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THE LATTICE OF CLOSED IDEALS AND a^*-EXTENSIONS OF AN ABELIAN l-GROUP

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An l-ideal A of an l-group G is *closed* if $x \in A$ whenever $x = \vee a_i, 0 \leqslant a_i \in A$. The intersection of any collection of closed l-ideals of G is again a closed l-ideal of G. Hence the set $\mathscr{K}(G)$ of all closed l-ideals of G is a complete lattice under inclusion. In the present paper this lattice is studied, as well as l-group extensions which preserve it. A common generalization of the essential closure of an archimedean l-group and the Hahn closure of a totally-ordered abelian group is obtained.

Unless otherwise specified all l-groups will be assumed abelian. Set-theoretic union and intersection will be written $\cup$ and $\cap$, respectively. The lattice of all l-ideals of an l-group G will be denoted $\mathscr{L}(G)$; the join operation in $\mathscr{L}(G)$ will be written $\vee$ (to be differentiated by context from the l-group operation). The join operation in $\mathscr{K}(G)$ will be written $\bigcup$. A subset D of a partially ordered set S will be called a *dual ideal* if $x \in D$ whenever $x \geqslant y$ for some $y \in D$.

$G(g)$ will denote the smallest l-ideal of G containing $g \in G$. $\bar{A}$ will denote the smallest closed l-ideal of G containing $A \in \mathscr{L}(G)$. We have $\bar{A}^+ = \{\bigvee a_i \mid 0 \leqslant a_i \in A\}$. ([5], Lemma 3.2).

$A \in \mathscr{L}(G)$ is a *regular* subgroup of G if it is maximal in $\mathscr{L}(G)$ without some $g \in G$; in this case A is also called a *value* of g. If A is the only value of some $g \in G$, then A is a *special* subgroup of G. Each special subgroup of G is closed. ([4], Prop. 4.1). If each $g \in G$ has only finitely many values, then G is *finite-valued*. An l-ideal of G is *prime* if it is the intersection of a chain of regular subgroups of G. An l-ideal of G which contains a closed prime subgroup of G is itself a closed prime subgroup. ([5], Lemma 3.3).

We conclude the introduction by reviewing the important results in [10]. Let Λ be a root system (i.e., Λ is a partially ordered set and no two noncomparable elements of Λ have a lower bound in Λ). Let $V(\Lambda, R)$ denote the group of all real-valued functions on Λ whose support satisfies the *ACC*. $\lambda \in \Lambda$ is a *maximal component* of $v \in V(\Lambda, R)$ if λ belongs to the support of v but no element of Λ exceeding λ belongs to the support of v. Define $v > 0$ if and only if $v(\lambda) > 0$ for each maximal component λ of v. Then $V(\Lambda, R)$ is an l-group. If $\lambda \in \Lambda$, then $V_\lambda = \{v \in V(\Lambda, R) \mid v(\alpha) = 0 \text{ for all } \alpha \geqslant \lambda\}$ is a closed regular subgroup of $V(\Lambda, R)$; moreover, these are the only closed regular subgroups of $V(\Lambda, R)$.

The set of all regular subgroups of an l-group G forms a root system, to be denoted by $\Gamma(G)$. A subset Δ of $\Gamma(G)$ is *plenary* if Δ is a dual ideal in $\Gamma(G)$ and $\cap \Delta = 0$. It will sometimes be convenient to identify Δ notationally with $\{G_\delta, \delta \in \Delta\}$; here the G_δ denote the regular subgroups of G belonging to Δ. If Δ is a plenary subset of $\Gamma(G)$, then there exists a v-isomorphism $\sigma: G \to V(\Delta, R)$ (i.e., σ is an l-isomorphism, and $g\sigma$ has a maximal component at $\delta \in \Delta$ if and only if δ is a value of $g \in G$).

Throughout this paper G and H will denote l-groups.

1. Lattice properties of $\mathscr{K}(G)$.

THEOREM 1.1. *$\mathscr{K}(G)$ is complete Brouwerian lattice. If $\{K_\alpha\} \subseteqq \mathscr{K}(G)$, then $\bigsqcup K_\alpha = \overline{\bigvee K_\alpha}$, and $(\bigsqcup K_\alpha)^+ = \{\bigvee x_i \mid 0 \leqslant x_i \in \cup K_\alpha\}$.*

Proof. We have noted that $\mathscr{K}(G)$ is a complete lattice. Since $\bigsqcup K_\alpha$ is an l-ideal it contains $\bigvee K_\alpha$ and hence $\bigsqcup K_\alpha = \overline{\bigvee K_\alpha}$. Let $x \in (\overline{\bigvee K_\alpha})^+$. Then $x = \bigvee x_i$ where $0 \leqslant x_i \in \bigvee K_\alpha$. Each x_i is the join in G of (finitely many) positive elements of $\cup K_\alpha$. ([9], p. 519). Hence x is the join in G of positive elements of $\bigsqcup K_\alpha$. Thus $(\overline{\bigvee K_\alpha})^+ \subseteqq \{\bigvee x_i \mid 0 \leqslant x_i \in \cup K_\alpha\}$. The converse containment is trivial.

