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# GROUPS IN WHICH AUT($G$) IS TRANSITIVE ON THE ISOMORPHISM CLASSES OF $G$

ALBERT D. POLIMENI

**Let $G$ be a finite group and let $A(G)$ denote the group of automorphisms of $G$. $G$ is called a $T_1$-group if whenever $L_1$ and $L_2$ are isomorphic subgroups of $G$ there is a $\phi \in A(G)$ such that $L_1^\phi = L_2$. It is the object of this paper to determine structural properties of $T_1$-groups of odd order. In particular, if $G$ is a $T_1$-group of odd order, then it is shown that $G$ is a split extension of a Hall subgroup $H$, which is a direct product of homocyclic groups, by a groups $K$ whose Sylow subgroups are cyclic. If $G$ is a supersolvable group of odd order then it is shown that $G$ is a $T_1$-group if and only if $G = HK$ as in the previous sentence and $K$ is cyclic with elements which induce power automorphisms on $H$. Finally, it is shown that if $G$ is a $T_1$-group of odd order, then every subnormal subgroup of $G$ is normal if and only if $G$ is supersolvable.**

1. Preliminaries. Gaschutz and Yen [3] call a group $G$ a $T(p)$-group, $p$ prime, if $A(G)$ acts transitively on the elements of $G$ of order $p$. $G$ is called a $T$-group if it is a $T(p)$-group for all primes $p$. The main theorem of their paper is

THEOREM A. *If $G$ is a $T$-group of odd order, then $G$ is an extension of a nilpotent Hall subgroup by a cyclic group.*

The important steps in their proof are:

(a) If $M$ is a minimal characteristic subgroup of the solvable $T(p)$-group $G$, then $G/M$ is a $T(p)$-group.

(b) Any Sylow $p$-subgroup of a solvable $T(p)$-group is also a $T(p)$-group.

(c) If $p$ is an odd prime and $G$ is a solvable $T(p)$-group, then $G$ has $p$-length $\leqq 1$.

(d) A $T$-group of odd order has a normal Sylow subgroup.

(e) If $G$ is a $T$-group of odd order, then the nonnormal Sylow subgroups of $G$ are cyclic.

Let $p$ be a prime. We call a group $G$ a $T_1(p)$-group if $A(G)$ acts transitively on the subgroups of $G$ of order $p$. Certainly a $T_1$-group is a $T_1(p)$-group for each prime $p$. It is of interest to point out that the Tits simple group (it has order $2^{11}\cdot 3^3\cdot 5^2\cdot 13$) is a $T_1(3)$-group; in fact, its subgroups of order 3 are conjugate. However, it has extraspecial Sylow 3-subgroups, which are not $T_1(3)$-groups.

It turns out that one can show that properties (a)—(e) are satisfied if $T_1$ replaces $T$, the proofs being similar to those given by Gaschutz and Yen. For the sake of completeness we will include these arguments. In the discussions which follow we will make frequent use of the Feit-Thompson result [2], that all groups of odd order are solvable.

LEMMA 1. *Let $G$ be a group of odd order. Then properties* (a)—(e) *above hold if $T$ is replaced by $T_1$.*

*Proof.* (a) Let $M$ be a minimal characteristic subgroup of the solvable $T_1(p)$-group $G$, then $M$ is an elementary abelian $q$-group for some prime $q$.

*Case* 1. $p \neq q$. Let $xM$ and $yM$ be elements of $G/M$ of order $p$, so we can assume $|x| = |y| = p$. Thus there is a $\phi \in A(G)$ such that $\langle x\rangle^\phi = \langle y\rangle$. The map $\phi$ induces an automorphism $\bar{\phi}$ of $G/M$ and $\langle xM\rangle^{\bar{\phi}} = \langle yM\rangle$.

