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ENERGY BOUNDS AND VIRIAL THEOREMS FOR ABSTRACT WAVE EQUATIONS

L. E. BOBISUD AND JAMES CALVERT

The abstract wave equation $u'' = A^2u + f(t, u)$ is considered on a Banach or Hilbert space, where A generates a (C_0) group. Under suitable conditions on f, a representation of the solution of the initial-value problem is used to establish bounds on the growth of the energy $1/2\|Au(t)\|^2 + 1/2\|u'(t)\|^2$. For $f \equiv 0$ it is shown that neither the potential energy $1/2\|Au(t)\|^2$ nor the kinetic energy $1/2\|u'(t)\|^2$ tends to zero as $t \to \infty$, and necessary and sufficient conditions for the kinetic and potential energies to be equal for large time are given.

Hille [8] has shown that the Cauchy problem for the abstract wave equation $u'' = A^2u$ on a Banach space $\mathfrak{X}$ is well posed if and only if A generates a strongly continuous group; hence we shall suppose throughout that A is the generator of such a group. Hersh [7] has given a representation theorem for solutions of abstract initial-value problems in terms of distributional solutions of the Cauchy problem for certain related partial differential equations in two variables—in our case, in terms of solutions of $U_{tt} = U_{xx}$. Here we shall exploit his simple, explicit formula for the solution of the Cauchy problem for $u' = A^2u + f$ to establish rather easily some results concerning the energy of the solution.

In the first part we establish estimates on the growth of the energy of a solution to the nonhomogeneous problem. The results here complement and extend some results in [5] and [6].

In part two we consider only the homogeneous linear problem, and show first that neither the kinetic nor the potential energy can tend to zero as time increases. We then specialize to Hilbert space, where we establish a necessary and sufficient condition for the kinetic and potential energies to be equal and constant for large t and data in some subspace; a similar result may be found in [4]. By utilizing a consequence of an abstract formulation of Huyghens' principle, we are able to establish very easily a theorem of Duffin [2] on the equality of kinetic and potential energies of solutions of the three-dimensional wave equation with compactly supported data.

I. **Energy bounds for abstract wave equations.** We shall consider first the energy $1/2\|Au(t)\|^2 + 1/2\|u'(t)\|^2$ for the abstract wave equation

$$(1) \qquad u'' = A^2u + f(t) \,, \quad u(0) = u_0 \in D(A^2) \,, \quad u'(0) = u_1 \in D(A)$$

on a Banach space $\mathfrak{X}$. We assume throughout that $f(t)$ is strongly continuous from $[0, \infty)$ to $D(A)$ and that $Af(t)$ is strongly continuous; instead, we cound assume that f is strongly continuously differentiable [5, II. 2]. The first part of Theorems 1 and 2 is proved differently in [5] for A invertible and $f \equiv 0$.

THEOREM 1. *Let A generate the (C_0) group $T(t)$ satisfying $\| T(t) \| \leqq M$, and let $\int_0^\infty \| f(t) \| dt < \infty$. Then u has bounded energy; specifically,*

$$\frac{1}{\sqrt{6}\,M}\left\{\| Au_0 \|^2 + \| u_1 \|^2 - \left[\sqrt{6}\,M\int_0^\infty \| f(t) \| dt\right]^2\right\}^{1/2}$$
$$\leqq \{\| Au(t) \|^2 + \| u'(t) \|^2\}^{1/2}$$
$$\leqq \sqrt{6}\,M\left\{\| Au_0 \|^2 + \| u_1 \|^2 + \left[\int_0^\infty \| f(t) \| dt\right]^2\right\}^{1/2} .$$

Conversely, let A generate a (C_0) group $T(t)$, and suppose that for zero initial data the energy is bounded above by a nondecreasing function of $\int_0^\infty \| f(t) \| dt$. Then $T(t)$ is uniformly bounded.

