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ON THE SINGULARITIES OF THE FUNCTION GENERATED BY THE BERGMAN OPERATOR OF THE SECOND KIND

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Let $\psi(\lambda, y) = P_2(f)$ be Bergman's integral operator of the second kind with domain of definition

$$W = \{(\lambda, y) \mid 3^{1/2} \mid \lambda \mid < y,\ \lambda \leqq 0,\ y > 0\}\ .$$

Let $f(q) = (q - A)^{-1}$, $A \in W$. In this paper it is shown that $\psi(\lambda, y)$ has singular points $z = 2A$, $2A(1 - w)$, where $w = A^{-1}\lambda$ and $z = \lambda + iy$.

Let

$$\psi(z, z^*) = P_2(f) = \int_l E(z, z^*, t) f\left(\frac{z}{2}(1 - t^2)\right) \frac{dt}{\sqrt{1 - t^2}}$$

be Bergman's integral operator of the second kind. $P_2(f)$ maps functions f analytic in one variable in the neighborhood of the origin into solutions of the linear partial differential equation

$$\psi_{zz^*} + N\left(\frac{z + z^*}{2}\right)(\psi_z + \psi_{z^*}) = 0\ , \quad z = \lambda + iy\ , \quad z^* = \lambda - iy\ ,$$

$N(\lambda) = -(1/12\lambda)(1 + b_1(-\lambda)^{2/3} + \cdots)$ is analytic for $-\infty < \lambda < 0$ and singular at $\lambda = 0$. $E(z, z^*, t)$, called the generating function of the operator, is analytic in the three variables z, z^*, and t providing $|z + z^*| < |t^2 z|$, l is some rectifiable Jordan curve in the upper complex t-plane connecting the points -1 and 1, [1], [3].

In a previous paper [7] we obtained some results on the singularities of $P_2(f)$ where f is meromorphic and z, z^* were treated as independent complx variables. In this paper we let $z^* = \bar{z}$ (conjugate of z) and $N(\lambda) = -1/12\lambda$ (Tricomi case). With these assumptions,

$$(1) \qquad \begin{aligned} \psi(\lambda, y) &= \int_{-1}^{1} E(u) \frac{f(q)}{\sqrt{1 - t^2}} dt\ , \quad \text{where} \quad u = \frac{t^2 z}{2\lambda}\ , \\ q &= \frac{1}{2} z(1 - t^2)\ , \\ z &= \lambda + iy\ , \end{aligned}$$

$E(u) = H(\lambda)(F^{(1)}(u) + F^{(2)}(u))$, $\quad F^{(1)}(u) = C_1 u^{-1/6} F_1(1/6, 2/3, 1/3, 1/u)$, $F^{(2)}(u) = C_2 u^{-5/6} F_2(5/6, 4/3, 5/3; 1/u)$, F_j is the hypergeometric function $j = 1, 2$, $H(\lambda) = C_3 \lambda^{-1/6}$, C_j are constants, $j = 1, 2, 3$, $(\lambda, y) \in W = \{(\lambda, y) \mid 3^{1/2} \mid \lambda \mid < y,\ \lambda \leqq 0,\ y > 0\}$,

l (the path of integration $= \{t \mid t = e^{i\theta}, 0 \leqq \theta \leqq \pi\}$, [4, p. 107].

THEOREM. *Let* $f(q) = (A - q)^{-1}$, $A = \lambda_0 + iy_0 \in W$, $\lambda/A = w = s + i\sigma$, $z = \lambda + iy$, $S_1 = \{(w, z) \mid z = 2A,\ \pi/2 \geqq \arg w \geqq \alpha_1,\ \pi/2 > \alpha_1 > \pi/3,\ 0 < \delta_1 \leqq |w| \leqq 1/4 - \delta_2,\ 1/4 > \delta_1, \delta_2 > 0,\ 1/4 > \delta_1 + \delta_2\}$,

$$S_2 = \{(w, z) \mid z = 2A(1 - w),\ \textit{same conditions on } w \textit{ as in } S_1\},$$

$S_3 = \{(0, 2y_0)\}$. *Let* $T = S_1 \cup S_2 \cup S_3$. *Then* T *is a singular set for at least one of the branches of* $\psi(w, z)$ *defined in* (1).