*Case* 2. $p = q$. Let $xM$ be an element of $G/M$ of order $p$. Since $p = q$ and $G$ is a $T_1(p)$-group $M = \{g \in G \mid g^p = 1\}$, thus $x$ must have order $p^2$. We will be finished if we can show that every cyclic subgroup of order $p^2$ can be mapped (mod $M$) by an automorphism of $G$ to a fixed cyclic subgroup of order $p^2$. Let $P$ be a Sylow $p$-subgroup of $G$ and let $u$ be an element of the second center of $P$ of order $p^2$. Let $x$ be an element of $G$ of order $p^2$, then we can find an inner automorphism of $G$ which will map $x$ into $P$. So we assume $x \in P$. By the $T_1(p)$-hypothesis on $G$ there is a $\phi \in A(G)$ such that $\langle x^p\rangle^\phi = \langle u^p\rangle$. It can be shown without too much trouble that $P \subseteqq C_G(M)$, so we can choose $c \in C_G(M)$ so that $P^{\phi c} = P$. If $(x^p)^\phi = u^{lp}$, then since $u$ is in the second center of $P$

$$((x^{\phi c})u^{-l})^p = (x^{\phi c})^p u^{-lp}[u^{-l}, x^{\phi c}]^{p(p-1)/2} = [u^{-lp}, x^{\phi c}]^{(p-1)/2} = 1 .$$

Thus $x^{\phi c}u^{-l} \in M$. Hence there is an $\alpha \in A(G/M)$ such that $\langle xM\rangle^\alpha = \langle uM\rangle$. So $G/M$ is a $T_1(p)$-group.

(b) Let $P$ be a Sylow $p$-subgroup of $G$ and let $M$ be as in part (a).

*Case* 1. $p \neq q$. Then $G/M$ is a $T_1(p)$-group and $PM/M$ is a Sylow $p$-subgroup of $G/M$. Thus by induction $PM/M$ is a $T_1(p)$-group. Hence $P$ is a $T_1(p)$-group.

*Case* 2. $p = q$. Let $C = C_G(M)$, then $P \subseteqq C$. Let $x, y \in P$ such that $|x| = |y| = p$, then there is a $\phi \in A(G)$ such that $\langle x\rangle^\phi = \langle y\rangle$. There is a $c \in C$ such that $P^{\phi c} = P$; let $\alpha$ be the restriction of $\phi I_c$ to

$P$, where $I_c$ denotes the inner automorphism of $G$ determined by $c$. Then $\alpha \in A(P)$ and $\langle x\rangle^{\alpha} = \langle x\rangle^{\phi c} = \langle y\rangle^{c} = \langle y\rangle$ since $y \in M$. Thus $P$ is a $T_1(p)$-group.

(c) Let $P$ be a Sylow $p$-subgroup of $G$. Since $P$ is a $T_1(p)$-group a result of E. Shult [8] states that $P$ is a homocyclic group. Hence the Sylow $p$-subgroups of $G$ are abelian, so that $G$ has $p$-length $\leqq 1$.

(d) Since $G$ is a $T_1$-group it is a $T_1(r)$-group for all primes $r$ dividing $|G|$. Let $M$ be as in part (a), then $G/M$ is a $T_1(r)$-group for each prime divisor $r$ of $|G/M|$. We show by induction that if $G$ is a group of odd order which is a $T_1(r)$-group for each prime divisor $r$ of $|G|$, then $G$ has a normal Sylow subgroup. Thus $G/M$ has a normal Sylow subgroup $QM/M$, say a Sylow $p$-subgroup.

*Case* 1. $p = q$. Then $Q$ is a normal Sylow $p$-subgroup of $G$.