Let $K \in \mathscr{K}(G)$ and $\{K_\alpha\} \subseteqq \mathscr{K}(G)$. To show $\mathscr{K}(G)$ is Brouwerian is to show $K \cap (\bigsqcup K_\alpha) = \bigsqcup (K \cap K_\alpha)$. Clearly $K \cap (\bigsqcup K_\alpha) \supseteqq \bigsqcup (K \cap K_\alpha)$. Let $0 \leqq x \in K \cap (\bigsqcup K_\alpha)$. Write $x = \bigvee x_i$ where $0 \leqq x_i \in \cup K_\alpha$. Since $0 \leqq x_i \leqq x$ and K is convex, $x_i \in K$; thus $x_i \in \cup (K \cap K_\alpha)$. Hence $x \in \bigsqcup (K \cap K_\alpha)$.

EXAMPLE. An l-group for which $\mathscr{K}(G)$ is not a sublattice of $\mathscr{L}(G)$. Let G be the l-group of all eventually constant sequences. Let S_1 (resp. S_2) be the set of sequences in G whose odd (resp. even) entries are zero. Then $S_1, S_2 \in \mathscr{K}(G)$ but $S_1 \vee S_2$ is the set of eventually zero sequences and is not closed in G.

Let L be a complete Brouwerian lattice. For $x \in L$ let x' denote the largest element of L such that $x \wedge x' = 0$. The collection $P(L) = \{x' \mid x \in L\}$ is a Boolean algebra (under the induced order). ([2], p. 130). In particular, if $x \in P(L)$ then $x = (x')'$. Hence $L = P(L)$ if and only if L is a Boolean algebra.

$\mathscr{L}(G)$ is a complete Brouwerian lattice. If $C \in \mathscr{L}(G)$, then $C \in P(\mathscr{L}(G))$ if and only if $C = \{g \in G \mid |g| \wedge |a| = 0 \text{ for all } a \in C'\}$. Thus $C \in \mathscr{K}(G)$ whenever $C \in P(\mathscr{L}(G))$. It follows that $P(\mathscr{K}(G)) = P(\mathscr{L}(G))$.

THEOREM 1.2. (*Bigard*, [1], *Thm.* 5.6). *G is archimedean if and only if $\mathscr{K}(G) = P(\mathscr{L}(G))$.*

COROLLARY 1.3. *G is archimedean if and only if $\mathscr{K}(G)$ is a*

Boolean algebra.

REMARK. Whether or not G is archimedean is also determined by $\mathscr{L}(G)$. This follows from the following observations. G is archimedean if and only if each principal l-ideal $G(g)$ of G is archimedean. The principal l-ideals of G are the compact elements of $\mathscr{L}(G)$. (An element x of a lattice L is *compact* if $x \leqslant \bigvee \{x_\alpha \mid \alpha \in A\}$ for $x_\alpha \in L$ implies $x \leqq \bigvee \{x_\alpha \mid \alpha \in F\}$ for some finite subset F of A.) An l-group with a strong unit is archimedean if and only if the intersection of its maximal l-ideals is 0 [14]. The maximal l-ideals of $G(g)$ are just those elements of $\mathscr{L}(G)$ which are maximal with respect to being properly contained in $G(g)$.

DEFINITION. Let L be a lattice. An element $x \in L$ is called

(1) *meet-irreducible* if $x = \bigwedge x_\alpha$ implies $x = x_\alpha$ for some α.

(2) *finite meet-irreducible* if $x = \bigwedge_{i=1}^{n} x_i$ (n finite) implies $x = x_i$ for some i.

The meet-irreducible elements of $\mathscr{L}(G)$ are the regular subgroups of G; the finite meet-irreducible elements of $\mathscr{L}(G)$ are the prime subgroups of G. ([9], pp. 1.13, 1.14.)

PROPOSITION 1.4. *Let $K \in \mathscr{K}(G)$. K is (finite) meet-irreducible in $\mathscr{K}(G)$ if and only if K is (finite) meet-irreducible in $\mathscr{L}(G)$. In particular, the closed regular subgroups of G are distinguishable in $\mathscr{K}(G)$.*

Proof. Suppose $K = A \cap B$, where $A, B \in \mathscr{L}(G)$, and that K is finite meet-irreducible in $\mathscr{K}(G)$. Let $x \in \bar{A} \cap \bar{B}$. Write $x = \bigvee a_i$, $0 \leqslant a_i \in A$, and $x = \bigvee b_j$, $0 \leqslant b_j \in B$. Then $x = \bigvee a_i \wedge \bigvee b_j = \bigvee_{i,j} (a_i \wedge b_j)$ is the join of elements of $A \cap B$, and thus $x \in \overline{A \cap B}$. Hence $K = \bar{A}$ or $K = \bar{B}$, and therefore $K = A$ or $K = B$.