Proof. By substitution one checks that the solution of (1) is given by

$$u(t) = \frac{1}{2}[T(t) + T(-t)]u_0 + \frac{1}{2}\int_{-t}^{t} T(s)u_1 ds + \frac{1}{2}\int_0^t\int_{-(t-\tau)}^{t-\tau} T(s)f(\tau)dsd\tau ; \tag{2}$$

this is the solution obtained in [7]. Since A is closed and $d/dt\,T(t)v = AT(t)v = T(t)Av$ for $v \in D(A)$, we obtain

$$Au(t) = \frac{1}{2}[T(t) + T(-t)]Au_0 + \frac{1}{2}[T(t) - T(-t)]u_1 + \frac{1}{2}\int_0^t [T(t - \tau) - T(-(t - \tau))]f(\tau)d\tau , \tag{3}$$

$$u'(t) = \frac{1}{2}[T(t) - T(-t)]Au_0 + \frac{1}{2}[T(t) + T(-t)]u_1 + \frac{1}{2}\int_0^t [T(t - \tau) + T(-(t - \tau))]f(\tau)d\tau . \tag{4}$$

Taking norms yield the upper bound in the first part of the theorem. For the lower bound, one obtains by adding and subtracting (3) and (4) and appropriate applications of $T(t)$ and $T(-t)$ that

$$Au_0 = \frac{1}{2}[T(t) + T(-t)]Au(t) - \frac{1}{2}[T(t) - T(-t)]u'(t) + \frac{1}{2}\int_0^t [T(\tau) - T(-\tau)]f(\tau)d\tau ,$$

$$u_1 = -\frac{1}{2}[T(t) - T(-t)]Au(t) + \frac{1}{2}[T(t) + T(-t)]u'(t) - \frac{1}{2}\int_0^t [T(\tau) - T(-\tau)]f(\tau)d\tau ,$$

whence the lower bound follows by taking norms.

For the converse, observe from (3) and (4) and the postulated energy bound that

$$(5) \qquad \left\| \int_0^t T(t-\tau)f(\tau)d\tau \right\| \leqq C\left(\int_0^\infty \| f(\tau) \| d\tau\right) ,$$

where C is a nondecreasing function of its argument. Let $\{\delta_n(t)\}$ be a sequence of nonnegative $C_0^\infty(-1, 1)$ functions converging as $n \to \infty$ to the Dirac δ-function; let $g \in D(A)$. Then

$$\int_0^t T(t-\tau)\delta_n(\tau - \bar{t})g d\tau \longrightarrow T(t-\bar{t})g$$

as $n \to \infty$ for $t > \bar{t} + 1$. Also

$$\int_0^\infty \delta_n(\tau - \bar{t}) \| g \| d\tau = \| g \| ,$$

so (5) implies that

$$\| T(s)g \| \leqq C(\| g \|)$$

for $s \geqq 1$ and $g \in D(A)$. Since A generates the (C_0) group $T(t)$, we have that $\| T(t) \| \leqq Me^{\omega|t|}$ for some constants $M > 0, \omega \geqq 0$. Thus $\| T(t)g \| \leqq \max(C(\| g \|), Me^{\omega}) \equiv K(\| g \|)$ for $t \geqq 0$; K is nondecreasing.

Let $h \in \mathfrak{X}$ and $t \in (-\infty, \infty)$ be given. Let

$$\varepsilon = \min(\| h \|, e^{-\omega t}K(2\| h \|)/M) ,$$

and choose $g \in D(A)(\overline{D(A)} = \mathfrak{X})$ such that $\| g - h \| < \varepsilon$. Then

$$\| T(t)h \| \leqq \| T(t)(g-h) \| + \| T(t)g \| \leqq \varepsilon Me^{\omega t} + K(\| g \|) \leqq 2K(2\| h \|) , \qquad t \geqq 0 .$$

A similar argument works for $t \leqq 0$. Thus the family $\{T(t): -\infty < t < \infty\}$ is bounded on each $h \in \mathfrak{X}$, and hence $\sup_{-\infty<t<\infty} \| T(t) \| < \infty$ by the uniform boundedness theorem.

THEOREM 2. *Let A generate a (C_0) group $T(t)$ satisfying $\| T(t) \| \leqq$*

$Me^{\omega|t|}$, $\omega \geqq 0$, and suppose $\|f(t)\| \leqq \bar{M}e^{\gamma t}$ for $\gamma \geqq 0, t > 0$; then

$$(6) \quad \begin{aligned}[\|Au(t)\|^2 + \|u'(t)\|^2]^{1/2} &\leqq C_1 e^{\omega t}\{\|Au_0\| + \|u_1\|\} \\ &\quad + C_2\int_0^t e^{\omega(t-\tau)}\|f(\tau)\|\,d\tau \\ &\leqq C_3(u_0, u_1, f)\begin{cases} e^t \max(\gamma, \omega) , & \omega \neq \gamma \\ (1+t)e^{\omega t} , & \omega = \gamma \end{cases}\end{aligned}$$

for certain constants C_1, C_2 independent of the data.