*Case* 2. $p \neq q$. Then $Q = MQ_1$, where $Q_1$ is a Sylow $p$-subgroup of $G$. All complements of $M$ in $Q$ are conjugate under $M$. Let $N = N_G(Q_1)$. By the Frattini argument we get that $G = MN$. Now $M \cap N = M \cap Z(Q)$ is a characteristic subgroup of $G$, so $M \cap N = 1$ or $M$. If $M \cap N = 1$, then $M$ will be a normal Sylow $p$-subgroup of $G$. If $M \cap N = M$, then $G = N$ and hence $Q_1$ is a normal Sylow subgroup of $G$. Thus if $G$ is a $T_1$-group of odd order it must have a normal Sylow subgroup.

(e) Since $G$ is a $T_1$-group it is also a $T_1(r)$-group for each prime $r$. Suppose $G$ is a $T_1(r)$-group for each prime $r$ and is minimal with respect to having a noncyclic nonnormal Sylow subgroup $Q$, say $Q$ is a Sylow $q$-subgroup.

1. $G$ has a unique minimal characteristic subgroup $M$. For suppose $M_1$ and $M_2$ are distinct minimal characteristic subgroups of $G$. By (a) and the minimality of $G$, since $QM_i/M_i$ are noncyclic for $i = 1, 2$, we know that $QM_1/M_1$ and $QM_2/M_2$ are normal Sylow subgroups of $G/M_1$ and $G/M_2$, respectively. Thus $QM_i \trianglelefteq G$ for $i = 1, 2$ and hence $Q = QM_1 \cap QM_2 \trianglelefteq G$, a contradiction.

2. $G = MQ$. $QM/M$ is a noncyclic Sylow subgroup of $G/M$, hence $QM/M \trianglelefteq G/M$. Thus $QM/M = 0_q(G/M)$, so $QM$ is characteristic in $G$. Then $QM$ is a $T_1(r)$-group for each prime $r$. If $QM \neq G$ then $Q \trianglelefteq QM$ by the minimality of $G$, and so $Q \trianglelefteq G$, a contradiction. Hence $G = MQ$.

3. $Q$ is abelian. See the proof of part (c).

4. $Q$ is cyclic and we have a contradiction. $M$ is a module under the action of $Q$. If $M$ were not a faithful $Q$-module then for some $v \in M$, $v^x = v$ for each $x \in Q$. Hence since $G$ is a $T_1(p)$-group we

would get $g^x = g$ for each $g \in M$ and $x \in Q$. Then $Q \trianglelefteq G$, a contradiction. Let $M = M_1 \times M_2 \times \cdots \times M_m$ be a decomposition of $M$ into irreducible submodules. If $u_i \in M_i$, $u_i \neq 1$, then $M_i = \langle u_i^x \mid x \in Q \rangle$ for $i = 1, 2, \cdots, m$. Let $G_i = \{x \in G \mid [x, u_i] = 1\}$, $i = 1, 2, \cdots, m$, then we see that $\bigcap_{i=1}^m G_i = M$. Let $u = u_1 u_2 \cdots u_m$ and let $G_0 = \{x \in G \mid [x, u] = 1\}$, then $[u, x] = 1$ if and only if $[u_i, x] = 1$ for $i = 1, 2, \cdots, m$. Thus $G_0 = M$. $A(G)/M$ transitively permutes the set $X$ of subgroups of order $p$ in $M$. Consequently $M$ breaks up into $A(G)$-orbits $U_1, U_2, \cdots, U_k$, each element in one orbit being the power of an element in any other. Thus $A(G)$ induces the same group of permutations in each $U_i$. Since $QM = G$ is a normal subgroups of $A(G)$, all $QM$-orbits on $U_i$ have the same cardinality, which does not depend on $i$. Since $G_0 = M$, $u$ belongs to a $QM$-orbit of length $|Q|$. Hence $u_1$ belongs to a $QM$-orbit of length $|Q|$. But this asserts that the irreducible $Z_p(Q)$-module $M_1$ is faithful. Thus $Z_p(Q)$ is a simple ring and $Q$ must be cyclic.