Now suppose K is meet-irreducible in $\mathscr{K}(G)$. Then, in particular, K is finite meet-irreducible in $\mathscr{L}(G)$ by the previous paragraph. K is thus a closed prime subgroup of G. Hence the members of $\mathscr{L}(G)$ that contain K all belong to $\mathscr{K}(G)$. Thus K is meet-irreducible in $\mathscr{L}(G)$.

The converse implications are trivial.

We note that all the preceding arguments in this section, except in the remark following Corollary 1.3, apply equally well to non-abelian l-groups with $\mathscr{L}(G)(\mathscr{K}(G))$ replaced by the lattice of all (closed) convex l-subgroups of G.

PROPOSITION 1.5. *The following are equivalent:*

(1) $\mathscr{K}(G) = \mathscr{L}(G)$.

(2) *$\Gamma(G)$ has no proper plenary subset.*

(3) *Each member of $\Gamma(G)$ is closed.*

Proof. An l-ideal A of G belongs to each plenary subset of $\Gamma(G)$ if and only if A is a closed regular subgroup of G. ([10], Thm. 5.2, [5], Cor. 3.12, and [4], Prop. 4.1). Thus (2) and (3) are equivalent. (1) implies (3) since $\Gamma(G) \subseteq \mathscr{L}(G)$. (3) implies (1) since each member of $\mathscr{L}(G)$ is an intersection of members of $\Gamma(G)$.

It is shown in ([9], p. 2.44) that G is finite-valued if and only if the elements of $\Gamma(G)$ are special subgroups of G. Since each special subgroup is closed, these conditions imply the conditions of Proposition 1.5. That the converse fails is shown in the following example.

EXAMPLE. Let X be an infinite compact Hausdorff space with a base of closed open subsets. Let $S(X)$ be the set of all continuous real-valued functions on X having finite range. The maximal ideals of $S(X)$ are of the form $M_x = \{f \in S(X) \mid f(x) = 0\}$; there are infinitely many of these. Since $S(X)$ is hyper-archimedean ([9], p. 2.17) these are the only prime ideals of $S(X)$.

Now, let $\Lambda = \{(x, n) \mid x \in X \text{ and } n = 1, 2\}$. Define $(x, 1) < (x, 2)$ for all $x \in X$, and let these be the only strict inequalities holding in Λ. Let G be the l-subgroup of $V(\Lambda, R)$ consisting of those functions $f\colon \Lambda \to R$ such that f has finite range, $f(x, 1) = 0$ for all but finitely many $x \in X$, and the restriction of f to $X \times 2$ is continuous.

Let $x \in X$. The ideal $A_x = \{f \in G \mid f(x, 1) = f(x, 2) = 0\}$ is the polar of a totally-ordered ideal of G, and hence is a minimal prime subgroup of G and is closed. Each l-ideal of G which contains some A_x is hence a closed prime subgroup of G. Let P be a prime ideal of G. Then $P \supseteq A_x$ for some x or $P \supseteq \{f \in G \mid f(x, 2) = 0 \text{ for all } x\} = \Sigma$. G/Σ is l-isomorphic to $S(X)$. Thus if $P \supseteq \Sigma$ then P corresponds to one of the prime ideals of $S(X)$, say $P = B_x = \{f \in G \mid f(x, 2) = 0\}$. But $B_x \supseteq A_x$. Hence each prime subgroup of G is closed, and thus each member of $\Gamma(G)$ is closed.

On the other hand, the function $g \in G$ such that, for all x, $g(x, 1) = 0$ and $g(x, 2) = 1$ has infinitely many values. (Each B_x is a value of g.)

Note also that Σ and G/Σ are both projectable, but G is not even though each prime subgroup of G exceeds a unique minimal prime.

2. a^*-extensions.

Let G be an l-subgroup of H. If $A \in \mathscr{L}(G)$ we write $\tilde{A} = \{x \in H \mid |x| \leqslant y \text{ for some } y \in A\}$. Then $\tilde{A} \in \mathscr{L}(H)$; indeed, it is the smallest l-ideal of H that contains A.