Proof. We consider first the case where $E(u) = H(\lambda)F^{(1)}(u)$.

Domain considerations. (3), (4) imply $\psi(w, z)$ is analytic function of the two complex variables w, z for disc neighborhoods satisfying $0 < |w| < 1/4$, $|A/2| < |z| < |A|$, where we have extended λ to the complex variable Aw. Note (1) implies we must specify branch cuts in our definition of $\psi(w, z)$. Since $z = \lambda + iy$ (see (1)), we must also consider the extension of λ, y to complex values subject to the above inequalities. Thus we can also obtain nonempty neighborhoods $N_\delta(\lambda)$, $N_\delta(y)$ such that $\psi(\lambda, y)$ is an analytic function in λ, y, where λ, y now have been extended to complex values.

In what follows we treat $\psi(w, z)$ as an analytic function in z for fixed w.

Consider the function obtained from (1) where we have used the series definition for $F_1(u)$,

$$(2)\qquad f(\lambda, y) = \int_{-1}^{1} \sum_{p=0}^{\infty} \frac{a_p t^{-2p}(2\lambda)^p}{z^p} \cdot \sum_{p=0}^{\infty} \left(\frac{z}{2A}\right)^p (1 - t^2)^p\, t^{-1/3} \frac{dt}{\sqrt{1 - t^2}},$$

$a_p = (\Gamma(p + 1/6)\Gamma(p + 2/3)/\Gamma(p + 1/3)\Gamma(p + 1))$, Γ is the Gamma function, $|z| < A$, $|2\lambda| < |z|$. From (2) we obtain two series,

$$(3)\qquad \begin{aligned} &\sum_{p=0}^{\infty} \left(\sum_{k=0}^{\infty} a_k \left(\frac{\lambda}{A}\right)^k t^{-2k-1/3}(1 - t^2)^{p+k-1/2}\right)\left(\frac{z}{2A}\right)^p \\ &+ \sum_{p=1}^{\infty} \left(\sum_{k=0}^{\infty} a_{p+k} \left(\frac{\lambda}{A}\right)^k t^{-(2(p+k+1/6))}(1 - t^2)^{k-1/2}\left(\frac{2\lambda}{z}\right)^p\right) \\ &|z| < |A|, \qquad |2\lambda| < |z|. \end{aligned}$$

We will limit ourselves to the first series in (3) for our analysis of the singularities of $P_2(f)$. When $|\lambda| \leqq |A/2| - \delta$, $|z| \leqq |A| - \delta$, $|A/2| > \delta > 0$, the operations of summation and integration (with respect to t) can be interchanged in the first series of (3), our integrals are in the improper Riemann sense. Integrating the first part of (3) by parts, then using the formula,

$$\int_{-1}^{1} t^{-1/3}(1-t^2)^{\nu+1/6}\frac{dt}{\sqrt{1-t^2}} = -\frac{1}{2}(1-e^{2\pi i/3})\frac{\Gamma(1/3)\Gamma(\nu+2/3)}{\Gamma(\nu+1)},$$

[2, p. 33], we obtain the function

$$f_1(w, z) = \sum_{p=0}^{\infty} \beta_p(w)\left(\frac{z}{2A}\right)^p, \tag{4}$$

where

$$\begin{aligned} &\beta_p(w) = \sum_{k=0}^{\infty} A_{pk}w^k, \quad A_{pk} = \frac{\Gamma(k+1/6)\Gamma(k+p+1/2)}{\Gamma(k+1/3)\Gamma(k+1)\Gamma(p+5/6)}, \\ &w = \frac{\lambda}{A}. \end{aligned} \tag{5}$$

