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ON THE SEMISIMPLICITY OF GROUP RINGS OF LINEAR GROUPS II

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In this paper we continue the study of the semisimplicity problem for group rings of linear groups. We consider the case in which the characteristics of the two fields involved are both equal to $p > 0$ and we obtain appropriate necessary and sufficient conditions in terms of the abstract structure of the group.

Let $K[G]$ denote the group ring of G over the field K. In this paper we study the semisimplicity problem for $K[G]$ with G a linear group. If $\operatorname{char} K = 0$ and if G is a linear group over any field, then it is trivial to see that $JK[G] = 0$. Thus the only case of interest occurs when $\operatorname{char} K = p > 0$. A study of this situation was initiated by A. E. Zalesskii in [4] and continued somewhat in [3]. Here we solve the problem in case G is a linear group over a field L and $\operatorname{char} L = \operatorname{char} K$. Before we can properly state the result it is necessary to describe a certain characteristic subgroup $\mathscr{L}(G)$ of G. Therefore, we postpone the statement until the next section. We follow the notation of [2] and [3].

1. Normal p-subgroups. Let G be a linear group over a field L of characteristic $p > 0$. That is, G is a subgroup of the group of units of L_u, the ring of $u \times u$ matrices over L. Of course G is also contained in $\tilde{L}_u$, where $\tilde{L}$ is the algebraic closure of L and thus without loss of generality we may assume that L is algebraically closed. Thus for the remainder of this work L will denote a fixed algebraically closed field of char $p > 0$ and any subgroup of L_u for any u will be called an L-linear group.

It is apparent from [4] that a necessary ingredient here must be a consideration of the normal p-subgroups of G. We start with a few elementary observations. If G is any group let $\boldsymbol{O}_p(G)$ denote its maximal normal p-subgroup. It is clear that $\boldsymbol{O}_p(G)$ always exists. If $G \subseteq L_u$ we let LG donote its L-linear span. Thus certainly LG is an L-subalgebra of L_u.

LEMMA 1.1. *Let G be an L-linear group. Then*

(i) *$\boldsymbol{O}_p(G)$ is a nilpotent group.*

(ii) *$G/\boldsymbol{O}_p(G)$ is an L-linear group.*

(iii) *If $\boldsymbol{O}_p(G) = \langle 1 \rangle$, then G can be represented as an L-linear group in such a way that LG is semisimple.*

(iv) *If LG is semisimple and $H \lhd G$ then LH is semisimple.*

Proof. Observe that LG is a finite dimensional L-algebra so JLG, its Jacobson radical, is nilpotent. We start by proving (iv). If $x \in G$ then since $H \lhd G$, x normalizes H and hence clearly x acts as an algebra automorphism on LH. Since JLH is characteristic in LH we have $x^{-1}(JLH)x = JLH$ so $(JLH)x = x(JLH)$. Thus since LG is spanned by all such x we obtain easily $(JLH)(LG) = (LG)(JLH)$. Now JLH is nilpotent and therefore by the above so is the ideal $(JLH)(LG)$. Thus $(JLH)(LG) \subseteqq JLG = 0$ and $JLH = 0$. This yields (iv).

Now let $\mu: LG \rightarrow LG/JLG$ be the natural map and let $P = \{g \in G \mid \mu(g) = 1\}$. Since $G \subseteqq LG$, P is a subgroup of $U = \{1 + \alpha \mid \alpha \in JLG\} \subseteqq LG$. Now JLG is nilpotent and char $L = p > 0$ so we see easily that U is a nilpotent p-group. Thus P is a nilpotent p-group and $P \subseteqq \boldsymbol{O}_p(G)$.

Now $\mu(LG) = LG/JLG$ is a finite dimensional L-algebra so it is contained in L_w for some integer w. Furthermore, LG/JLG contains the group $\bar{G} = G/P$ and is clearly spanned by it. This shows that $\bar{G}$ is an L-linear group with $L\bar{G}$ semisimple. If $\boldsymbol{O}_p(G) = \langle 1 \rangle$ then certainly $P = \langle 1 \rangle$ so $G = \bar{G}$ and (iii) is proved.

Observe that if we show that $P = \boldsymbol{O}_p(G)$ then (i) and (ii) will follow and to do this we need only show that $\bar{Q} = \boldsymbol{O}_p(\bar{G}) = \langle 1 \rangle$. Since $L\bar{G}$ is semisimple, part (iv) and $\bar{Q} \lhd \bar{G}$ implies that $L\bar{Q}$ is also semisimple. Let I be the subalgebra of $L\bar{Q}$ spanned by all $1 - x$ with $x \in \bar{Q}$. Then I is an ideal of $L\bar{Q}$ and I is a finite dimensional algebra (without 1) spanned by the nilpotent elements $1 - x$. As is well known (see for example the proof of Lemma 10.1 (ii) of [2]) this implies that I is nilpotent so $I \subseteqq JL\bar{Q} = 0$. If $x \in \bar{Q}$ then $1 - x \in I = 0$ so $x = 1$. Thus $\bar{Q} = \langle 1 \rangle$ and the lemma is proved.

Let G be any group and let H be a subgroup of G. We set

$$\boldsymbol{D}_G(H) = \{x \in G \mid [H: \boldsymbol{C}_H(x)] < \infty\} .$$

Clearly $\boldsymbol{D}_G(H)$ is a subgroup of G and if H is normal or characteristic in G then so is $\boldsymbol{D}_G(H)$. Furthermore,

$$\boldsymbol{D}_G(G) = \Delta(G) = \{x \in G \mid [G: \boldsymbol{C}_G(x)] < \infty\}$$

is the F. C. subgroup of G. Finally $\Delta^p(G)$ is defined to be the subgroup of $\Delta(G)$ generated by all p-elements, that is elements whose order is a power of p. We say that G is a Δ-group if $G = \Delta(G)$.

LEMMA 1.2. *Let G be an L-linear group.*

(i) *If $H \lhd G$ and $G = \boldsymbol{D}_G(H)$ then $[H: H \cap \boldsymbol{Z}(G)] < \infty$.*

(ii) *If $\boldsymbol{O}_p(G) = \langle 1 \rangle$ then $\Delta^p(G)$ is finite.*

Proof. Since LG is finite dimensional we can choose some finite number of group elements $x_1, x_2, \cdots, x_n$ which span LG. By assumption for each i, $[H: C_H(x_i)] < \infty$ and thus by Lemma 1.1 of [2], $[H: Z] < \infty$ where $Z = \bigcap_1^n C_H(x_i)$. Now $Z \subseteq LG$ is centralized by a spanning set so it is, therefore, centralized by all of LG and hence by all of G. This shows that $Z \subseteq Z(G)$ and thus (i) follows.

Suppose $O_p(G) = \langle 1 \rangle$ and set $H = \Delta^p(G)$. Then $H = \Delta(H)$ so by part (i) applied to H we conclude that $[H: Z(H)] < \infty$. Now $O_p(G) = \langle 1 \rangle$ and $Z(H) \lhd G$ so $O_p(Z(H)) = \langle 1 \rangle$ and since $Z(H)$ is abelian this says that $Z(H)$ has no elements of order p. Thus $\Delta^p(Z(H)) = \langle 1 \rangle$. On the other hand, since $[\Delta^p(G): Z(H)] < \infty$, Lemma 19.3 (v) of [2] implies that $[Z(H): \Delta^p(Z(H))] < \infty$. Thus $\Delta^p(Z(H)) = \langle 1 \rangle$ yields $|Z(H)| < \infty$ and hence $|H| < \infty$. This completes the proof.