LEMMA 2.1. *Let G be an l-subgroup of H.*
(a) *If $K \in \mathscr{K}(G)$ then $\overline{\overline{K}} \cap G = K$.*
(b) *If $K \in \mathscr{K}(H)$ then $(K \cap G)^{=} \subseteq K$ and $(K \cap G)^{=} \cap G = K \cap G$.*

Proof. (a). Clearly $\bar{\bar{K}} \cap G \supseteqq K$. Let $0 \leqslant g \in \bar{\bar{K}} \cap G$. Then $g = \bigvee_H h_i$ where $0 \leqslant h_i \leqslant k_i \in K$. Note $g \wedge k_i \geqslant h_i$. Suppose $h \in H$ and $h \geqslant g \wedge k_i$ for all i. Then $h \geqslant h_i$ for all i and hence $h \geqslant g$. Thus $g = \bigvee_H (g \wedge k_i)$. Since G is an l-subgroup of H and $g, g \wedge k_i \in G$, we have $g = \bigvee_G (g \wedge k_i)$. Thus g is a join in G of elements of K, and hence $g \in K$.

(b). Let $K \in \mathscr{K}(H)$. Then $K \cap G \subseteqq K$, whence $(K \cap G)^{-} \subseteqq K$ and $(K \cap G)^{=} \subseteqq K$. Thus $K \cap G \subseteqq (K \cap G)^{=} \subseteqq K$ and hence $(K \cap G)^{=} \cap G = K \cap G$.

Definition. Let G be an l-subgroup of H. H is an *a^*-extension* of G if the map $K \mapsto K \cap G$ is a one-to-one map of $\mathscr{K}(H)$ onto $\mathscr{K}(G)$.

If H is an a^*-extension of G and $K \in \mathscr{K}(H)$, then by Lemma 2.1 (b), $(K \cap G)^{=} = K$; thus both the map $K \mapsto K \cap G$ and its inverse preserve order. Hence if H is an a^*-extension of G the map $K \mapsto K \cap G$ is a lattice isomorphism of $\mathscr{K}(H)$ onto $\mathscr{K}(G)$.

H is an *a-extension* of G if the map $C \mapsto C \cap G$ is a one-to-one map of $\mathscr{L}(H)$ onto $\mathscr{L}(G)$. Each a-extension of G is an a^*-extension of G. ([3], Thm. 3.9). H is an *essential* extension of G if $C \cap G \neq 0$ for all $0 \neq C \in \mathscr{L}(H)$.

Lemma 2.2. *If H is an essential extension of G and $K \in \mathscr{K}(H)$, then $K \cap G \in \mathscr{K}(G)$.*

Proof. Suppose $g = \bigvee_G k_i$ where $0 \leqslant k_i \in K \cap G$. Then since H is abelian and an essential extension of G, $g = \bigvee_H k_i$. ([7], Lemma 5.4). Thus $g \in K$ and so $g \in K \cap G$. Hence $K \cap G \in \mathscr{K}(G)$.

Lemma 2.3. *If G is an l-ideal of H and G is archimedean, then $\bar{G}$ is archimedean.*

Proof. Suppose (by way of contradiction) that there exist $a, b \in \bar{G}$ with $0 < a \ll b$. Then $0 < 2b \in \bar{G}$ and thus $2b = \bigvee_H g_i$ where $0 < g_i \in G$. Now $b = (\bigvee g_i) - b = \bigvee (g_i - b) = \bigvee ((g_i - b) \vee 0)$ and $0 < a = a \wedge b = \bigvee (((g_i - b) \vee 0) \wedge a)$. Hence $((g_i - b) \vee 0) \wedge a > 0$ for some g_i.

For totally ordered groups it is the case that $0 < a \ll b$ and $0 < g_i$ imply $g_i \gg ((g_i - b) \vee 0) \wedge a$. (Consider the cases $g_i - b < 0$ and $g_i - b \geqslant 0$.) Hence this implication holds in the abelian l-group $\bar{G}$.

$((g_i - b) \vee 0) \wedge a$ is a join of positive elements of G. Hence there exists $0 < g \in G$ such that $g \ll g_i$, contradicting the hypothesis that G is archimedean.

We remark that Lemma 2.3 and its proof are valid more generally when H is any l-group that can be represented as a subdirect product of (possibly non-abelian) totally ordered groups.

LEMMA 2.4. *If $K \in \mathscr{K}(G)$ and $A \in \mathscr{K}(K)$, then $A \in \mathscr{K}(G)$. Conversely, if $A, K \in \mathscr{K}(G)$ and $A \subseteqq K$, then $A \in \mathscr{K}(K)$.*

Proof. Let $K \in \mathscr{K}(G)$ and $A \in \mathscr{K}(K)$. If $g = \bigvee_G a_i$ where $0 \leqslant a_i \in A$, then $g \in K$ and hence $g = \bigvee_K a_i$, whence $g \in A$.

Conversely, let $A, K \in \mathscr{K}(G)$ and $A \subseteqq K$. If $k \in K$ and $k = \bigvee_K a_i$, $0 \leqslant a_i \in A$, then since K is convex in $G, k = \bigvee_G a_i$; hence $k \in A$.