Conversely, let A generate the (C_0) group $T(t)$, and suppose that for zero initial data the energy inequality (6) is satisfied for all bounded strongly continuous $f \in D(A)$ with Af strongly continuous; then

$$\|T(t)\| \leqq Me^{\omega|t|}$$

for some constant M.

The proof is similar to that of Theorem 1 and will be omitted.

Small perturbations of the differential equation should cause small changes in the rate of growth of the energy. This is the content of the following theorem.

THEOREM 3. *Let A generate the (C_0) group $T(t)$ satisfying $\|T(t)\| \leqq Me^{\omega|t|}(\omega \geqq 0)$, and let $f(t)$ satisfy $\|f(t)\| \leqq \bar{M}e^{\gamma t}(\gamma \geqq 0)$. Let $g: R^+ \times \mathfrak{X} \to \mathfrak{X}$ satisfy*

$$(7) \qquad \|g(t, u)\| \leqq K\{\|Au\| + \|u\|\}$$

for some constant K and $u \in D(A)$, in addition to hypotheses which guarantee the existence of solutions on $[0, \infty)$ to the problem (see [5] for such conditions)

$$(8) \quad u'' = A^2u + g(t, u) + f(t) \quad u(0) = u_0 \in D(A^2), u'(0) = u_1 \in D(A) .$$

Then for any $\varepsilon > 0$ there exists K_0 such that if the constant K in (7) satisfies $0 < K < K_0$, the energy growth estimate

$$(9) \qquad \|u'(t)\| + \|Au(t)\| \leqq \text{const. } e^{t(\max(\gamma,\omega)+\varepsilon)}$$

is valid. Conversely, given $K > 0$, (9) is satisfied by the solution u of (8) for any $\varepsilon > 4MK \max(1, \omega^{-1})$.

REMARK. If B is a closed operator with $D(B) \supset D(A)$, then $\|Bu\| \leqq K\{\|Au\| + \|u\|\}$ for some constant K and $u \in D(A)$, and solutions of $u'' = (A^2 + B)u + f$ exist [5], so our theorem covers this case of linear perturbation. Also included is the case where $g(t, u)$ satisfies a Lipschitz condition in u uniformly in t.

Proof. We shall treat only the case $\omega > 0$, $\omega \neq \gamma$; the other cases are similar. The solution u of (8) satisfies the integral equation

$$\begin{aligned} u(t) &= \frac{1}{2}[T(t) + T(-t)]u_0 + \frac{1}{2}\int_{-t}^{t} T(s)u_1 ds \\ &\quad + \frac{1}{2}\int_0^t \int_{-(t-\tau)}^{t-\tau} T(s)f(\tau)ds d\tau \\ &\quad + \frac{1}{2}\int_0^t \int_{-(t-\tau)}^{t-\tau} T(s)g(\tau, u(\tau))ds d\tau , \end{aligned} \tag{10}$$

whence we get the estimates

$$\begin{aligned} \|u(t)\| \leqq Me^{\omega t}\left[\|u_0\| + \frac{1}{\omega}\|u_1\|\right] &+ \frac{M\bar{M}}{\omega|\omega - \gamma|}e^{t\max(\gamma,\omega)} \\ &+ \frac{MK}{\omega}\int_0^t e^{\omega(t-\tau)}\{\|Au(\tau)\| + \|u(\tau)\|\}d\tau , \end{aligned}$$

$$\begin{aligned} \|Au(t)\|, \|u'(t)\| \leqq Me^{\omega t}[\|Au_0\| + \|u_1\|] &+ \frac{M\bar{M}}{|\omega - \gamma|}e^{t\max(\gamma,\omega)} \\ &+ MK\int_0^t e^{\omega(t-\tau)}\{\|Au(\tau)\| + \|u(\tau)\|\}d\tau . \end{aligned}$$