Let G be any group. We define a characteristic subgroup $\mathscr{L}(G)$ of G as follows. Let $P = O_p(G)$ and set $G^* = D_G(P)$ so that $G^* \cap P = D_P(P) = \Delta(P)$. Then $\mathscr{L}(G)$ is the subgroup of G^* given by

$$G^* \supseteq \mathscr{L}(G) \supseteq \Delta(P), \qquad \mathscr{L}(G)/\Delta(P) = \Delta^p(G^*/\Delta(P)) .$$

LEMMA 1.3. *Let G be an L-linear group. Then with the above notation $[\mathscr{L}(G): \Delta(P)]$ is finite and $\mathscr{L}(G)$ is a characteristic Δ-subgroup of G.*

Proof. $\mathscr{L}(G)$ is clearly characteristic by its construction. Now $G^* \lhd G$ so $O_p(G^*) \subseteq O_p(G) = P$ and thus $O_p(G^*) = \Delta(P)$. Therefore, by Lemma 1.1 (ii), $G^*/\Delta(P)$ is an L-linear group and certainly $O_p(G^*/\Delta(P)) = \langle 1 \rangle$. Thus Lemma 1.2 (ii) implies that $\Delta^p(G^*/\Delta(P))$ is finite and we see that $[\mathscr{L}(G): \Delta(P)]$ is finite. Furthermore, since clearly $G^* = D_{G^*}(\Delta(P))$, Lemma 1.2 (ii) yields $[\Delta(P): \Delta(P) \cap Z(G^*)] < \infty$ and this and the above show that $\mathscr{L}(G)$ has a center of finite index. Therefore, $\mathscr{L}(G)$ is a Δ-group.

We can now state our main result. If H is a subgroup of G we say that H has locally finite index in G and write $[G: H] = \text{l.f.}$ if for all finitely generated subgroups S of G we have $[S: S \cap H] < \infty$.

THEOREM. *Let K be a field of characteristic $p > 0$ and let G be a linear group over a field of the same characteristic p. Then $JK[G] \neq 0$ if and only if there exists an element $h \in \mathscr{L}(G)$ of order p with $[G: C_G(h)] = l.f.$*

Observe that the above necessary and sufficient conditions concern the abstract structure of G and not how G is written as a linear group.

2. The case: $O_p(G) = \langle 1 \rangle$. The linear groups with $O_p(G) = \langle 1 \rangle$ were studied in [4] under the additional assumption that $K = L$, that is the two fields are the same, and the semisimplicity problem was solved in that case. Here we modify the original argument slightly to handle the case in which K and L are different.

If S is a subset of any group G we say that S has finite index in G and write $[G: S] < \infty$ if G can be written as a finite union, $G = \bigcup_1^n Sx_i$, of right translates of S.

LEMMA 2.1. *Let G be an L-linear group and let $T_1, T_2, \cdots, T_j$ be a finite number of L-subspaces of LG properly smaller than LG. Let S be a subset of G and suppose that*

$$G = S \cup \bigcup_1^j (G \cap T_i) .$$

Then either $[G: S] < \infty$ or $G = \bigcup_1^j (G \cap T_i)$ and G has a subgroup H of finite index with $LH \neq LG$.

Proof. We assume that $[G: S]$ is infinite and we consider all ways of writing G as a finite union

$$G = \bigcup_1^s Sx_i \cup \bigcup_1^t (G \cap M_i)$$

where $x_i \in G$ and the M_i are L-subspaces of LG each contained in some $T_{i'}$. By assumption such a decomposition exists. For each such union we associate an ordered pair (d, r) where $d = \max \dim M_i$ and r is the number of M_i of dimension d. We say $(d_1, r_1) < (d_2, r_2)$ if $d_1 < d_2$ or $d_1 = d_2$ and $r_1 < r_2$. This then is a well ordering and assume the above union is so chosen that (d, r) is minimal. By definition $d < \dim LG$. We may assume that $\dim M_i = d$ for $i = 1, 2, \cdots, r$. Note that the M_i terms must occur since $[G: S] = \infty$.

Fix $k \leqq r$ and $g \in G$. Then

$$(G \cap M_k)g \subseteqq G = \bigcup_1^s Sx_i \cup \bigcup_1^t (G \cap M_i)$$

so

$$\begin{aligned} G \cap M_k &\subseteqq \bigcup_1^s Sx_ig^{-1} \cup \bigcup_1^t (G \cap M_i)g^{-1} \\ &= \bigcup_1^s Sx_ig^{-1} \cup \bigcup_1^t (G \cap M_ig^{-1}) \end{aligned}$$

and thus

$$G \cap M_k \subseteqq \bigcup_1^s Sx_ig^{-1} \cup \bigcup_1^t (G \cap (M_ig^{-1} \cap M_k)) .$$

Thus replacing the term $G \cap M_k$ in the original union by the above yields a new such union with the subspace M_k replaced by the finitely many subspaces $M_i g^{-1} \cap M_k$ for $i = 1, 2, \cdots, t$. If $\dim (M_i g^{-1} \cap M_k) < \dim M_k$ for all i, we then get a new decomposition with some smaller parameter (d', r'). Since this cannot happen we conclude that for some i, $M_i g^{-1} \cap M_k = M_k$ or $M_i \supseteqq M_k g$. Since M_k has the largest dimension of all the subspaces we therefore have $M_i = M_k g$ for some $i \leqq r$.

We have therefore shown that G permutes by right multiplication the subspaces $M_1, M_2, \cdots, M_r$ and hence if H is the stabilizer of M_1 then $[G: H] < \infty$. If $LH = LG$ then $M_1 H = M_1$ implies that $M_1(LH) = M_1$ and then $M_1 G = M_1$. Again by the minimality of (d, r) and $[G: S] = \infty$ we have $G \cap M_1 \neq \emptyset$ so let $y \in G \cap M_1$. Then $M_1 G = M_1$ yields $M_1 \supseteqq yG = G$. Thus $M_1 \supseteqq LG$, a contradiction. This shows that $LH \neq LG$ and therefore LH is a proper subalgebra of LG.

Finally let $1 = g_1, g_2, \cdots, g_m$ be a set of right coset representatives for H in G. By renumbering the M_i's if necessary we may assume that $M_1 g_i = M_i$. Let $T_{i'}$ be chosen with $M_i \subseteqq T_{i'}$. Now $M_1 H = M_1$ yields $yH \subseteqq M_1$ so $yHg_i \subseteqq M_1 g_i = M_i$. Thus

$$G = yG = \bigcup_1^m yHg_i \subseteqq \bigcup_1^m M_i \subseteqq \bigcup_1^m T_{i'}$$

so clearly $G = \bigcup_1^j (G \cap T_i)$ and the lemma is proved.

For the remainder of this work we let K denote a fixed field of characteristic p. If G is a group and if $x, y \in G$ we use $x \sim_G y$ to indicate that x and y are conjugate in G.

LEMMA 2.2. *Let* $\alpha = \sum_{i=1}^k a_i g_i \in K[G], \alpha \neq 0$ *and suppose that* α *is nilpotent. Then for some* $i \neq j$ *and some integer* n *we have* $g_i^{p^n} \sim_G g_j^{p^n}$.