COROLLARY. *Suppose H is an a^*-extension of G, and $K \in \mathscr{K}(H)$. Then K is an a^*-extension of $K \cap G$.*

THEOREM 2.5. *If H is an a^*-extension of G, then H is an essential extension of G.*

Proof. Let $0 \neq C \in \mathscr{L}(H)$. We prove $C \cap G \neq 0$.

Case 1. Suppose C is not archimedean. Then there exist $0 < x$, $y \in C$ such that $x \ll y$. Then $H(x) < y$ and hence $\overline{H(x)} < y$. Thus $0 \neq \overline{H(x)} \cap G \subseteqq C \cap G$, and hence $P(\mathscr{L}(G)) = P(\mathscr{L}(G))$.

Case 2. Suppose C is archimedean. Then $\bar{C}$ is archimedean by Lemma 2.3, and $\bar{C}$ is an a^*-extension of $\bar{C} \cap G$ by the corollary to Lemma 2.4. Thus $X \rightarrow X \cap \bar{C} \cap G$ is a one-to-one correspondence between the polars in $\bar{C}$ and those in $\bar{C} \cap G$. Thus, since $\bar{C}$ is archimedean, $\bar{C}$ is an essential extension of $\bar{C} \cap G$. ([6], Thm. 3.7). Hence $0 \neq C \cap (\bar{C} \cap G) = C \cap G$.

THEOREM 2.6. *Let G be an l-subgroup of H. The following are equivalent:*

(1) *H is an a^*-extension of G.*

(2) *H is an essential extension of G, and $(K \cap G)^{=} = K$ for all $K \in \mathscr{K}(H)$.*

(3) *H is an essential extension of G, and $K_1 = K_2$ whenever $K_1 \cap G = K_2 \cap G$ for $K_1, K_2 \in \mathscr{K}(H)$.*

Proof. (1) implies (2). Immediate from Theorem 2.5 and Lemma 2.1 (b).

(2) implies (3). If $K_1 \cap G = K_2 \cap G$, then $(K_1 \cap G)^{=} = (K_2 \cap G)^{=}$ whence $K_1 = K_2$.

(3) implies (1). This follows from Lemmas 2.2 and 2.1 (a).

McCleary ([12], Cor. 5) has proved that if G is completely distributive, then each $K \in \mathscr{K}(G)$ is the intersection of a set of closed regular subgroups of G. On the other hand, Byrd and Lloyd ([5], Thm. 3.10) proved that G is completely distributive if and only if the collection of all closed regular subgroups of G has 0 intersection. These remarks are applicable, in particular, to $V((\Lambda, R)$, where Λ is any root system, since $\bigcap\{V_\lambda, \lambda \in \Lambda\} = 0$.

THEOREM 2.7. *Let Δ be a plenary subset of $\Gamma(G)$ and $\sigma: G \to V(\Delta, R)$ a v-isomorphism. $V(\Delta, R)$ is an a^*-extension of $G\sigma$ if and only if each G_δ, $\delta \in \Delta$, is a special subgroup of G.*

Proof. For convenience we identify G with $G\sigma$.

Suppose each G_δ is special. If $\delta \in \Delta$ there exists $g_\delta \in G$ such that the only maximal component of g_δ is δ. It follows that $V = V(\Delta, R)$ is an essential extension of G.

Let $K_1, K_2 \in \mathscr{K}(V)$ and suppose $K_1 \cap G = K_2 \cap G$. As noted above there exist subsets A, B of Δ (which without loss of generality are dual ideals of Δ) such that $K_1 = \bigcap\{V_\alpha \mid \alpha \in A\}$ and $K_2 = \bigcap\{V_\beta \mid \beta \in B\}$. Suppose $\delta \in A\backslash B$ and let $g = g_\delta$. Suppose there exists $\beta \in B$ such that $g \notin V_\beta$. Then $g(\gamma) \neq 0$ for some $\gamma \geqslant \beta$, and since δ is the only maximal component of $g, \delta \geqslant \gamma$. Thus $\delta \geqslant \beta$ and so $\delta \in B$, a contradiction. Hence $g \in V_\beta$ for all $\beta \in B$, and therefore $g \in K_2 \cap G$. But clearly $g \notin K_1 \cap G$. This contradicts $K_1 \cap G = K_2 \cap G$. Hence $A \subseteqq B$, and similarly $B \subseteqq A$. Thus $K_1 = K_2$, and V is an a^*-extension of G.

Conversely, suppose V is an a^*-extension of G. For $\beta \in \Delta$ let $M_\beta = \bigcap\{V_\delta \mid \delta \nless \beta\}$ and $N_\beta = \bigcap\{V_\delta \mid \delta \nleq \beta\}$. Then $M_\beta, N_\beta \in \mathscr{K}(V)$. By definition M_β(resp., N_β) is the set of all elements of V whose support lies strictly below β (resp., on or below β). Thus there exists $g \in G$ such that the only maximal component of g is β. $G_\beta = V_\beta \cap G$ is the only value of g in G. Thus G_β is special for all $\beta \in \Delta$.

