. Since P_A and ${}_AX$ are flat, both $P \otimes JX$ and $PJ \otimes X = NP \otimes X$ have the same image in $P \otimes X$ under the maps $P \otimes JX \to P \otimes X$ and $PJ \otimes X \to P \otimes X$ respectively. Thus $P \otimes K \subseteqq NP \otimes X$. Hence $P \otimes K' \subseteqq P \otimes X$ because N is left T-nilpotent. Therefore $P \otimes X \xrightarrow{1 \otimes \alpha} P \otimes Y$ is a projective cover.

To suppose N is left T-nilpotent is one way to weaken Anderson's condition that A be left perfect. Another way is as follows. Instead of assuming that every left A-module has a projective cover, we assume that a particular left A-module has a projective cover. But first we need the following lemma.

LEMMA 4.10. *${}_A(A/T)$ is projective in ${}_A\mathfrak{M}$ if and only if $A = T + r_A(T)$.*

Proof. If ${}_A(A/T)$ is projective then ${}_AT$ is a direct summand of ${}_AA$. Hence ${}_AA = {}_AT + {}_AU$. Clearly ${}_AU \subseteqq r_A(T)$. Thus $A = T + r_A(T)$.

Conversely, if $A = T + r_A(T)$, then $1 = t_0 + u$ where $t_0 \in T$, $u \in r_A(T)$. If $t \in T$, then $t = t1 = tt_0 + tu = tt_0$. That is, $T = At_0$; hence ${}_AT$ is finitely generated. Furthermore, for $t \in T$ we see that $t = tt_0 \in tT$. Hence by Lemma 4.1, ${}_A(A/T)$ is flat. Thus ${}_A(A/T)$ is projective [5, Corollary to Proposition 2.2].

THEOREM 4.11. *Let P_A be finitely generated projective with trace ideal T and $B = \mathrm{End}\,(P_A)$. Then the following statements are equivalent.*

(a) *P_A is a perfect projector and ${}_A(A/T)$ has a projctive cover.*

(b) *$F = {}_BP \otimes {}_A(\ )$ preserves minimal epimorphisms and ${}_A(A/T)$ has a projective cover.*

(c) *${}_A(A/T)$ is projective in ${}_A\mathfrak{M}$.*

Proof. (a) $\Rightarrow$ (c). Let $\alpha: {}_AX \to {}_A(A/T)$ be a projective cover of ${}_A(A/T)$ and let $K = \ker \alpha$. Then $F(\alpha): F(X) \to F(A/T)$ is a projective cover. However, $F(A/T) = 0$ which implies that $F(X) = 0$ since $\ker F(\alpha)' \subsetneqq F(X)$. Hence $F(K) = 0$, so $TK = 0$.

By the projectivity of ${}_AA$ we have the following commutative diagram.

$$\begin{tikzcd} & {}_AA \arrow[dl, dashed, "\beta"'] \arrow[d, "\rho"] & \\ {}_AX \arrow[r, "\alpha"] & A/T \arrow[r] & 0 \text{ (exact)} \end{tikzcd}$$

where ρ is the natural map. β is onto since $K' \subsetneqq X$. Now $K = (T)\beta = T(T)\beta = TK = 0$. Thus α is an isomorphism, hence ${}_A(A/T)$ is projective.

(c) $\Rightarrow$ (a). Let $\alpha\colon {}_AX \to {}_AY$ be a projective cover and let $K = \ker \alpha$. Since ${}_A(A/T)$ is projective, so is ${}_AT$. Hence P_A is a projector by Theorem 2.9. Thus ${}_BP \otimes {}_AX$ is projective.

By Lemma 4.10, $A = T + r_A(T)$. Hence $X = AX = TX + r_A(T)X$. Suppose that $TK + H = TX$. Then $TK + H + r_A(T)X = TX + r_A(T)X = X$. Thus $H + r_A(T)X = X$ since $TK \subsetneqq K' \subsetneqq X$. Now $TH + Tr_A(T)X = TX$, and so $TH = TX$. Therefore, $H = TX$, so that $TK' \subsetneqq TX$. Hence $F(\alpha)\colon F(X) \to F(Y)$ is a projective cover by Lemma 4.6.

(b) $\Leftrightarrow$ (c). This is essentially the proof of (a) $\Leftrightarrow$ (c).

COROLLARY 4.12. *Let A be semiperfect. Then the following statements are eqivalent.*

(a) *P_A is a perfect projector.*

(b) *${}_AP^*$ is a perfect injector.*

(c) *${}_A(A/T)$ is projective in ${}_A\mathfrak{M}$.*

Proof. (a) $\Rightarrow$ (b). Since A is semiperfect, ${}_A(A/T)$ has a projective cover. Thus ${}_A(A/T)$ projective (hence flat) by Theorem 4.11. So ${}_AP^*$ is a perfect injector by Theorem 4.2.

(b) $\Rightarrow$ (c). ${}_A(A/T)$ is flat by Theorem 4.2. Since A is semiperfect, ${}_A(A/T)$ has a projective cover. Thus ${}_A(A/T)$ is projective by Lemma 1.2.

(c) $\Rightarrow$ (a). This follows easily by Theorem 4.11.

EXAMPLE 4.13. We give an example to show that the condition that ${}_A(A/T)$ have a projective cover is not redundant.

Let F be a field, ${}_FV$ an infinite dimensional vector space over F, and A the ring of all linear transformations of ${}_FV$. Since ${}_FV$ is a generator, V_A is finitely generated projective and $F = \operatorname{End}(V_A)$ (see [2]). Let $e\colon {}_FV \to Fv$ be the projection map onto the one-dimensional subspace Fv of ${}_FV$. Clearly $V_A \cong eA$; hence, the trace ideal of V_A is $T = AeA$. *Properties of A, ${}_FV_A$, and T.*

(a) $T = \text{Socle}(A_A)$ (see [6]).

(b) ${}_A(A/T)$ is not projective since ${}_AT$ is not a direct summand of ${}_AA$, as ${}_AT$ is not finitely generated.

(c) For ${}_AX$, $\text{Hom}_A(X, A) = 0$ if and only if $V \otimes_A X = 0$.

(d) ${}_AA$ is self-injective (e.g. see [9]).

(e) ${}_FV \otimes_A(\ )$ preserves minimal epimorphisms.

(f) V_A is a projector, hence a perfect projector by (e).

Proof. (c) follows by the isomorphism

$$\begin{aligned}\text{Hom}_A(X, A) &= \text{Hom}_A(X, \text{Hom}_F(V, V)) \\ &\cong \text{Hom}_F(V \otimes_A X, V) .\end{aligned}$$

(e). Let $\alpha: {}_AX \to {}_AY$ be a minimal epimorphism with $K = \ker \alpha$. Then

$$0 \longrightarrow V \otimes_A K \longrightarrow V \otimes_A X \longrightarrow V \otimes_A Y \longrightarrow 0$$

is an exact sequence. But $V \otimes_A K = 0$ since $\text{Hom}_A(X, A) = 0$. That is, given $f: K \to A$, then f extends to $\hat{f}: X \to A$ by the self-injectivity of ${}_AA$. Now $\hat{f}(K)' \subseteq A$ which implies $\hat{f}(K) = 0$, as $J(A) = 0$. Hence $f = 0$.
(f). By Theorem 2.8, V_A is a projector since ${}_FV$ is projective.

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