Proof. Let S denote the subspace of $K[G]$ spanned by all Lie products $[\beta, \gamma] = \beta\gamma - \gamma\beta$ with $\beta, \gamma \in K[G]$. Then S is spanned by all Lie products $[x, y] = xy - yx$ with $x, y \in G$. Now $yx = x^{-1}(xy)x$ so $yx \sim_G xy$ and, therefore, we see that if $\delta \in S$ then the sum of the coefficients of δ over any conjugacy class of G is zero.

By assumption α is nilpotent so we can choose $n \geqq 0$ with $\alpha^{p^n} = 0$. Then Lemma 3.4 of [2] yields

$$0 = \alpha^{p^n} = \sum_{i=1}^k a_i^{p^n} g_i^{p^n} + \delta$$

for some $\delta \in S$. If $a_i \neq 0$ then since the sum of the coefficients in the conjugacy class of $g_i^{p^n}$ must be zero in the above and since the contribution of δ to this sum is zero, we conclude that some $j \neq i$ must exist with $g_j^{p^n} \sim_G g_i^{p^n}$.

LEMMA 2.3. *Let G be an L-linear group with LG semisimple. Since L is algebraically closed, LG is a finite direct sum of full matrix rings over L and we embed LG in L_u for some u by placing the matrix rings of LG in blocks along the diagonal of L_u. Then* tr, *the matrix trace map on L_u, yields a nondegenerate symmetric bilinear form $(\alpha, \beta) = \operatorname{tr} \alpha\beta$ on LG.*

Proof. The form $(\alpha, \beta) = \operatorname{tr} \alpha\beta$ is certainly bilinear and symmetric. We need only show that it is nondegenerate on LG. Let $\alpha \in LG$ with $(\alpha, LG) = 0$. Then

$$\operatorname{tr}(LG)\alpha(LG) = \operatorname{tr} \alpha(LG)(LG) = \operatorname{tr} \alpha(LG) = 0$$

so every element of the ideal $(LG)\alpha(LG)$ has trace zero. But any nonzero ideal of LG contains one of the full matrix ring and certainly all its elements cannot have trace 0. Thus α must be zero and the lemma is proved.

We now obtain our generalization of Zalesskii's result by modifying the proof of [4]. It is apparent that the proof could be greatly simplified if we only knew that the radical was a nil ideal.

LEMMA 2.4. *Let G be an L-linear group with $\boldsymbol{O}_p(G) = \langle 1 \rangle$. Then G has a normal subgroup G_0 of finite index and a representation of G_0 as an L-linear group in such a way that LG_0 is semisimple and if $[G_0 : H] < \infty$ then $LG_0 = LH$.*

Proof. Since $\boldsymbol{O}_p(G) = \langle 1 \rangle$, Lemma 1.1 (iii) implies that G can be represented as an L-linear group with LG semisimple. We now consider all normal subgroups H of G of finite index and all ways in which H can be represented as an L-linear group with LH semisimple and we choose G_0 to give the minimum possible dimension of LG_0.

Thus we have $G_0 \lhd G$, $[G : G_0] < \infty$ and G_0 is an L-linear group with LG_0 semisimple. Furthermore, let H be a subgroup of G_0 of finite index. Then $[G : H] < \infty$ so H_0, the intersection of the finitely many G-conjugates of H, is a normal subgroup of G of finite index. Since $H_0 \lhd G_0$ we have LH_0 semisimple by Lemma 1.1 (iv) and thus by the minimality of the dimension of LG_0 we have $LG_0 = LH_0$ and hence $LG_0 = LH$.

PROPOSITION 2.5. *Let G be an L-linear group with $\boldsymbol{O}_p(G) = \langle 1 \rangle$. Then $JK[G]$ is nilpotent.*

Proof. Let G_0 be the normal subgroup of G of finite index given in the preceding lemma and let us write LG_0 as described in Lemma 2.3. Thus $LG_0 \subseteqq L_u$ and tr yields a nondegenerate bilinear form on LG_0. We show now that $K[G_0]$ is semisimple.

Suppose by way of contradiction that $\alpha = \sum_{i=1}^k a_i g_i \in JK[G_0]$ with $\alpha \neq 0$ and with the group elements g_i distinct. If $x \in G_0$ then also $\alpha x = \sum_{i=1}^k a_i g_i x \in JK[G_0]$. Thus if G_1 is the finitely generated subgroup of G_0 given by $G_1 = \langle g_1, g_2, \cdots, g_k, x\rangle$ then $\alpha x \in JK[G_0] \cap K[G_1] \subseteqq JK[G_1]$ by Lemma 16.9 of [2]. We show now that for some $i \neq j$, $\operatorname{tr}(g_i x) = \operatorname{tr}(g_j x)$.

Suppose this is not the case and let $G\widetilde{F}(p)$ denote the algebraic closure of $GF(p)$. Since G_1 is a finitely generated subgroup of L_u we can find, by the Extension Theorem for Places, a place $\varphi\colon L \to G\widetilde{F}(p) \cup \{\infty\}$ such that φ is finite on all the matrix entries of the generators of G_1 and their inverses and furthermore for all $i \neq j$, $\varphi(\operatorname{tr}(g_i x)) \neq \varphi(\operatorname{tr}(g_j x))$. If $\mathcal{O}$ denotes the corresponding valuation ring in L then clearly $G_1 \subseteqq \mathcal{O}_u$ and φ can be extended to a homomorphism $\varphi\colon \mathcal{O}_u \to G\widetilde{F}(p)_u$ and therefore $\varphi(G_1)$ is finite.

Consider the natural map $\eta\colon K[G_1] \to K[\varphi(G_1)]$. Since η is an epimorphism, $\eta(JK[G_1]) \subseteqq JK[\varphi(G_1)]$ and thus

$$\eta(\alpha x) = \sum_{i=1}^{k} a_i \varphi(g_i x) \in JK[\varphi(G_1)] .$$

Now $\varphi(G_1)$ is finite so $JK[\varphi(G_1)]$ is nilpotent and therefore $\sum_{i=1}^k a_i\varphi(g_i x)$ is nilpotent. Thus Lemma 2.2 implies that for some $i \neq j$ and some integer n, $\varphi(g_i x)^{p^n} \sim_{\varphi(G_1)} \varphi(g_j x)^{p^n}$. Let $\widetilde{\operatorname{tr}}$ denote the trace map in $G\widetilde{F}(p)_u$. Since similar matrices have the same trace and since the fields have characteristic $p > 0$ we conclude that

$$[\widetilde{\operatorname{tr}}\, \varphi(g_i x)]^{p^n} = \widetilde{\operatorname{tr}}\, [\varphi(g_i x)^{p^n}] = \widetilde{\operatorname{tr}}\, [\varphi(g_j x)^{p^n}] = [\widetilde{\operatorname{tr}}\, \varphi(g_j x)]^{p^n}$$

and thus $\widetilde{\operatorname{tr}}\, \varphi(g_i x) = \widetilde{\operatorname{tr}}\, \varphi(g_j x)$. But certainly $\widetilde{\operatorname{tr}} \circ \varphi = \varphi \circ \operatorname{tr}$ so we obtain

$$\varphi(\operatorname{tr}(g_i x)) = \widetilde{\operatorname{tr}}\, \varphi(g_i x) = \widetilde{\operatorname{tr}}\, \varphi(g_j x) = \varphi(\operatorname{tr}(g_j x))$$

a contradiction.