REMARK. A lattice L is *meet-generated* by $S \subseteqq L$ if each element of L is the meet of some subset of S. If, in addition, no two dual ideals of S have the same meet, then S *freely* meet-generates L. It can be shown that the equivalent conditions of Theorem 2.8 are in turn equivalent to the condition: $\mathscr{K}(G)$ is freely meet-generated by Δ.

3. a^*-closures.

DEFINITION. An l-group H is *a^*-closed* if it admits no proper a^*-extension. H is an *a^*-closure* of G if H is an a^*-extension of G and H is a^*-closed.

The arguments leading up to the first theorem of this section need no commutativity hypothesis. Hence the a^*-closure of an archimedean l-group would be that of the theorem even if this paper admitted non-abelian l-groups (with the lattice of closed convex l-subgroups playing the role of $\mathscr{K}(G)$).

Suppose G is archimedean and H is an a^*-extension of G. Then by Corollary 1.3 H is archimedean, and, furthermore, by ([6], Thm. 3.7) H is an essential extension of G. Conversely, if H is archimedean and an essential extension of G, then by Theorem 1.2 and ([8], Thm. 3.4) H is an a^*-extension of G. Thus for archimedean l-groups the a^*-extensions are the archimedean essential extensions. It was proved in [6] that each archimedean l-group G admits a unique essential closure relative to the class of all archimedean l-groups. Thus we have

THEOREM 3.1. *Each archimedean l-group G has an a^*-closure. Furthermore, if H_1 and H_2 are l-groups each of which is an a^*-closure of G, then there exists an l-isomorphism τ of H_1 onto H_2 such that $\tau|_G = 1_G$.*

This closure is the l-group of all almost-finite extended real-valued functions on the Stone space associated with the Boolean algebra $P(\mathscr{L}(G))$. ([6], Thm. 3.6). Since the members of $P(\mathscr{L}(G))$ are closed l-ideals of G, we conclude that if G is archimedean then $|G| \leqslant |R^{(2^{\mathscr{K}(G)})}|$. This fact will be useful later.

The proofs of the next two lemmas make repeated use of Theorem 2.6.

LEMMA 3.2. *Suppose F is an l-subgroup of G and G is an l-subgroup of H. If H is an a^*-extension of G and G is an a^*-extension of F, then H is an a^*-extension of F, and conversely.*

Proof. Suppose H is an a^*-extension of G and G is an a^*-extension of F. The map $K \to K \cap F$ where $K \in \mathscr{K}(H)$ is the composition $K \to K \cap G \to (K \cap G) \cap F$. Thus H is an a^*-extension of F.

Conversely, let H be an a^*-extension of F. Then H is an essential extension of F and hence of G. Let $K_1, K_2 \in \mathscr{K}(H)$ be such that $K_1 \cap G = K_2 \cap G$. Then $K_1 \cap F = K_2 \cap F$ and hence $K_1 = K_2$. Thus H is an a^*-extension of G.

Let $0 < g \in G$. Then $g \in H$, and since H is an essential extension of F, there exists $0 < f \in F$ such that $f \leqslant ng$ for some positive integer n. Thus $f \in G(g)$, and hence G is an essential extension of F.

Let $K_1, K_2 \in \mathscr{K}(G)$ and suppose $K_1 \cap F = K_2 \cap F$. We apply Lemma 2.1 (a). $\overline{\widetilde{K}}_1 \cap F = (\overline{\widetilde{K}}_1 \cap G) \cap F = K_1 \cap F = K_2 \cap F = (\overline{\widetilde{K}}_2 \cap G) \cap F = \overline{\widetilde{K}}_2 \cap F$. Hence $\overline{\widetilde{K}}_1 = \overline{\widetilde{K}}_2$ and so $K_1 = K_2$.

LEMMA 3.3. *If $\{H_\alpha \mid \alpha \in A\}$ is a chain of l-groups each of which is an l-subgroup of the members of the chain that contain it, and each of which is an a^*-extension of G, then $H = \cup H_\alpha$ is an a^*-extension of G.*

Proof. Each H_α is an essential extension of G. Let $0 < x \in H$. Then $x \in H_\alpha$ for some α. Hence $H_\alpha(x) \cap G \neq 0$. But $H(x) \supseteq H_\alpha(x)$. Thus $H(x) \cap G \neq 0$, and hence H is an essential extension of G.