We note that if $G$ is abelian then $G$ is a $T_1$-group if and only if $G$ is a direct product of homocyclic groups. Finally, if $H_1$ and $H_2$ are $T_1$-groups such that $(|H_1|, |H_2|) = 1$, then their direct product is also a $T_1$-group. Thus it suffices to consider nonabelian $T_1$-groups of odd order which have no Hall direct factors.

**II. The main theorems.** We see from Lemma 1 that if $G$ is a $T_1$-group of odd order then $G$ has a normal Sylow subgroup and the nonnormal Sylow subgroups are cyclic. We call a group 1-metacyclic if each of its Sylow subgroups is cyclic. Hence we have

THEOREM 1. *If $G$ is a $T_1$-group of odd order, then*:

(1) $G = HK$, *where* $H \cap K = 1$.

(2) *$H$ is a normal Hall subgroup of $G$ which is a direct product of homocyclic groups.*

(3) *$K$ is a 1-metacyclic Hall subgroup of $G$.*

*Proof.* Let $H$ be the product of the normal Sylow subgroups of $G$ and let $K$ be a complement of $H$ in $G$. $G$ is a $T_1(p)$-group for each prime $p$ hence each Sylow subgroup of $G$ is homocyclic. Moreover, part (e) of Lemma 1 tells us that $K$ is 1-metacyclic.

It is important to point out that we cannot conclude that $K$ is cyclic in Theorem 1. For let $H$ be an elementary abelian group of order $5^3$. The group $SL_3(5)$ has a subgroup $K$ of order $31 \cdot 3$ which is Frobenius. Let $G$ be the splitting extension of $H$ by $K$, then $G$ is a $T_1$-group in which the isomorphism classes are actually conjugacy classes. So the best we can say is that $K$ is 1-metacyclic.

It is an easy matter to see that 1-metacyclic groups are $T_1$-groups (for the structure of 1-metacyclic groups see [5], p. 146).

LEMMA 2. *Let $G$ be a* 1-*metacyclic group. Then $G$ is a $T_1$-group.*

*Proof.* Let $L_1$ and $L_2$ be isomorphic subgroups of $G$ and suppose $G = K_1K_2$, where $K_1 = G'$, $K_1$ and $K_2$ are cyclic, and $(|K_1|, |K_2|) = 1$. Then $L_1 = (L_1 \cap K_1)T$ and $L_2 = (L_2 \cap K_1)S$, where $|T| = |S|$ is a divisor of $|K_2|$. Since $L_1 \cong L_2$ and $K_1$ is cyclic, we have $L_1 \cap K_1 = L_2 \cap K_1$. By a Hall theorem there is an element $x \in K_1$ such that $S^x \subseteqq K_2$. Without loss of generality one may assume $T \leqq K_2$, so we have $S^x = T$ as $K_2$ is cyclic. Thus $L_2^x = (L_2 \cap K_1)^x S^x = (L_2 \cap K_1)T = L_1$. So $G$ is a $T_1$-group.

Henceforth, if a group satisfies conditions 1–3 of Theorem 1 then we say it satisfies conditions 1–3.

LEMMA 3. *Let $G$ be a group satisfying conditions* 1–3 *and let $L_1$ and $L_2$ be isomorphic subgroups of $G$ such that $L_1 \cap H = L_2 \cap H$. Then $L_1$ and $L_2$ are conjugate.*

*Proof.* We go by induction on $|G|$. Suppose $L_1 = H_1T$ and $L_2 = H_1S$, where $H_1 = L_1 \cap H$ and $S \leqq K$. Then there is an element $x \in H$ such that $T^x \leqq K$, so $L_1^x = H_1T^x$. Let $N = N_G(H_1)$, then $N = HR$ where $R \leqq K$ and $\langle T^x, S \rangle \leqq R$. Moreover, $N$ satisfies conditions 1–3. If $R \neq K$, then by induction there is an element $y \in N$ such that $L_1^{xy} = L_2$. If $N = G$ then $H_1 \trianglelefteq G$, so choose $y \in K$ so that $T^{xy} = S$ (this choice is possible by Lemma 2). Then $L_1^{xy} = H_1^y T^{xy} = H_1S = L_2$.