(5) can be rewritten as

$$\beta_p(w) = \alpha_p c_p(w), \qquad \alpha_p = \frac{\Gamma(1/6)\Gamma(1/2+p)}{\Gamma(1/3)\Gamma(5/6+p)}, \tag{6}$$

$c_p(w) = F_3(1/6, 1/2+p, 1/3; w)$, F_3 a hypergeometric function. Using the asymptotic formula for F_3 for large p [6, pp. 235, 241 (23)], we can write $c_p(w)$ as

$$\begin{aligned} &c_p(w) = a_p(w)e^{i\pi/6}R_1 + a_p(w)(1-w)^{-1/3-p}R_2, \\ &a_p(w) = \left(\Gamma\left(\frac{1}{6}\right)\right)^{-1}\Gamma\left(\frac{1}{3}\right)(pw)^{-1/6}, \end{aligned} \tag{7}$$

p sufficiently large, $w \in T_1 = \{w \mid 0 < \delta_1 \leqq |w| \leqq 1/4 - \delta_2,\ 1/4 > \delta_2,\ \delta_1 > 0,\ 1/4 > \delta_1 + \delta_2,\ \pi/2 \geqq \arg w \geqq \alpha_1,\ \pi/2 > \alpha_1 > \pi/3\}$,

$$R_j(p, w) = 1 + R_0^{(j)}(p, w),$$

$\lim_{p\to\infty} pR_0^{(j)}(p, w) = h_j(w) \neq 0$ uniformly for $w \in T_1$, $j = 1, 2$. Using (6), (7), we can rewrite (4) as

$$\begin{aligned} &f_1(w, z) = \sum_{p=0}^{p=p_0\geqq 1} \alpha_p c_p(w)\left(\frac{z}{2A}\right)^p + \sum_{p=p_0+1}^{\infty} c_1(p, w)z_1^p + \sum_{p=p_0+1}^{\infty} c_2(p,w)z_2^p, \\ &z_1 = \frac{z}{2A}, \qquad z_2 = (1-w)^{-1}\frac{z}{2A}, \end{aligned} \tag{8}$$

and

$$c_1(p, w) = \alpha_p a_p(w)e^{i\pi/6}R_1, \qquad c_2(p, w) = \alpha_p a_p(w)(1-w)^{-1/3}R_2, \tag{9}$$

see (6) for the definition of α_p, (7) for $a_p(w)$.

From (9) we obtain

$$\rho = \lim_{p\to\infty} |c_j(p, w)|^{-1/p} = 1, \tag{10}$$

the radius of convergence of the second and third series in (8), and $-\varepsilon < \arg c_j(p, w) < \varepsilon$, $0 < \varepsilon < \pi/2$, p sufficiently large, $w \in T_1$, $j = 1$, 2.

Proof of (10). From (7) we obtain

$$1 + \varepsilon \geqq |R_j(p, w)| \geqq 1 - \varepsilon > 0 ,$$

$1 > \varepsilon > 0$, p sufficiently large, $w \in T_1$. So we can take the pth root (say principle branch) of $c_j(p, w)$, $j = 1, 2$, cf (9).

Using the asymptotic formula $(\Gamma(p + A)/\Gamma(p + B)) \sim p^{A-B}$, we conclude the first part of (10). Since $\lim_{p\to\infty} (1 + R_0^{(j)}(p, w) = 1$, $w \in T_1$, see (7), the second part of (10) follows.

(11) $$z = 2A \quad \text{and} \quad z = 2A(1 - w) , \quad w \in T_1 ,$$

are singular points of (8).