We have, therefore, shown that for each $x \in G_0$ there exists some $i \neq j$ with $\operatorname{tr} g_i x = \operatorname{tr} g_j x$. For each $i \neq j$ let T_{ij} be the L-subspace of LG_0 given by

$$T_{ij} = \{\delta \in LG_0 \mid \operatorname{tr}(g_i - g_j)\delta = 0\} .$$

Since tr yields a nondegenerate bilinear form we see that $T_{ij} \neq LG_0$ and by the above we have

$$G = \bigcup_{i \neq j} G \cap T_{ij} .$$

But then Lemma 2.1 with $S = \emptyset$ implies that G_0 has a subgroup H of finite index with $LH \neq LG_0$, a contradiction. This shows that $K[G_0]$ is semisimple. Since $[G\colon G_0] < \infty$, Lemma 16.8 of [2] implies that $JK[G]$ is nilpotent and result follows.

3. A local situation. We now study a group G with a rather special structure. We say G has property (*) if G has a normal series $G \supseteqq W \supseteqq P \supseteqq Z$ satisfying

1. G/W is infinite cyclic.
2. $\bar{G} = G/P$ is an L-linear group with $\boldsymbol{O}_p(\bar{G}) = \langle 1 \rangle$.
3. P is an abelian p-group.
4. $[P\colon Z] < \infty$ and W centralizes Z.

We say that G has property (**) if G satisfies all of the above and in addition

5. $P \cap \Delta(G) = \langle 1 \rangle$.

Our aim is essentially to completely determine $JK[G]$ if G satisfies (*). We start by assuming that G satisfies (**) and prove that $JK[G]$ is nilpotent. For the remainder of this section we assume that G satisfies (**) and is given as above. We start by introducing some more notation.

LEMMA 3.1. *There exists a subgroup G_0 of G of finite index with $G \supseteqq G_0 \supseteqq P$ and such that*

(i) *$\bar{G}_0 = G_0/P$ has a representation as an L-linear group with $L\bar{G}_0$ semisimple and with $L\bar{G}_0 = L\bar{H}$ for all subgroups $\bar{H} \subseteqq \bar{G}_0$ of finite index.*

(ii) *G_0 centralizes the quotient P/Z.*

(iii) *If $W_0 = G_0 \cap W$ then G_0/W_0 is infinite cyclic.*

Proof. The existence of a group G_0 satisfying (i) is an immediate consequence of Lemma 2.4. Furthermore, it is clear that this same property holds for any subgroup of G_0 of finite index which contains P. Now G_0 acts on finite group P/Z and P centralizes this quotient. Thus we may certainly replace G_0 by $C_{G_0}(P/Z)$ if necessary and then this new G_0 also satisfies (ii). Finally

$$G_0/W_0 = G_0/(W \cap G_0) \cong G_0W/W$$

is a subgroup of finite index in the infinite cyclic group G/W and the result follows.

We will show that $K[G_0]$ is semisimple. Thus by way of contradiction we assume now that G_0 is given as above and $JK[G_0] \neq 0$.

LEMMA 3.2. *There exists a nonzero element* $\gamma = \alpha\beta \in JK[G_0] \cap K[W_0]$ *satisfying*

(i) $\alpha = \hat{Q}$, *the sum of all the elements of* Q, *where* Q *is a finite subgroup of* P.

(ii) $\beta = \sum_{i=1}^{n} a_i g_i$ *where the* g_i *are in distinct cosets of* P *in* W_0.

(iii) γ centralizes $K[P]$.

Proof. By assumption $JK[G_0] \neq 0$ and since G_0/W_0 is infinite cyclic Theorem 17.7 of [2] implies that

$$I = JK[G_0] \cap K[\mathrm{W}_0]$$

is a nonzero ideal of $K[W_0]$. Choose $\gamma \in I, \gamma \neq 0$ such that Supp γ is contained in the smallest number n of cosets of P. By multiplying γ by a group element if necessary we may assume that one of these cosets is the identity coset. Thus

$$\gamma = \sum_{i=1}^{n} \alpha_i g_i$$

where $\alpha_i \in K[P]$ and $g_1 = 1, g_2, \cdots, g_n$ are in distinct cosets of P.

Let Q be the subgroup of P generated by the support of all the α_i. Then Q is a finitely generated and hence finite subgroup of the abelian p-group P. Therefore, as is well known, the unique minimal ideal of $K[Q]$ consists of all K-multiples of $\hat{Q}$ and thus $\hat{Q}$ is a multiple of α_1 in $K[Q]$. By multiplying γ on the left by this suitable factor we may clearly assume that $\alpha_1 = \hat{Q}$. Let $h \in Q$. Then $(1 - h)\alpha_1 = 0$ so $(1 - h)\gamma \in I$ has support contained in a smaller number of cosets. This implies that $(1 - h)\gamma = 0$ for all $h \in Q$ and thus we have for all i, $\alpha_i = a_i\hat{Q}$ for some $a_i \in K$. This yields

$$\gamma = \alpha\beta = \hat{Q}\left(\sum_{i=1}^{n} a_i g_i\right)$$

and (i) and (ii) are proved.

Finally let $h \in P$. Since P is abelian and $g_1 = 1$ we see $h^{-1}\gamma h - \gamma \in I$ has support in fewer cosets of P. By the minimality of n we conclude that $h^{-1}\gamma h - \gamma = 0$ for all $h \in P$ and (iii) follows.

We now define an even smaller subgroup of G. Again we fix the above notation for the remainder of this section. Let

$$T = \{h \in Q \mid h \neq 1, \boldsymbol{C}_{G_0}(h) \not\subseteq W_0\} .$$

Now define the subgroup G_1 by

$$G_1 = \bigcap_{h \in T} W_0 \boldsymbol{C}_{G_0}(h)$$

with the understanding that $G_1 = G_0$ if $T = \varnothing$.

LEMMA 3.3. *Let G_1 be as above. Then*
(i) $G_0 \supseteqq G_1 \supseteqq W_0$, $[G_0: G_1] < \infty$ *and G_1/W_0 is infinite cyclic.*
(ii) *If $h \in T$ then $G_1 = W_0 \boldsymbol{C}_{G_1}(h)$.*

Proof. By definition we have $G_0 \supseteqq G_1 \supseteqq W_0$. Moreover, G_1/W_0 is the intersection of finitely many nonidentity subgroups of the infinite cyclic group G_0/W_0. Thus G_1/W_0 is infinite cyclic and $[G_0: G_1] < \infty$.
Finally let $h \in T$. Then $W_0 \subseteqq G_1 \subseteqq W_0 \boldsymbol{C}_{G_0}(h)$ so

$$G_1 = W_0(G_1 \cap \boldsymbol{C}_{G_0}(h)) = W_0 \boldsymbol{C}_{G_1}(h)$$

and the lemma is proved.

The reason for working with G_1 rather than G_0 will be apparent in the following result.

LEMMA 3.4. *Let $x \in G_1 - W_0$ and let α be as above. Suppose that for infinitely many integers s (positive or negative) there exists an integer $r = r(s) \geqq 1$ with*

$$\alpha \alpha^{x^{-s}} \alpha^{x^{-2s}} \cdots \alpha^{x^{-rs}} = 0 .$$

Then for some $h \in T$ we have $x \in \boldsymbol{C}_{G_1}(h)$.

Proof. The assumption on α clearly implies that for each such s the group $QQ^{x^{-s}}Q^{x^{-2s}} \cdots Q^{x^{-rs}}$ is not a direct product of the indicated factors. Since there are infinitely many such s there are certainly infinitely many positive or infinitely many negative ones. Therefore, by Lemmas 3 and 4 of [1], there exists $h \in Q$, $h \neq 1$ and a positive integer m with x^{-m} or x^m in $\boldsymbol{C}_{G_1}(h)$ and hence $x^m \in \boldsymbol{C}_{G_1}(h)$. Now $x \in G_1 - W_0$ and G_1/W_0 is infinite cyclic so $x^m \notin W_0$ and by definition of T we must have $h \in T$.