THEOREM 2. *Let $G$ be a group of odd order satisfying conditions* 1–3. *Then $G$ is a $T_1$-group if and only if $A(G)$ permutes the isomorphism classes of $H$ transitively.*

*Proof.* Assume $A(G)$ permutes the isomorphism classes of $H$ transitively and let $L_1$ and $L_2$ be isomorphic subgroups of $G$. Then $L_1 = (L_1 \cap H)T$ and $L_2 = (L_2 \cap H)S$, where it is assumed $S \leqq K$. There is a $\phi \in A(G)$ such that $(L_1 \cap H)^\phi = L_2 \cap H$, hence $L_1^\phi = (L_2 \cap H)T^\phi$. Thus by Lemma 2 there is an $x \in G$ such that $L_1^{\phi x} = L_2$. Thus $G$ is a $T_1$-group.

THEOREM 3. *Let $G$ be a supersolvable group of odd order. Then $G$ is a $T_1$-group if and only if $G$ satisfies conditions* 1–3 *with $K$ cyclic and the elements of $K$ induce power automorphisms on $H$.*

*Proof.* Suppose $G$ is a $T_1$-group. Then we know $G$ satisfies conditions 1–3, so we need to show that each element of $K$ induces a power automorphism on $H$. Let $c \in K$ and let $p$ be a prime divisor of $|H|$. Since $G$ is supersolvable it has a normal subgroup of order

$p$. Thus since $G$ is a $T_1$-group every subgroup of order $p$ is normal in $G$. Let $A_1 = \langle x \in H \mid |x| \text{ is a prime}\rangle$ and suppose $|A_1| = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, a prime factorization. Let $B_i$ be the $p_i$-Sylow subgroup of $A_1$ and suppose $B_i = \langle a_1\rangle \times \langle a_2\rangle \times \cdots \times \langle a_s\rangle$, where $|a_j| = p_i$ for $1 \leqq j \leqq s$. We know that $a_j^c = a_j^{l_j}$ for some $l_j \in Z, 1 \leqq j \leqq s$, and $(a_1a_2 \cdots a_s)^c = (a_1a_2 \cdots a_s)^{t_i}$ for some $t_i \in Z$. Then $l_j \equiv t_i \pmod{p_i}$, $1 \leqq j \leqq s$, so if $b \in B_i$, say $b = a_1^{m_1} \cdots a_s^{m_s}$, then $b^c = a_1^{m_1c} \cdots a_s^{m_sc} = a_1^{t_im} \cdots a_s^{t_im_s} = b^{t_i}$. So $c$ induces a power automorphism on $B_i$, $1 \leqq i \leqq r$. Now let $x \in A_1$, such that $x = x_1x_2 \cdots x_r$, where $|x_i| = p_i$ for $i = 1, \cdots, r$. Then $x^c = x^t$ for some $t \in Z$, hence $t \equiv t_i \pmod{p_i}$, $1 \leqq i \leqq r$. So if $y \in A_1$, $y = y_1y_2 \cdots y_r$ where $y_i \in B_i$, then $y^c = y_1^c \cdots y_r^c = y_1^t \cdots y_r^t = y^t$. So $c$ induces a power automorphism on $A_1$. Let $A_k = \langle x \in H \mid |x| = p^l \leqq p^k$ for some prime $p \mid |H|\rangle$, $k \geqq 1$, and assumes $c$ induces a power automorphism on $A_k$. Then $A_k$ is a characteristic subgroup of $G$ and $G/A_k$ is a $T_1(p)$-group for each prime $p \mid |H|$. Since $G/A_k$ is supersolvable it has a normal subgroup $\langle aA_k\rangle$ of order $p$, where $p$ is a prime divisor of $|G/A_k|$. We can assume $|a| = p^{k+1}$. Then $a^c = a^sb$ for some $b \in A_k$ and some $s \in Z$, so $a^{pc} = a^{ps}\ b^p \in \langle a^p\rangle$ as $\langle a^p\rangle$ is normalized by $c$. So we can assume $b^p = 1$. Suppose $x^c = x^t$ for all $x \in A_k$, then $a^{pc} = a^{ps} = a^{pt}$. So $s \equiv t \pmod{p^k}$, hence $y^s = y^t$ if $|y| = p^l \leqq p^k$. If $|c| = n$, then $x^{c^n} = x^{t^n} = x$ for all $x \in A_k$, hence from $a^c = a^sb$ one can find that $a = a^{s^n}b^{ns^{n-1}}$. Thus we get $b^n = a^{s-s^{n+1}}$, and since $(n, p) = 1$, we get $b \in \langle a\rangle$. Thus $c$ normalizes $\langle a\rangle$ and hence every cyclic subgroup of order $p^{k+1}$, for each prime $p$ dividing $|H|$. Using methods similar to those used for $A_1$ one can show that $c$ induces a power automorphism on $A_{k+1}$. Thus $c$ induces a power automorphism on $H$.