Suppose $K_1, K_2 \in \mathscr{K}(H)$ and $K_1 \cap G = K_2 \cap G$. Then for each $\alpha \in A$, we have $K_1 \cap H_\alpha, K_2 \cap H_\alpha \in \mathscr{K}(H_\alpha)$ since H is an essential extension of $H_\alpha \supseteq G$. Moreover, $(K_1 \cap H_\alpha) \cap G = K_1 \cap G = K_2 \cap G = (K_2 \cap H_\alpha) \cap G$. Since H_α is an a^*-extension of G, we conclude $K_1 \cap H_\alpha = K_2 \cap H_\alpha$. Thus $K_1 = K_1 \cap H = K_1 \cap (\bigcup H_\alpha) = \bigcup (K_1 \cap H_\alpha) = \cup (K_2 \cap H_\alpha) = K_2$. Thus H is an a^*-extension of G.

LEMMA 3.4. *Let $K \in \mathscr{K}(G), A \in \mathscr{L}(G)$ and $A \supseteq K$. If $A/K \in \mathscr{K}(G/K)$, then $A \in \mathscr{K}(G)$.*

Proof. Suppose $g \in G$ and $g = \bigvee_G a_i$, $0 \leqslant a_i \in A$. Then ([4], Lemma 4.4) since $K \in \mathscr{K}(G)$, $g + K = \bigvee (a_i + K)$. Thus $g + K \in A/K$ and hence $g \in A$. Hence $A \in \mathscr{K}(G)$.

We note that the example at the end of §1 can be used to show that the converse of Lemma 3.4 fails. Referring to that example, we have $B_x, \Sigma \in \mathscr{K}(G)$ and $B_x \supseteq \Sigma$, but B_x/Σ is not closed in G/Σ unless x is an isolated point of X. X can be chosen so that it has no isolated points. R. Byrd has sent us a similar example illustrating the failure of the converse for Lemma 3.4.

LEMMA 3.5. *Let $g \in G$ with $g \neq 0$. There exist $A, B \in \mathscr{K}(G)$ with $A \subsetneqq B$ such that $g \in B \backslash A$ and B/A is archimedean.*

Proof. Since g belongs to an l-ideal of G if and only if $|g|$ does, we can assume $g > 0$.

Let $S = \{z \in G \mid 0 \leqslant z \ll g\}$. Then S is a convex subsemigroup of G and the subgroup A generated by S is an l-ideal of G. If $x \in G$ and $x = \bigvee a_i$, $0 \leqslant a_i \in A$, then $na_i \leqslant g$ and hence $n \bigvee a_i = \bigvee na_i \leqslant g$; thus $x \in A$. Hence $A \in \mathscr{K}(G)$.

We show A is the intersection of the maximal l-ideals of $G(g)$. Let $0 < a \in A$ and let M be a maximal l-ideal of $G(g)$. Since $a \ll g$ we have $n(M + a) = M + na < M + g$ for all integers n. $G(g)/M$ is l-isomorphic to an l-subgroup of R. Hence $a \in M$.

Now suppose $x > 0$ is an element of each maximal ideal M of $G(g)$. Let n be an integer. Then $M + g > M + nx$. The maximal ideals of $G(g)$ are precisely the values of g-nx in $G(g)$. Thus $M^+g - ux > M$ for all values of $g - nx$ in $G(g)$, and hence $g - nx \geqslant 0$. Thus $x \ll g$ and $x \in A$.

Since the intersection of all the maximal l-ideals of $G(g)/A$ is zero, $G(g)/A$ is a subdirect product of copies of R, and hence is archimedean. Let B be the l-ideal of G such that $B \supseteqq A$ and B/A is the least member of $\mathscr{K}(G/A)$ containing $G(g)/A$. By Lemma 2.4 B/A is archimedean and by Lemma 3.4 $B \in \mathscr{K}(G)$. Since $g \in B \backslash A$, the proof is complete.

REMARK. The above argument contains a proof of the fact that for abelian l-groups with strong unit the intersection of all maximal l-ideals is a closed l-ideal.

THEOREM 3.6. *Each l-group G has an a^*-closure.*

Proof. The divisible hull of G is an a-extension of G and hence an a^*-extension of G. Thus without loss of generality G is a rational vector space.

Let A index the set of ordered pairs (K^α, K_α) of elements of $\mathscr{K}(G)$ such that $K^\alpha \supset K_\alpha$ and K^α/K_α is archimedean. For each $\alpha \in A$ choose some fixed C_α such that G is the group direct sum of K^α and C_α. Define $\eta\colon G \to \Pi K^\alpha/K_\alpha$ by $\eta(g) = (\cdots g_\alpha \cdots)$ where $g = g_\alpha + c_\alpha$ with $g_\alpha \in K^\alpha$ and $c_\alpha \in C_\alpha$. Then η is a group homomorphism, and by Lemma 3.5 $\operatorname{Ker} \eta = 0$. Thus η is injective.

