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A NOTE ON AN INEQUALITY FOR REARRANGEMENTS

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If $a = (a_1, \cdots, a_n)$ is a n-tuple of positive real numbers let $a^+ = (a_1^+, \cdots, a_n^+)$ denote the rearrangement of a in increasing order and $a^- = (a_1^-, \cdots, a_n^-)$ the rearrangement of a in decreasing order. In this note a characterization is given for those functions $f: R_+ \to R$ such that

$$\sum_1^n f\left(\frac{a_k^+}{b_k^+}\right) \leqq \sum_1^n f\left(\frac{a_k}{b_k}\right) \leqq \sum_1^n f\left(\frac{a_k^+}{b_k^-}\right) \tag{1}$$

holds for all $a, b \in R_+^n$. Here n is a fixed integer > 1.

In [3] Minc proved that

$$\prod_1^n (a_k^+ + b_k^+) \leqq \prod_1^n (a_k + b_k) \leqq \prod_1^n (a_k^+ + b_k^-) .$$

This inequality was generalized by London [1, Theorem 1], who (in an equivalent form) proved that (1) holds if the function $f(e^x - 1)$, $x \geqq 0$, is convex and $f(x) \geqq f(0)$, $x > 0$.

London [1, Theorem 2] also proved (1) if f is convex and $f(x) \geqq f(0)$, $x > 0$, which in fact is contained in the previous case.

The proofs in [1] are based on an interesting representation theorem of Mirsky.

The purpose of this note is to characterize those functions f, for which (1) holds.

The left inequality in (1) is in fact a special case of a theorem of Lorentz [2, Theorem 1]. This theorem especially gives, that if $\Phi = \Phi(u, v)$ belongs to $C^{(2)}(\boldsymbol{R}_+ \times \boldsymbol{R}_+)$ then

$$\sum_1^n \Phi(a_k^+, b_k^+) \leqq \sum_1^n \Phi(a_k, b_k) \quad \text{all} \quad a, b \in \boldsymbol{R}_+^n \qquad (n > 1) \tag{2}$$

if and only if

$$\frac{\partial^2 \Phi}{\partial u \partial v} \leqq 0 . \tag{3}$$

From this it is fairly easy to deduce that (2) and (3) are both equivalent to

$$\sum_1^n \Phi(a_k, b_k) \leqq \sum_1^n \Phi(a_k^+, b_k^-) \quad \text{all} \quad a, b \in \boldsymbol{R}_+^n \qquad (n > 1) . \tag{4}$$

We shall give an independent proof of these equivalences, which differs from that given in [2].

Let $a, b \in \boldsymbol{R}_+^n$ and choose $\varepsilon \in \boldsymbol{R}_+$ such that $\varepsilon < \min(\min a_k, \min b_k)$. If A is a finite set, let $|A|$ denote the number of distinct elements in A.

By adding the identities

$$\Phi(a_k, b_k) = \Phi(\varepsilon, \varepsilon) - \Phi(a_k, \varepsilon) - \Phi(\varepsilon, b_k) + \int_\varepsilon^{a_k}\int_\varepsilon^{b_k} \frac{\partial^2 \Phi}{\partial u \partial v} du dv ,$$
$$k = 1, \cdots, n ,$$

we have

$$\begin{aligned} \sum_1^n \Phi(a_k, b_k) &= n\Phi(\varepsilon, \varepsilon) - \sum_1^n \Phi(a_k, \varepsilon) - \sum_1^n \Phi(\varepsilon, b_k) \\ &\quad + \int_\varepsilon^\infty \int_\varepsilon^\infty |\{k \mid a_k \geqq u,\ b_k \geqq v\}| \frac{\partial^2 \Phi}{\partial u \partial v} du dv . \end{aligned} \tag{5}$$

Set

$$N(a, b; u, v) = |\{k \mid a_k \geqq u,\ b_k \geqq v\}| .$$

Then

$$N(a^+, b^-; u, v) \leqq N(a, b; u, v) \leqq N(a^+, b^+; u, v)$$

and so (2) and (4) follow from (3) and (5).

To prove, (4) implies (3), set

$$D(u, v) = N(a, b; u, v) - N(a^+, b^-; u, v) .$$

In view of (5), (4) is then equivalent to

$$\int_0^\infty \int_0^\infty D(u, v) \frac{\partial^2 \Phi}{\partial u \partial v} du dv \leqq 0 . \tag{6}$$

If (4) holds for some $n > 1$, then it also holds for $n = 2$. To see this we only need to apply (4) to the n-tuples $a = (a_1, a_2, \varepsilon, \cdots, \varepsilon)$ and $b = (b_1, b_2, \delta, \cdots, \delta)$, where $\varepsilon > \max(a_1, a_2)$ and $0 < \delta < \min(b_1, b_2)$.

Choose $0 < u_1 < u_2$, $0 < v_1 < v_2$ and set $a_1 = u_1$, $a_2 = u_2$, $b_1 = v_1$ and $b_2 = v_2$. Then $D(u, v) = 1$ if $u_1 < u \leqq u_2$, $v_1 < v \leqq v_2$ and $= 0$ elsewhere. Therefore (3) follows from (6).

Analogously we can prove that (2) implies (3).

In [2] there are also necessary and sufficient conditions on Φ in order for (2) ((4)) to hold, when Φ is only continuous. It is easy to see that (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) in this case too, if (3) is interpreted in the distribution sense, that is the left-hand side is a negative measure. A formal proof goes via regularization of Φ.

We now return to the inequality (1). This corresponds to

$$\Phi(u, v) = f\left(\frac{u}{v}\right),$$

which gives

$$\frac{\partial^2 \Phi}{\partial u \partial v} = -\frac{1}{v^2}\left\{f''\left(\frac{u}{v}\right) \cdot \frac{u}{v} + f'\left(\frac{u}{v}\right)\right\} \leqq 0,$$

that is

$$f''(x) \cdot x + f'(x) \geqq 0, \qquad x > 0.$$

This means that (1) holds if and only if the function $f(e^x)$, $-\infty < x < +\infty$, is convex.

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