We know that $K$ is 1-metacyclic and induces power automorphisms on $H$, hence $K/C_K(H)$ is isomorphic to an abelian subgroup of $A(H)$. Thus $K' \leqq C_K(H)$, so $K' \trianglelefteq G$. Then letting $K_1$ be a complement of $K'$ in $K$ we arrive at $G = (HK')K_1$ where $HK'$ is a normal Hall subgroup of $G$ which is a direct product of homocyclic groups and $K_1$ is cyclic.

Conversely, if $G$ is a group satisfying conditions 1–3 with the elements of $K$ inducing power automorphisms on $H$ then one can extend each element of $A(H)$ to $G$. For if $\phi \in A(H)$ define $\alpha\colon G \to G$ by $(hk)^\alpha = h^\phi k$, where $h \in H$ and $k \in K$. The essential feature is to show that $\alpha$ is a homomorphism. Let $h_1k_1, h_2k_2 \in G$ and suppose $k_1hk_1^{-1} = h^m$ for all $h \in H$, where $m \in Z$. Then

$$\begin{aligned}[(h_1k_1)(h_2k_2)]^\alpha &= [h_1(k_1h_2k_1^{-1})k_1k_2]^\alpha = [(h_1h_2^m)(k_1k_2)]^\alpha = (h_1h_2^m)^\phi(k_1k_2)\\ &= h_1^\phi(h_2^\phi)^m(k_1k_2) = h_1^\phi(k_1h_2^\phi k_1^{-1})k_1k_2 = (h_1^\phi k_1)(h_2^\phi k_2)\ .\end{aligned}$$

Thus $\alpha$ is an extension of $\phi$ to $G$. Thus $A(G)$ permutes the isomorphism classes of $H$ transitively, so by Theorem 2 $G$ is a $T_1$-group.

**III. $t$-groups.** A group $G$ is called a $t$-group [4] if every subnormal subgroup of $G$ is normal. W. Gaschutz has studied $t$-groups and shown that if $G$ is a solvable $t$-group with $G/L$ the largest nilpotent factor group, then:

(a) $G/L$ is Dedekind,

(b) $L$ is an abelian Hall subgroup of odd order,

(c) the inner automorphisms induce powers on $L$.

It is clear that a solvable $t$-group is supersolvable.