Since $G_1 = W_0 \boldsymbol{C}_{G_1}(h)$ by Lemma 3.3 (ii) we can write $x = wy$ with $w \in W_0$ and $y \in \boldsymbol{C}_{G_1}(h)$. Therefore, $y \in \boldsymbol{C}_{G_1}(h)$, $x^m = (wy)^m \in \boldsymbol{C}_{G_1}(h)$ and since W_0 centralizes P/Z we have $h^w = hz$ for some $z \in Z$. It then follows easily by induction on i that

$$h^{(wy)^i} = hz^y z^{y^2} \cdots z^{y^i}$$

and therefore

$$h = h^{(wy)^m} = hz^y z^{y^2} \cdots z^{y^m}$$

so we have $z^y z^{y^2} \cdots z^{y^m} = 1$. We now conjugate this last expression by y^{-1} and obtain

$$zz^y \cdots z^{y^{m-1}} = 1 = z^y z^{y^2} \cdots z^{y^m} .$$

Thus since P is abelian we have $z = z^{y^m}$.

Since G satisfies (**) we know that W centralizes Z and thus we have $C_{G_1}(z) \supseteqq \langle W_0, y^m \rangle$. Furthermore, G_1/W_0 is infinite cyclic and $y \notin W_0$ since $x \notin W_0$ so clearly $[G_1: C_{G_1}(z)] < \infty$. Hence $[G: C_G(z)] < \infty$ and we have $z \in P \cap \Delta(G)$. Again by assumption (**), $P \cap \Delta(G) = \langle 1 \rangle$ so $z = 1$. Finally $h^x = h^{wy} = hz^y = h$ so $x \in C_{G_1}(h)$ and the result follows.

Let $^-$ denote the natural map $G_0 \to \bar{G}_0 = G_0/P$ and we extend this to the map $K[G_0] \to K[\bar{G}_0]$. Thus for $\beta = \sum_{i=1}^n a_i g_i$ as given before we have $\bar{\beta} = \sum_{i=1}^n a_i \bar{g}_i$. We now represent $\bar{G}_0$ as an L-linear group as in Lemma 3.1 (i) so that $L\bar{G}_0$ is semisimple.

LEMMA 3.5. *We can embed $L\bar{G}_0$ in the matrix ring L_u in such a way that* tr, *the matrix trace map on L_u, yields a nondegenerate symmetric bilinear form $L\bar{G}_0$. Futhermore, if for each $i \neq j$ we define T_{ij} by*

$$T_{ij} = \{\bar{\delta} \in L\bar{G}_0 \mid \operatorname{tr} (\bar{g}_i - \bar{g}_j)\bar{\delta} = 0\}$$

then T_{ij} is a proper L-subspace of $L\bar{G}_0 = L\bar{G}_1$.

Proof. The first part follows immediately from Lemma 3.1 (i) and Lemma 2.3. The second part about T_{ij} follows from the nondegeneracy of the bilinear form and the fact that $\bar{g}_i \neq \bar{g}_j$ by Lemma 3.2 (ii). Finally $L\bar{G}_0 = L\bar{G}_1$ by Lemma 3.1 (i).

LEMMA 3.6. *Let $x \in G_1 - W_0$ and let β be as above. Suppose that $\bar{\beta}\bar{x}^s \in K[\bar{G}_1]$ is nilpotent for all integers s (positive or negative) with possibly finitely many exceptions. Then for some $i \neq j$ we have $\bar{x} \in T_{ij}$.*

Proof. Since $x \in G_1 - W_0$ and $\bar{G}_1/\bar{W}_0$ is infinite cyclic we see that $\langle \bar{x} \rangle$ is infinite. We consider $\langle \bar{x} \rangle$ as an L-linear subgroup of $\bar{G}_0$. Let V denote the finite set of exceptional integers in the above and let

$$S = \{\bar{x}^v \mid v \in V\} .$$

Then S is a finite subset of $\langle \bar{x} \rangle$ so clearly $[\langle \bar{x} \rangle : S] = \infty$. Now let s be an integer not in V. Since

$$\bar{\beta}\bar{x}^s = \sum_{i=1}^n a_i \bar{g}_i \bar{x}_i^s$$

is nilpotent we conclude from Lemma 2.2 that for some $i \neq j$ and some integer $t \geqq 0$

$$(\bar{g}_i\bar{x}^s)^{p^t} \sim_{\bar{G}_1} (\bar{g}_j\bar{x}^s)^{p^t} .$$

Thus since similar matrices have the same trace and since char $L = p > 0$ we have

$$(\operatorname{tr} \bar{g}_i\bar{x}^s)^{p^t} = \operatorname{tr} (\bar{g}_i\bar{x}^s)^{p^t} = \operatorname{tr} (\bar{g}_j\bar{x}^s)^{p^t} = (\operatorname{tr} \bar{g}_j\bar{x}^s)^{p^t} .$$

Hence $\operatorname{tr} \bar{g}_i\bar{x}^s = \operatorname{tr} \bar{g}_j\bar{x}^s$ and $\bar{x}^s \in T_{ij}$.

We have therefore shown that

$$\langle\bar{x}\rangle = S \cup \bigcup_{i \neq j} (\langle\bar{x}\rangle \cap T_{ij})$$

and since $[\langle\bar{x}\rangle : S] = \infty$, Lemma 2.1 implies that

$$\langle\bar{x}\rangle = \bigcup_{i \neq j} (\langle\bar{x}\rangle \cap T_{ij}) .$$

This shows that $\bar{x} \in T_{ij}$ for some $i \neq j$ and the lemma is proved.

We now come to the main result of this section.

PROPOSITION 3.7. *Let G be a group satisfying* (**). *Then $JK[G]$ is nilpotent.*

Proof. We use all the above notation and show first that $JK[G_0] = 0$. If this is not the case then all of the above lemmas and notation apply.

Let $x \in G_1 - W_0$ and let $s \neq 0$ be an integer (positive or negative). Since G_1/W_0 is infinite cyclic, the element x^{-s} has infinite order modulo W_0. Since $\gamma \in JK[G_0] \cap K[W_0]$, Lemma 21.3 of [2] implies that for some integer $r = r(s) \geqq 1$ we have

$$\gamma\gamma^{x^{-s}}\gamma^{x^{-2s}} \cdots \gamma^{x^{-rs}} = 0 .$$

Now $\gamma = \alpha\beta$ so this yields

$$\alpha\beta\alpha^{x^{-s}}\beta^{x^{-s}}\alpha^{x^{-2s}} \cdots \alpha^{x^{-rs}}\beta^{x^{-rs}} = 0 .$$

By Lemma 3.2 (iii) γ centralizes $K[P]$ and hence since $P \lhd G$, $\gamma^{x^{-is}}$ also centralizes $K[P]$.