Set $\sigma = \max(\gamma, \omega) + \varepsilon$ for convenience; then for the three quantities above we have the bounds

$$\begin{aligned} &\|u(t)\|, \|Au(t)\|, \|u'(t)\| \\ &\leqq Ce^{\sigma t} + MK\max(1, \omega^{-1})\int_0^t e^{\omega(t-\tau)}\{\|Au(\tau)\| + \|u(\tau)\|\}d\tau \end{aligned} \tag{11}$$

for a constant C depending on the data and γ, ω. We insist that $K < \varepsilon/4M\max(1, \omega^{-1})$ and define

$$S = C\left[\frac{1}{2} - \frac{2MK}{\varepsilon}\max(1, \omega^{-1})\right]^{-1} .$$

Since $S > 2C$, we see from (11) that

$$\|u(t)\|, \|Au(t)\|, \|u'(t)\| < Se^{\sigma t} \tag{12}$$

holds for positive t near zero. Suppose $\bar{t}$ is the first positive time such that one of the strict inequalities (12) fails. Thus (12) holds on $[0, \bar{t})$, whence (11) yields for $t \in [0, \bar{t})$

$$\|u(t)\|, \|Au(t)\|, \|u'(t)\| \leqq Ce^{\sigma t} + \frac{1}{\varepsilon}2MK\max(1, \omega^{-1})Se^{\sigma t} < \frac{1}{2}Se^{\sigma t};$$

thus $\|u(\bar{t})\|, \|Au(\bar{t})\|, \|u'(\bar{t})\| \leqq (1/2)Se^{\sigma\bar{t}}$ by continuity. But then (12) holds at $\bar{t}$. This contradiction proves that (12) must hold on $[0, \infty)$, establishing the theorem.

The following theorem sharpens a result of [6] in the case $\omega > 0$, $\gamma > 0$; it is, however, weaker if $\gamma = 0$ or $\omega = 0$.

THEOREM 4. *Let A generate a (C_0) group $T(t)$ satisfying $\|T(t)\| \leqq Me^{\omega|t|}$, and let B be a closed linear operator on X such that $D(B) \supset D(A)$; let u be a strong solution of*

$$u'' = (A^2 + B)u + f\ , \quad u(0) = u_0 \in D(A^2)\ , \quad u'(0) = u_1 \in D(A)\ ,$$

where f satisfies $\|f(t)\| \leqq \bar{M}e^{\gamma t}$. Suppose the estimates

$$\|u(t)\|, \|Au(t)\| \leqq \text{const. } e^{(\omega+|\gamma-\omega|)t}$$

are satisfied on $[0, \infty)$. Then for any $\varepsilon > 0$

$$\|u'(t)\| \leqq \text{const. } e^{(\omega+|\gamma-\omega|+\varepsilon)t}$$

holds for $t \geqq 0$.

Outline of proof: Set $C_\alpha = A - \alpha I$ for $\alpha = |\gamma - \omega| + \varepsilon$; then u satisfies

$$u'' = C_\alpha^2 u + [2\alpha A + B - \alpha^2 I)u + f\ . \tag{13}$$

C_α generates the (C_0) group $T_\alpha(t) = e^{-\alpha t}T(t)$, where $\|T_\alpha(t)\| \leqq Me^{\omega|t|}e^{-\alpha t}$. Also B satisfies the estimate $\|Bu\| \leqq K\{\|Au\| + \|u\|\}$ for some constant K and $u \in D(A)$. After converting (13) to the integral equation

$$u(t) = \frac{1}{2}[T(t) + T(-t)]u_0 + \frac{1}{2}\int_{-t}^{t} T(s)u_1 ds$$
$$+ \int_0^t\int_{-(t-\tau)}^{t-\tau} T(s)[f(\tau) + \{2\alpha A + B - \alpha^2 I\}u(\tau)]dsd\tau\ ,$$

standard estimates yield the result, as in the preceding proof.

II. **Virial theorems for $u'' = A^2u$.** Studying the solutions of $u'' = A^2u$ on a Hilbert space where A generates a unitary group, Shinbrot [9] and Goldstein [3] have established conditions under which the potential energy $1/2\|Au(t)\|^2$ and the kinetic energy $1/2\|u'(t)\|^2$ approach a common limit. This virial theorem fails for solutions of the one-dimensional wave equation on $[0, 1]$ with zero boundary data, where the lim inf of both the kinetic and potential energy can be zero. A weaker, related result is true, however; namely, that neither the kinetic nor the potential energy can have limit zero as time increases.