Proof of (11). (10) satisfies the hypotheses of a theorem of Dienes [5, p. 227]. From this theorem we conclude $z = 2A$ and $z = 2A(1 - w)$ are singular points respectively of the second and third series in (8). Further, $c_j(p = \xi = \rho e^{i\psi}, w)$ (see (9)) is an analytic function in ξ in the half-plane $x_1 \geqq 1$, $\xi = x_1 + iy_1$, and

$$|c_j(1 + \rho e^{i\psi}, w)| < e^{\varepsilon\rho} , \ \varepsilon > 0 ,$$

and arbitrarily small, $\rho > 0$ and sufficiently large, and $-\pi/2 \leqq \psi \leqq \pi/2$, $w \in T_1$, $j = 1$, 2. This follows from a definition of the remainder term $R_0^{(j)}(p, w)$ of (7), see [6, p. 235]. Hence by a theorem of Le Roy and Lindelöf [5, p. 340], we conclude the only possible singular points of the second series in (8) are the points on the ray $\varphi = \varphi_0$, $\varphi_0 = \arg 2A$, joining $2A$ to infinity and the only possible singular points of the third series in (8) are the points on the ray $\varphi = \theta_0$, $\theta_0 = \arg 2A(1 - w)$, $w \in T_1$, joining $2A(1 - w)$ to infinity. Further, $\arg 2A \neq \arg (2A(1 - w))$, $w \in T_1$. Hence the singular points $z = 2A$, $z = 2A(1 - w)$, $w \in T_1$, of the second and third series respectively are not removed upon addition of these two series in (8). This completes the proof of (11).

(12) $(0, 2y_0)$ is a singular point of $\psi(w, z)$.

Proof. Let $w = \lambda/A = \lambda_0 = 0$. (3) then reduces to the first series, and (4) reduces to the hypergeometric function $F_4(1, 1/2, 5/6; (y/2y_0))$ times a constant. F_4 is singular at the point $y = 2y_0$, so (12) holds.

From (11), (12) we conclude T is a singular set (see Theorem for the definition of T) of $\psi(w, z)$ for the case F_1.

Proof. We note the second series in (3) when integrated with respect to t gives rise to a function $f_2(w, z)$ which is regular at the points in T.

For the case F_2 (see (1)) we use the formula

$$\int_{-1}^{1} t^{-5/3}(1-t^2)^{\nu+5/6}\frac{dt}{\sqrt{1-t^2}} = \frac{1}{2}(1-e^{-(2\pi i/3)})\frac{\Gamma(-1/3)\Gamma(\nu+4/3)}{\Gamma(\nu+1)}$$

[2, p. 33].

Proceeding as above, we then conclude $T - \{(0, 2y_0)\}$ is a singular set for the case F_2. (1) thus can be written as the sum of two functions,

$$\psi(w, z) = \frac{1}{z^{5/6}}(g(w, z) = z^{2/3}P_1(w, z) + P_2(w, z)) , \tag{13}$$

where P_j is singular at the points in $T - \{(0, 2y_0)\}$, $j = 1, 2$. This follows from the linearity of the operator $P_2(f)$.

(14) At least one of the branches of $g(w, z)$ of (15) is singular for points in $T - \{(0, 2y_0)\}$.

Proof of (14). $z^{2/3}$ can be one of the three branches,

$$\alpha_1 = R^{2/3}e^{i2/3\theta} , \quad \alpha_2 = R^{2/3}e^{i(2/3\theta+2/3\pi)} , \quad \alpha_3 = R^{2/3}e^{i(2/3\theta+4/3\pi)}, \ \pi > \theta > -\pi .$$

We form the sum

$$\sum_{i=1}^{3} g_i(w, z) = \sum_{i=1}^{3} \alpha_i P_1(w, z) + 3P_2(w, z) .$$

We note $\sum_{i=1}^{3} \alpha_i P_1(w, z) = 0$, $|w| < 1/4$; $|A/2| < |z| < |A|$ (see (3)). So if all the branches of $\psi(w, z)$ in (13) were regular at the points in $T - \{(0, 2y_0)\}$, then $P_2(w, z)$ would be regular at the same points, a contradiction. For $w = \lambda/A = \lambda_0 = 0$, $P_2(0, z) = 0$, hence $(0, 2y_0)$ is a singular point for all branches (13) (see (12)). This completes the proof of our Theorem.

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