We use this latter fact to rearrange the terms in the above product. First since the product is

$$\gamma\gamma^{x^{-s}} \cdots \gamma^{x^{-(r-1)s}}\alpha^{x^{-rs}}\beta^{x^{-rs}}$$

we can shift the $\alpha^{x^{-rs}}$ factor past all the $\gamma^{x^{-is}}$ and obtain

$$\alpha^{x^{-rs}}\gamma\gamma^{x^{-s}} \cdots \gamma^{x^{-(r-2)s}}\alpha^{x^{-(r-1)s}}\beta^{x^{-(r-1)s}}\beta^{x^{-rs}} .$$

We next shift the $\alpha^{x^{-(r-1)s}}$ term all the way to the left and continuing this process we clearly obtain

$$(\alpha\alpha^{x^{-s}}\alpha^{x^{-2s}}\cdots\alpha^{x^{-rs}})(\beta\beta^{x^{-s}}\beta^{x^{-2s}}\cdots\beta^{x^{-rs}}) = 0 .$$

Let σ denote the above first factor and τ the second. Suppose that $\sigma \neq 0$. Now P is an abelian p-group and char $K = p$ so $JK[P]$ is the unique maximal ideal of $K[P]$. This implies that every element of $K[P] - JK[P]$ is a unit in $K[P]$. If we now write τ as $\tau = \Sigma\tau_i y_i$ with $\tau_i \in K[P]$ and the y_i in distinct cosets of P, $\sigma\tau = 0$ and $\sigma \neq 0$ therefore implies that $\tau_i \in JK[P]$ and hence $\tau \in (JK[P])K[G_0]$. But this ideal is precisely the kernel of the homomorphism $K[G_0] \rightarrow K[\bar{G}_0]$ and therefore $\bar{\tau} = 0$. Thus

$$0 = \bar{\tau} = \bar{\beta}\bar{\beta}^{\bar{x}^{-s}}\bar{\beta}^{\bar{x}^{-2s}}\cdots\bar{\beta}^{\bar{x}^{-rs}} = (\bar{\beta}\bar{x}^s)^{r+1}\bar{x}^{-rs}$$

and $(\bar{\beta}\bar{x}^s)^{r+1} = 0$.

We have therefore shown that for each $s \neq 0$ either

$$\alpha\alpha^{x^{-s}}\alpha^{x^{-2s}}\cdots\alpha^{x^{-rs}} = 0$$

for some $r = r(s) \geqq 1$ or $\bar{\beta}\bar{x}^s$ is nilpotent. If the first fact occurs for infinitely many s then by Lemma 3.4, $x \in C_{G_1}(h)$ for some $h \in T$. If this first fact occurs for only finitely many s, then $\bar{\beta}\bar{x}^s$ is nilpotent for all but finitely many s and Lemma 3.6 yields $\bar{x} \in T_{ij}$ for some $i \neq j$.

Observe that the above holds for any $x \in G_1 - W_0$. Thus we see that

$$\bar{G}_1 = S \cup \bigcup_{i\neq j}(\bar{G}_1 \cap T_{ij})$$

where

$$S = \bar{W}_0 \cup \bigcup_{h\in T}\overline{C_{G_1}(h)} .$$

We apply Lemma 2.1 and there are two possible conclusions. First there exists a subgroup $\bar{H}$ of $\bar{G}_1$ of finite index with $L\bar{H} \neq L\bar{G}_1$. But $[\bar{G}_0: \bar{G}_1] < \infty$ so $[\bar{G}_0: \bar{H}] < \infty$ and Lemma 3.1 (i) then yields $L\bar{G}_1 = L\bar{G}_0 = L\bar{H}$, and contradiction. Secondly we have $[\bar{G}_1: S] < \infty$ and this says that $\bar{G}_1$ is a finite union of cosets of the subgroups $\bar{W}_0$ and $\overline{C_{G_1}(h)}$ for all $h \in T$. Then by Lemma 1.2 of [2] we see that one of these subgroups must have finite index in $\bar{G}_1$. Since $\bar{G}_1/\bar{W}_0$ is infinite cyclic we, therefore, have for some $h \in T$, $[\bar{G}_1: \overline{C_{G_1}(h)}] < \infty$. Moreover, $[G: G_1] < \infty$ and $C_{G_1}(h) \supseteqq P$ so this yields $[G: C_{G_1}(h)] < \infty$. Thus $h \neq 1$ and $h \in P \cap \Delta(G)$, a contradiction since G satisfies (**).

We have therefore shown that $JK[G_0] = 0$. Since $[G: G_0] < \infty$, Lemma 16.8 of [2] implies that $JK[G]$ is nilpotent and the proposition is proved.

4. The main theorem. In this section we prove our result. However, we first need a few additional facts about groups satisfying condition (*).

LEMMA 4.1. *Let G satisfy* (*) *and suppose that $P \cap \Delta(G)$ is finite. Then $JK[G]$ is nilpotent.*

Proof. Let $Q = P \cap \Delta(G) \lhd G$ and consider G/Q. Then G/Q has a normal series

$$G/Q \supseteqq W/Q \supseteqq P/Q \supseteqq ZQ/Q$$

and it is trivial to see that G/Q has property (*). In addition G/Q satisfies (**) as follows. Let $h \in P$ with $hQ/Q \in \Delta(G/Q)$. Then the G conjugates of h are contained in only finitely many cosets of Q. Since Q is finite this implies that $h \in P \cap \Delta(G) = Q$ and $hQ/Q = 1$. Thus $P/Q \cap \Delta(G/Q) = \langle 1 \rangle$ and Proposition 3.7 implies that $JK[G/Q]$ is nilpotent.

Consider the natural map $K[G] \to K[G/Q]$. Since Q is a finite p-group the kernel of this map is the nilpotent ideal $(JK[Q])K[G]$. Moreover, we have

$$JK[G]/(JK[Q])K[G] \cong JK[G/Q]$$

and since both $JK[G/Q]$ and $(JK[Q])K[G]$ are nilpotent, the lemma is proved.

LEMMA 4.2. *Let Q be a periodic normal subgroup of a group G with $Q \subseteqq \Delta(G)$. Let $g, y \in G$ and suppose that $gQ/Q \in \Delta(G/Q)$. Then there exists an integer $m \geqq 1$ such that y^m centralizes g.*

Proof. Since $hQ/Q \in \Delta(G/Q)$ it follows that some power $y^{m'}$ of y with $m' \geqq 1$ centralizes gQ/Q and thus $(y^{m'}, g) \in Q$. Moreover, since Q is a periodic normal subgroup of G contained in $\Delta(G)$, there exists a finite normal subgroup H of G with $(y^{m'}, g) \in H$. This implies that $y^{m'}$ normalizes the finite coset Hg and therefore some possibly bigger power y^m of y centralizes g.

At this point we could completely determine the structure of $JK[G]$ if G satisfies (*). However, we will content ourselves with observing the following key fact. If $\alpha \in K[G]$ we let

$$p\text{-Supp } \alpha = \{h \in \text{Supp } \alpha \mid h \neq 1 \text{ has order a power of } p\} .$$

PROPOSITION 4.3. *Let G satisfy (*) and let $x \in G$. Suppose that $\alpha \in JK[G]$ with $1 \in \text{Supp}\,\alpha$. Then there exists $h \in p\text{-Supp}\,\alpha$ and an integer $n \geqq 1$ such that x^n centralizes h and $hP/P \in \Delta^p(W/P)$.*

Proof. Let $Q = P \cap \Delta(G) \lhd G$ and consider G/Q. Then G/Q has a normal series

$$G/Q \supseteqq W/Q \supseteqq P/Q \supseteqq ZQ/Q$$

and it is trivial to see that G/Q also satisfies (*). Suppose $z \in Z$ with $zQ/Q \in \Delta(G/Q)$ and choose $y \in G$ with $G = \langle W, y\rangle$. Then Lemma 4.2 applies and we conclude that y^m centralizes z for some $m \geqq 1$. Since $z \in Z$ we therefore have $C_G(z) \supseteqq \langle W, y^m\rangle$ and hence

$$[G\colon C_G(z)] < \infty ,\, z \in P \cap \Delta(G) = Q \quad \text{and} \quad zQ/Q = 1 .$$

We have shown that the group G/Q satisfies (*) and in addition $ZQ/Q \cap \Delta(G/Q) = \langle 1\rangle$. Since $[P/Q\colon ZQ/Q] < \infty$ we therefore conclude that $P/Q \cap \Delta(G/Q)$ is finite and hence by Lemma 4.1, $JK[G/Q]$ is nilpotent.