THEOREM 5. *Let A generate a norm-preserving group $T(t)$ on the*

Banach space $\mathfrak{X}$. Let u be the solution of

$$u'' = A^2u\ , \quad u(0) = u_0\ , \quad u'(0) = u_1\ , \tag{14}$$

where $u_0 \in D(A^2)$, $u_1 \in D(A)$, and $||u_0|| + ||u_1|| > 0$. Then $||u'(t)|| \nrightarrow 0$, $||Au(t)|| \nrightarrow 0$ as $t \to \infty$ unless $u_1 = 0$ and $A^2u_0 = 0$.

Proof. Assume to the contrary that $u'(t) \to 0$ for certain initial data u_0, u_1. From (4) we have that

$$T(t)[Au_0 + u_1] - T(-t)[Au_0 - u_1] \longrightarrow 0\ ,$$

whence (3) implies that

$$Au(t) - \begin{Bmatrix} T(t)\quad [Au_0 + u_1] \\ T(-t)[Au_0 - u_1] \end{Bmatrix} \longrightarrow 0\ .$$

From (4) we also get

$$\begin{aligned} 2||u'(t)|| &= 2||T(t)u'(t)|| = ||T(2t)[Au_0 + u_1] - [Au_0 - u_1]|| \longrightarrow 0\ , \\ 2||u'(t)|| &= 2||T(-t)u'(t)|| = ||T(-2t)[Au_0 - u_1] - [Au_0 + u_1]|| \longrightarrow 0; \end{aligned}$$

from this it follows that $T(t)[Au_0 + u_1] \longrightarrow [Au_0 - u_1]$, $T(-t)[Au_0 - u_1] \to [Au_0 + u_1]$. Thus $Au(t)$ converges to both $Au_0 + u_1$ and $Au_0 - u_1$, which is impossible unless $u_1 = 0$.

Suppose then that $u_1 = 0$. The argument above shows that $T(t)Au_0 \to Au_0$, so we have that

$$\begin{aligned} ||T(h)Au_0 - Au_0|| &= ||T(t+h)Au_0 - T(t)Au_0|| \\ &\leqq ||T(t+h)Au_0 - Au_0|| + ||Au_0 - T(t)Au_0|| \longrightarrow 0 \end{aligned}$$

as $t \longrightarrow \infty$. Thus $T(h)Au_0 = Au_0$ for all h, and so

$$A^2u_0 = \lim_{h\to 0} \frac{T(h)Au_0 - Au_0}{h} = 0\ .$$

This proves the theorem, for a similar argument holds if $Au(t) \to 0$.

In the case A a skew-adjoint operator on a Hilbert space, $A^2u_0 = 0$ implies that $Au_0 = 0$, and hence the total energy vanishes.

Huyghens' principle for the hyperbolic equation

$$u_{tt} = L[u]\ ,$$

where L is an elliptic operator in n space variables, asserts that for compactly supported data the solution will be zero at any fixed point in space for all sufficiently large time [2]. This principle is known to be valid for $L = \Delta_n$, the n-dimensional Laplacian, for $n = 2m + 3$, $m = 0, 1, \cdots$. We shall give an abstract Hilbert space formulation of Huyghens' principle in order to derive information about the group generated by a square root of the Laplacian.

Let $\mathscr{H}$ be a Hilbert space, and let $\{H_j\}_{j=0}^{\infty}$ be a sequence of linear submanifolds of $\mathscr{H}$ such that $H_0 \subset H_1 \subset H_2 \cdots$. We shall say that Huyghens' principle is valid for the abstract wave equation (14) relative to $\{H_j\}$ provided:

1. $D(A) \cap H_j$ is dense in H_j, $j = 0, 1, \cdots$, and $A(D(A) \cap H_j) \subset H_j$
2. for each $j = 0, 1, \cdots$ there exists a T_j such that $|t| > T_j$ implies $(u(t), h_j) = 0$ for $h_j \in H_j$ if $u_0 \in D(A^2) \cap H_0$, $u_1 \in D(A) \cap H_0$.