By Lemma 2.4 $|\mathscr{K}(K^\alpha)| \leqslant |\mathscr{K}(G)|$ and by Lemma 3.4 $|\mathscr{K}(K^\alpha/K_\alpha)| \leqslant |\mathscr{K}(K^\alpha)|$. Thus $|K^\alpha/K_\alpha| \leqslant |R^{2^{\mathscr{K}(G)}}|$ for all $\alpha \in A$. (See the paragraph following Theorem 3.1.) Now since $A \subseteqq \mathscr{K}(G) \times \mathscr{K}(G)$ we conclude that there is some cardinal number $\aleph$ *dependent only on* $|\mathscr{K}(G)|$ such that $|G| \leqslant \aleph$. If H is an a^*-extension of G, then since $|\mathscr{K}(G)| = |\mathscr{K}(H)|$, we have $|H| \leqslant \aleph$.

It follows now by Lemmas 3.2 and 3.3 and the usual transfinite arguments that G has an a^*-closure.

THEOREM 3.7. *Suppose the closed regular subgroups of G form a plenary subset Δ of $\Gamma(G)$. Then each a^*-closure of G is l-isomorphic to an l-subgroup of $V(\Delta, R)$. If each member of Δ is a special subgroup of G, then each a^*-closure of G is l-isomorphic to $V(\Delta, R)$.*

Proof. Let H be an a^*-closure of G. By Theorem 1.4 $\{G_\delta, \delta \in \Delta\}$ is the set of meet-irreducible elements of $\mathscr{K}(G)$. Let H_δ be the element of $\mathscr{K}(H)$ such that $H_\delta \cap G = G_\delta$. Then $\{H_\delta, \delta \in \Delta\}$ is the set of closed regular subgroups of H, and $\bigcap H_\delta = 0$ since $\bigcap G_\delta = 0$. Thus $\{H_\delta, \delta \in \Delta\}$ is a plenary subset of $\Gamma(H)$, and there exists a v-isomorphism

$\sigma: H \rightarrow V(\Delta, R)$. Thus H is l-isomorphic to an l-subgroup of $V(\Delta, R)$. The last assertion of the theorem follows from Theorem 2.7.

COROLLARY 3.8. *$V(\Lambda, R)$ is a^*-closed for any root system Λ.*

A stronger form of uniqueness than that given by Theorem 3.7 exists when the members of Δ are special, and we proceed to establish this.

LEMMA 3.9. *Let G and H be divisible l-groups with G an l-subgroup of H, and let $\{G_\delta, \delta \in \Delta\}$ be a plenary subset of $\Gamma(G)$. Suppose there exists a plenary subset $\{H_\delta, \delta \in \Delta\}$ of $\Gamma(H)$ such that $H_\delta \cap G = G_\delta$ and $H^\delta \cap G = G^\delta$ for all $\delta \in \Delta$. (Here $H^\delta(G^\delta)$ denotes the intersection of all l-ideals of $H(G)$ which properly contain $H_\delta(G_\delta)$.) If $\sigma: G \rightarrow V(\Delta, R)$ is a v-isomorphism then there exists a v-isomorphism $\tau: H \rightarrow V(\Delta, R)$ such that $g\tau = g\sigma$ for all $g \in G$.*

Proof. Note that under the hypothesis the natural map $G^\delta/G_\delta \rightarrow H^\delta/H_\delta$ is a well-defined l-isomorphism into H^δ/H_δ. Now the proof of ([9], Lemma 4.11) applies.

THEOREM 3.10. *Suppose the special subgroups of G form a plenary subset Δ of $\Gamma(G)$. Then G has an a^*-closure which is l-isomorphic to $V(\Delta, R)$. Moreover, if H_1 and H_2 are a^*-closures of G, there exists an l-isomorphism μ of H_1 onto H_2 such that $\mu|_G = 1_G$.*

Proof. Let $\sigma: G \rightarrow V(\Delta, R)$ be a v-isomorphism. H_1 and H_2 are divisible since the divisible hull of an l-group is an a^*-extension of it. Moreover, since σ extends uniquely to a v-isomorphism of the divisible hull of G into $V(\Delta, R)$, we can assume G is divisible. Δ is the set of closed regular subgroups of G. The closed regular subgroups of G and the l-ideals that cover them are distinguishable in $\mathscr{K}(G)$. Thus, for $i = 1, 2$, there exists by Lemma 3.9 a v-isomorphism $\tau_i: H_i \rightarrow V(\Delta, R)$ such that $g\tau_i = g\sigma$ for all $g \in G$. By Theorem 2.7 and Lemma 3.2 τ_i is surjective. Now $\mu = \tau_1\tau_2^{-1}$ is an l-isomorphism of H_1 onto H_2 and $g\mu = g$ for all $g \in G$.

COROLLARY 3.11. *If G is finite-valued, then $V(\Gamma, R)$ is the unique a^*-closure of G.*

COROLLARY 3.12. *If G is totally ordered, then $V(\Gamma, R)$ is the unique a^*-closure of G.*

Thus the a^*-closure of a totally-ordered abelian group coincides with its Hahn closure.

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