Write α as

$$\alpha = \sum_{i=1}^{t} \alpha_i g_i$$

with $\alpha_i \in K[Q]$ and with $g_1 = 1, g_2, \cdots, g_t$ in distinct cosets of Q in G. Since $1 \in \text{Supp}\,\alpha$ we can assume that $1 \in \text{Supp}\,\alpha_i$ for all i and hence $g_i \in \text{Supp}\,\alpha$.

Suppose first that $\alpha_1 \in JK[Q]$. Since $1 \in \text{Supp}\,\alpha_1$ it follows that there exists $h \in \text{Supp}\,\alpha_1 \subseteqq \text{Supp}\,\alpha$ with $h \neq 1$. Then h has order a power of p and $h \in \Delta(G)$ so certainly x^n centralizes h for some n. Finally $hP/P = 1 \in \Delta^p(W/P)$.

Now assume that $\alpha_1 \notin JK[Q]$ and let $\sim$ denote the natural map $K[G] \to K[G/Q]$. Since Q is an abelian p-group we see that the kernel of $\sim$ is $(JK[Q])K[G]$ and therefore for each i, $\tilde{\alpha}_i = a_i\tilde{1}$ for some $a_i \in K$ and by assumption $a_1 \neq 0$. Then

$$\tilde{\alpha} = \sum_{i=1}^{t} a_i \tilde{g}_i \in JK[G/Q]$$

has $\tilde{1}$ in its support. Furthermore, $JK[G/Q]$ is nilpotent so Theorem 20.2 (i) and Lemma 3.5 of [2] imply that for some $i \neq 1$, $\tilde{g}_i \in \Delta^p(G/Q)$ and $\tilde{g}_i$ has order a power of p. Since Q is a p-group we see that g_i has order a power of p and by Lemma 4.2, x^n centralizes g_i for some

$n \geqq 1$. Now g_i has finite order a power of p and G/W is infinite cyclic so $g_i \in W$. Moreover, g_iQ/Q has only finitely many conjugates in G/Q so certainly g_iP/P has only finitely many conjugates in W/P. Thus $g_iP/P \in \Delta^p(W/P)$ and the proposition is proved.

The following is well known.

LEMMA 4.4. *Let G be a group and let H be a normal Δ-subgroup of G. Suppose that there exists an element $h \in H$ of order p with $[G: C_G(h)] = l.f.$ Then $JK[G] \cap K[H] \neq 0$.*

Proof. Let h and H be given as above and let $H^* = \langle h \rangle^H$ be the normal closure of $\langle h \rangle$ in H. Then H^* is a finite normal subgroup of H whose order is divisible by p. We show that $JK[H^*] \subseteqq JK[G]$. Since $JK[H^*] \neq 0$ and $JK[H^*] \subseteqq K[H]$ this will yield the result.

Since H^* is finite, it clearly suffices by Lemma 17.6 of [2] to show that if S is a finitely generated subgroup of G with $S \supseteqq H^*$ then $JK[H^*] \subseteqq JK[S]$. Now by definition $[S: C_S(h)] < \infty$ so since $C_S(h)$ clearly normalizes H^* we have $[S: N_S(H^*)] < \infty$. Let N denote the core of $N_S(H^*)$ in S, that is the intersection of all conjugates of $N_S(H^*)$. Then $[S: N] < \infty$ and $N \lhd S$. Since $H^* \subseteqq S \cap H \lhd S$ and $S \cap H \subseteqq N_S(H^*)$ we have $H^* \subseteqq S \cap H \subseteqq N$ and clearly $H^* \lhd N$. By Lemma 19.4 of [2], $JK[H^*] \subseteqq JK[N]$ and by Theorem 16.6 of [2], $JK[N] \subseteqq JK[S]$. Thus $JK[H^*] \subseteqq JK[S]$ and the result follows.

We can now prove our main theorem.

Proof of the Theorem. Let G be an L-linear group. Suppose first that there exists an element $h \in \mathscr{L}(G)$ of order p with $[G: C_G(h)] =$ l.f. Then by Lemmas 1.3 and 4.4 we have $JK[G] \cap K[\mathscr{L}(G)] \neq 0$ and hence $JK[G] \neq 0$.

Conversely let us assume that $JK[G] \neq 0$. There are three cases to consider.

Case 1. $O_p(G) = \langle 1 \rangle$.

By definition, $\mathscr{L}(G) = \Delta^p(G)$ here and by Proposition 2.3, $JK[G]$ is nilpotent. Thus by Theorem 20.2 there exists an element $h \in \Delta^p(G)$ of order p. Since $h \in \Delta^p(G)$ we have $[G: C_G(h)] < \infty$ and hence $[G: C_G(h)] =$ l.f.

Case 2. G has a finite normal nonidentity p-subgroup.

Let this subgroup be Q. Then $Q \subseteqq O_p(G)$ so $Q \subseteqq \Delta(O_p(G)) \subseteqq \mathscr{L}(G)$. Let h be an element of order p in Q. Then again $h \in \Delta(G)$ implies that $[G: C_G(h)] < \infty$ and hence $[G: C_G(h)] =$ l.f.

Case 3. $P = \boldsymbol{O}_p(G) \neq \langle 1 \rangle$ and G has no finite normal nonidentity p-subgroups.

Set $G^* = \boldsymbol{D}_G(P)$. Since $JK[G] \neq 0$ and P is nilpotent by Lemma 1.1 (i), it follows from results of [5], that $JK[G] \cap K[G^*] \neq 0$. Thus we may choose $\alpha \in JK[G] \cap K[G^*]$ with $1 \in \operatorname{Supp} \alpha$. We set $T = p\text{-}\operatorname{Supp} \alpha \cap \mathscr{L}(G)$.

Since P is nilpotent and $P \neq \langle 1 \rangle$ we have $\Delta(P) \neq \langle 1 \rangle$ and hence by assumption $\Delta(P)$ is infinite. On the other hand, Lemma 1.2 (i) implies that $[\Delta(P) \colon (\Delta(P) \cap \boldsymbol{Z}(G^*))] < \infty$. Thus we can choose $h_0 \in \Delta(P) \cap \boldsymbol{Z}(G^*)$ to be an element of order p. We show now that in the notation of [3]

$$G = \sqrt{\boldsymbol{C}_G(h_0)} \cup \bigcup_{h \in T} \sqrt{\boldsymbol{C}_G(h)} .$$