EXAMPLE. Let $\mathscr{H} = \mathscr{L}^2(R^3)$, $A^2 = \Delta_3$, $H_j = \mathscr{L}^2(B_j)$, where B_j is the ball of radius $C(j + 1)$ for some constant $C > 0$. Define the Fourier transform by

$$F\{f\}(\xi) = \int e^{-i(x,\xi)} f(x)dx ,$$

where $x = (x_1, x_2, x_3)$, $\xi = (\xi_1, \xi_2, \xi_3)$, and $(x, \xi) = x_1\xi_1 + x_2\xi_2 + x_3\xi_3$. Then we can define

$$Af = iF^{-1}\{(\xi_1^2 + \xi_2^2 + \xi_3^2)^{1/2}F\{f\}\} ,$$
$$D(A) = \{f \in \mathscr{L}^2(R^3)\colon f' \text{ exists and } f' \in \mathscr{L}^2(R^3)\} ;$$

it is easy to see that $A^2 = \Delta_3$. A simple computation using the identity $\int F\{f\}g = \int fF\{g\}$ shows that A is skew-adjoint and thus generates a (C_0) unitary group on $\mathscr{H}$ (Stone's theorem). The validity of the abstract formulation of Huyghens' principle for the wave equation $u'' = \Delta_3 u$ relative to $\{H_j\}$ follows from the known classical result [1]. This example is readily generalized to $A^2 = \Delta_{2m+3}$, $m = 0, 1, \cdots$.

THEOREM 6. *Let A generate a (C_0) unitary group $T(t)$ on the Hilbert space $\mathscr{H}$, and let $\{H_j\}_{j=0}^{\infty}$ be an increasing sequence of linear manifolds in $\mathscr{H}$. Then $(T(t)h_0, h_j) = 0$ for $|t| > T_j$ and all $h_0 \in H_0$, $h_j \in H_j$ if Huyghens' principle is valid for* (14) *relative to $\{H_j\}$.*

Conversely, suppose $D(A) \cap H_j$ is dense in H_j for $j = 0, 1, \cdots$, and there exist T_j such that $|t| > T_j$ implies $(T(t)h_0, h_j) = 0$ for each $h_0 \in H_0$, $h_j \in H_j$. Then $(u(t), h_j) = C_j$ for some constant C_j (depending on the data for u) and all $h_j \in H_j$ if $u_0 \in D(A^2) \cap H_0$ and $u_1 \in D(A) \cap H_0$.

Proof. Let h_j be an element of the dense set $D(A) \cap H_j$; then

$$\begin{aligned} -2(u(t), Ah_j) &= 2(Au(t), h_j) \\ &= (T(t)[Au_0 + u_1] + T(-t)[Au_0 - u_1], h_j) = 0 \end{aligned}$$

for $|t| > T_j$ since A is skew-adjoint by Stone's theorem. Taking $u_0 = 0$ yields

$$(T(t)u_1, h_j) = (T(-t)u_1, h_j) .$$

For $|t| > T_j$ we also have $(d/dt)(u(t), h_j) = (u'(t), h_j) = 0$, whence for $u_0 = 0$

$$(T(t)u_1, h_j) = -(T(-t)u_1, h_j) ,$$

so we must have $(T(t)u_1, h_j) = 0$ for $|t| > T_j$ and all $u_1 \in D(A) \cap H_0$, $h_j \in D(A) \cap H_j$. Since these sets are dense in H_0 and H_j respectively, the first part of the theorem follows by a simple approximation argument.

For the partial converse, observe that $(T(t)h_0, h_j) = 0$ for all $h_0 \in H_0$, $h_j \in H_j$, and $|t| > T_j$ implies that $(d/dt)(u(t), h_j) = 0$ for $|t| > T_j$ and data in H_0.

The converse cannot in general be strengthened to conclude that $(u(t), h_j) = 0$. To see this, let $A = (d/dx)$ on $\mathscr{L}^2(-\infty, \infty)$ and $H_j = \mathscr{L}^2(-(j+1), (j+1))$. A generates the group $T(t)f(x) = f(x+t)$, and Huyghens' principle is not valid (readily seen from the classical solution of d'Alembert). Nevertheless, for $w \in H_0$,

$$(T(t)w, h_j) = \int_{-\infty}^{\infty} w(x+t)\overline{h_j(x)}dx = 0$$

for $|t| > j + 2$.

Duffin [2] has recently established a virial theorem for solutions of the classical three-dimensional wave equation with compactly supported data; see also [4]. Using the Paley-Wiener theorem, Duffin shows that for sufficiently large time the kinetic energy $1/2\|u'(t)\|^2$ and the potential energy $1/2\|Au(t)\|^2$ are constant and equal. We shall derive this result as a corollary to the following theorem for abstract wave equations; the result itself is similar to one of [4].