THEOREM 4. *Let $G$ be a $T_1$-group of odd order. Then $G$ is a $t$-group if and only if $G$ is supersolvable.*

*Proof.* Assume $G$ is supersolvable and let $R$ be a subnormal subgroup of $G$. Now $G$ satisfies conditions 1–3, so $R = (R \cap H)T$ where $T$ is cyclic and $|T| \mid |K|$. We can assume without loss that $T \leqq K$. Choose a subgroup $S$ of $K$ maximal with respect to $(R \cap H)S$ being subnormal in $G$. Then there is a subnormal series in $G$ of the form

$$1 < R \cap H < (R \cap H)T < (R \cap H)S < H_1S < \cdots < H_mS = HS < G .$$

We have $(R \cap H)S \trianglelefteq H_1S \trianglelefteq H_2S$ and $(|R \cap H)S/R \cap H|, |H_1S: (R \cap H)S|) = 1$. So $(R \cap H)S/R \cap H$ is characteristic in $H_1S/R \cap H$. Hence $(R \cap H)S \trianglelefteq H_2S$. Continuing this process we get $(R \cap H)S \trianglelefteq SH$. But $(|R \cap H)S/R \cap H|, |HS: (R \cap H)S|) = 1$, so $(R \cap H)S/R \cap H$ is characteristic in $HS/R \cap H$. Thus $(R \cap H)S \trianglelefteq G$. Now $R/R \cap H$ is characteristic in $(R \cap H)S/R \cap H$ since $(R \cap H)S/R \cap H$ is cyclic. Thus $R \trianglelefteq G$, and $G$ is a $t$-group.

It is clear that there exist $t$-groups of odd order which are not $T_1$-groups.

**IV. The general case.** At this point I would like to make some remarks which may prove useful in trying to characterize $T_1$-groups of odd order. Given a $T_1$-group of odd order we know that it satisfies conditions 1–3. Then:

1. $G' = (G' \cap H)K'$.

2. If $P$ is the Sylow $p$-subgroup of $H$ and $P \not\subseteq G'$, then $P \cap G' = 1$ and $P \subseteqq Z(G)$. Thus we can write $H = H_1H_2$ where $H_1$ and $H_2$ are Hall subgroups of $H$, $H_1 \subseteqq Z(G)$, and $H_2 \subseteqq G'$. Moreover, $G = H_1H_2K$ is a $T_1$-group if and only if $H_2K$ is a $T_1$-group. Thus it suffices to characterize $T_1$-groups $G$ for which $H \subseteqq G'$.

3. The following lemma, which appears in other papers [see, for example, 7], will be useful in the solution of this problem.

LEMMA B. *Let $G$ be a group satisfying conditions* 1–3. *Then an element $\phi \in A(H)$ is extendible to $G$ if and only if $\phi$ normalizes the image of $K$ in $A(H)$.*

Thus we see that if $L$ is a subgroup of $A(H)$ which permutes the isomorphism classes of $H$ transitively and $L$ normalizes the image of $K$ in $A(H)$, then $G$ is a $T_1$-group.

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## REFERENCES

1. L. Dickson, *Linear Groups*, Dover Pub., Inc., New York, 1958.
2. W. Feit and J. Thompson, *Solvability of groups of odd order*, Pacific J. Math., **13**, No. 3, (1963).
3. W. Gaschutz and T. Yen, *Groups with an automorphism group which is transitive on the elements of prime order*, Math. Zeit., **86** (1964), 123–127.
4. W. Gaschutz, *Gruppen, in einen das Normalteilersein transitiv ist.* J. Reine Angew. Math., **198** (1957), 87–92.
5. M. Hall, *The Theory of Groups*, Macmillan Co., New York, 1959.
6. M. Hestenes, *Singer Groups*, Canad. J. Math., Vol. **XXII**, No. 3, (1970), 492–513.
7. J. Lewin, *A finitely presented group whose group of automorphisms is finitely generated*, J. London Math. Soc., **42** (1967), 610–613.
8. E. Shult, *On finite automorphic algebras*, Illinois J. Math., **13** (1969), 625–653.

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STATE UNIVERSITY COLLEGE AT FREDONIA