Let $x \in G$ and suppose first that xG^*/G^* has infinite order. We consider the group $\tilde{G} = \langle G^*, x \rangle$ and show that it satisfies condition (*). First we have the normal series

$$\tilde{G} \supseteqq G^* \supseteqq \Delta(P) \supseteqq Z$$

where $Z = \Delta(P) \cap \boldsymbol{Z}(G^*)$. By assumption $\tilde{G}/G^*$ is generated by xG^*/G^* and is therefore infinite cyclic. This yields condition (1). Now $G^* \cap P = \Delta(P)$, and since $\tilde{G}/G^*$ is infinite cyclic we have $\tilde{G} \cap P = G^* \cap P = \Delta(P)$. Thus since G/P is an L-linear group by Lemma 1.1 (ii) so is $\tilde{G}/\Delta(P) \cong \tilde{G}P/P \subseteqq G/P$. Again since $\tilde{G}/G^*$ is infinite cyclic, $\boldsymbol{O}_p(\tilde{G}) = \boldsymbol{O}_p(G^*) \lhd G$ so $\boldsymbol{O}_p(\tilde{G}) \subseteqq P \cap \tilde{G} = \Delta(P)$ and therefore $\boldsymbol{O}_p(\tilde{G}/\Delta(P)) = \langle 1 \rangle$ so condition (2) is satisfied. Moreover, Lemma 1.2 (i) clearly yields (4). Finally $\Delta(P)$ has a center of finite index so by Lemma 2.1 of [2], $\Delta(P)'$ is finite. Then this is a finite normal p-subgroup of G so by assumption $\Delta(P)' = \langle 1 \rangle$, $\Delta(P)$ is abelian and condition (3) holds.

Thus $\tilde{G}$ satisfies (*). Now $\alpha \in JK[G] \cap K[\tilde{G}] \subseteqq JK[\tilde{G}]$ by Lemma 16.9 of [2] so Proposition 4.3 implies that there exists $h \in p\text{-}\operatorname{Supp} \alpha$ and an integer $n \geqq 1$ such that x^n centralizes h and $h\Delta(P)/\Delta(P) \in \Delta^p(G^*/\Delta(P))$. Note that the latter condition really says that $h \in \mathscr{L}(G)$. Thus $h \in T$ and

$$x \in \bigcup_{h \in T} \sqrt{\boldsymbol{C}_G(h)} \subseteqq \sqrt{\boldsymbol{C}_G(h_0)} \cup \bigcup_{h \in T} \sqrt{\boldsymbol{C}_G(h)} .$$

Now let $x \in G$ with xG^*/G^* of finite order. Then $x^n \in G^*$ for some $n \geqq 1$ and hence by the choice of h_0, $x^n \in \boldsymbol{C}_G(h_0)$. Therefore, in this case also we have

$$x \in \sqrt{\boldsymbol{C}_G(h_0)} \subseteqq \sqrt{\boldsymbol{C}_G(h_0)} \cup \bigcup_{h \in T} \sqrt{\boldsymbol{C}_G(h)} .$$

Thus we have show that

$$G = \sqrt{C_G(h_0)} \cup \bigcup_{h \in T} \sqrt{C_G(h)} \; .$$

Therefore, since G is a linear group, Proposition 7 of [3] implies that for some $g \in \{h_0\} \cup T$ we have $[G: C_G(g)] = \text{l.f.}$ Now by definition $\{h_0\} \cup T \subseteq \mathscr{L}(G)$ and hence $g \neq 1$ is an element of $\mathscr{L}(G)$ of order a power of p. Finally if h is an element of order p in $\langle g \rangle$, then $h \in \mathscr{L}(G)$ and $C_G(h) \supseteq C_G(g)$ so $[G: C_G(h)] = \text{l.f.}$ and the theorem is proved.

5. Comments. The preceding proof is complicated by having to handle a number of small details. In each case if our knowledge of the situation was only a little more complete, a simplification of the proof would occur. For example, the unpleasantness of the place argument in Proposition 2.5 could be avoided if we knew that $JK[G]$ was a nil ideal. In addition much of the work in § 3 would be simpler if we could assume that $P \subseteq \Delta(W)$ or in other words if we knew that for an L-linear group G, $\Delta(P) \subseteq \Delta(G^*)$ where $P = O_p(G)$ and $G^* = D_G(P)$.

Actually even a greater simplification would occur if only we could handle the equation

$$G = \bigcup_{i=1}^{n} \sqrt{H_i} \cup \bigcup_{j=1}^{m} (G \cap T_j)$$

where the H_i are centralizer subgroups of G and the T_j are proper L-subspaces of LG where G is an L-linear group. We would of course want to conclude from the above that either $[G: H_i] = \text{l.f.}$ for some i or else that some subgroup of finite index has smaller linear span than G. However, this does not appear to be true at least in this generality. For example we have

EXAMPLE 5.1. Consider the 2×2 linear group over the complex numbers C given by

$$G = \left\{ \begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix} \middle| \, a, b \in C \text{ and } b \text{ is a root of unity} \right\} .$$

Then G has a normal subgroup H

$$H = \left\{ \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \middle| \, a \in C \right\}$$

isomorphic to C^+, the additive group of C. Note that C^+ has no proper subgroups of finite index and thus if $\tilde{G}$ is a subgroup of G of finite index then $\tilde{G} \supseteq H$ and it follows easily that $C\tilde{G} = CG$.

Let

$$T = \left\{ \begin{bmatrix} d & 0 \\ a & d \end{bmatrix} \middle| a, d \in C \right\} .$$

Then T is a proper C-subspace of CG and $H \subseteqq T$. Now suppose $x \in G - H$. Then $x = \begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix}$ for some $b \neq 1$ and thus clearly the matrix x is similar to $\begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}$. Since b is a root of unity, this implies that x has finite order and hence certainly $x \in \sqrt{C_G(g)}$ where $g = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

We have therefore shown that

$$G = \sqrt{C_G(g)} \cup (G \cap T)$$

and certainly $[G:C_G(g)]$ is not locally finite since $C_G(g) \cap H = \langle 1 \rangle$. Thus we see that we cannot conclude from such a decomposition of G what we would like to.

Finally it would appear from the main result here and also the result for solvable groups given in [5] (or see [3] for a description of this fact) that $JK[G] \neq 0$ must imply in general that G has a nonidentity normal Δ-subgroup. However, this is unfortunately not the case as we see below.

Let p be a prime and let $A = Z_p$ be the cyclic group of order p if $p > 2$ and $A = Z_4$ if $p = 2$.

LEMMA 5.2. *Let H be an infinite p-group and let G be the Wreath product $G = A \wr H$. If N is a normal Δ-subgroup of G then N is contained in the normal abelian subgroup of G which in ΣA.*

Proof. Write $G = WH$ where $W = \Sigma A$ is the direct sum of copies of A, one for each element of H. If $N \not\subseteq W$ choose $x \in N - W$ with $x^p \in W$. Then $N \supseteqq (x, W)$ but we see easily since H is infinite that $[(x, W): C_{(x,W)}(x)] = \infty$, a contradiction.

EXAMPLE 5.3. Let G_1 be an infinite locally finite p-group and define $G_1 \subseteqq G_2 \subseteqq G_3 \subseteqq \cdots$ inductively by $G_{n+1} = A \wr G_n$. Then $G = \bigcup_{n=1}^{\infty} G_n$ is a locally finite p-group. If $N \neq \langle 1 \rangle$ is a normal Δ-subgroup of G choose n so that $N \cap G_n \neq \langle 1 \rangle$. Then $N \cap G_{n+1}$ is a normal Δ-subgroup of $G_{n+1} = A \wr G_n$ not contained in ΣA, a contradiction by the above lemma.

Thus G has no nonidentity normal Δ-subgroup. On the other hand, if K is a field of characteristic p then $JK[G]$ is the augmentation ideal of $K[G]$, since G is a locally finite p-group. Therefore, $JK[G] \neq 0$.

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