THEOREM 7. *Let A generate the unitary (C_0) group $T(t)$ on the Hilbert space $\mathscr{H}$. Let H be a subspace of $\mathscr{H}$ such that $D(A) \cap H$ is dense in H and $A(D(A) \cap H) \subset H$. Then, for solution u of* (14) *with arbitrary data $u_0 \in D(A^2) \cap H$, $u_1 \in D(A) \cap H$, the kinetic and potential energies will be equal for $t > S$ if and only if*

$$Re(T(2t)h, h') = 0$$

for all $h, h' \in H$ and $t > S$.

Proof. From (3), (4), and the parallelogram law we have

$$\|Au(t)\|^2 = \frac{1}{4}\|Au_0 + u_1\|^2 + \frac{1}{4}\|Au_0 - u_1\|^2 + \frac{1}{2}Re(T(t)[Au_0 + u_1], T(-t)[Au_0 - u_1])$$

$$= \frac{1}{2}\|Au_0\|^2 + \frac{1}{2}\|u_1\|^2$$

$$+ \frac{1}{2}Re(T(t)[Au_0 + u_1],\ T(-t)[Au_0 - u_1])\ ,$$

$$\|u'(t)\|^2 = \frac{1}{2}\|Au_0\|^2 + \frac{1}{2}\|u_1\|^2$$

$$- \frac{1}{2}Re(T(t)[Au_0 + u_1],\ T(-t)[Au_0 - u_1])\ ;$$

thus the total energy $1/2\|u'(t)\|^2 + 1/2\|Au(t)\|^2$ is constant for all time. Clearly, the kinetic and potential energies will be equal for $t > S$ for all allowable data in (14) if and only if $Re(T(t)[Au_0 + u_1], T(-t)[Au_0 - u_1]) = 0$ for all $u_0 \in D(A^2) \cap H$ and $u_1 \in D(A) \cap H$. It is obvious that $Re(T(2t)h, h') = 0$ for all $h, h' \in H$ and $t > S$ implies that the kinetic energy equals the potential energy for $t > S$. Conversely, if the kinetic and potential energies are equal for $t > S$, then setting $u_0 = 0$ yields $Re(T(2t)u_1, u_1) = 0$ for $t > S$ and all $u_1 \in D(A) \cap H$. Since $D(A) \cap H$ is dense in H, an approximation argument guarantees that $Re(T(2t)h, h) = 0$ for $t > S$ and $h \in H$. The polarization identity for the sesquilinear form $\varphi(h, h') = (T(2t)h, h')$ shows that this is equivalent to $Re(T(2t)h, h') = 0$ for $t > S$ and $h, h' \in H$.

EXAMPLE. Let $\mathscr{H} = \mathscr{L}^2(-\infty, \infty)$, $H = \mathscr{L}^2(-C, C)$, $A = d/dx$, $T(t)f(x) = f(x + t)$; then $S = 2C$.

COROLLARY (Duffin). *Let* $\mathscr{H} = \mathscr{L}^2(R^{2m+3})$, $H = \mathscr{L}^2(B)$ *for a ball* $B \subset R^{2m+3}$, $A^2 = \Delta_{2m+3}$, $m = 0, 1, \cdots$. *Then the kinetic and potential energies for the wave equation* $u'' = \Delta_{2m+3}u$, $u(0) = u_0 \in D(A^2) \cap H$, $u'(0) = u_1 \in D(A) \cap H$ *are equal for all sufficiently large time.*

This follows from Theorem 6 and the example preceding it if we take $j = 0$ in the theorem and $C =$ radius of the ball B in the example.

The following theorem is similar to results of Shinbrot [9] and Goldstein [3]; a proof can be given readily along the lines of the proof of Theorem 7.

THEOREM 8. *Let the skew-adjoint operator* A *on the Hilbert space* $\mathscr{H}$ *generate the group* $T(t)$ *and let* $u(t)$ *be the solution of* (14) *with* $u_0 \in D(A^2)$, $u_1 \in D(A)$. *Then*

$$\lim_{|t|\to\infty} \|Au(t)\|^2 = \lim_{|t|\to\infty} \|u'(t)\|^2 = \frac{1}{2}(\|Au_0\|^2 + \|u_1\|^2)$$

if and only if

$$\lim_{|t|\to\infty} Re(T(t)[Au_0 + u_1], T(-t)[Au_0 - u_1]) = 0 .$$

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UNIVERSITY OF IDAHO