

RINGS OF QUOTIENTS OF RINGS WITHOUT NILPOTENT ELEMENTS

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The first theorem gives a number of characterizations of when a ring with zero right singular ideal has a strongly regular right quotient ring. This result (and also Theorem 2) is a generalization of a similar theorem of F. W. Anderson for a certain class of lattice-ordered rings and a theorem of G. Renault for reduced rings (i.e., rings without nilpotent elements). As a corollary one obtains a characterization of when a semiprime ring has a strongly regular right quotient ring similar to Utumi's characterization of when a ring has a regular right quotient ring. Also, some theorems on commutative regular rings are extended to strongly regular rings and regular rings that satisfy a polynomial identity. For instance, a reduced ring is regular if and only if each of its prime homomorphic images is regular. This theorem has been obtained independently by Herstein, by Snider, and by Wong. Using rings of quotients some theorems of R. Wiegand are also generalized. It is shown that the endomorphism ring S of an ideal I of a strongly regular ring R is strongly regular, and some characterizations of when R is self-injective are obtained.

1. Rings of quotients. We first review some definitions and facts and fix some notation. Good references are [8], [21], [17], and [22]. Let A be a subset of the right R -module M . Then $r_R(A) = r(A) = \{x \in R \mid Ax = 0\}$ is the right annihilator of A in R . Similarly, $l(A)$, where appropriate, will denote the left annihilator of A . A ring R is called reduced if it has no nonzero nilpotent elements; i.e., $x \in R$ and $x^2 = 0$ implies $x = 0$. If A is a subset of a reduced ring R , then $l(A) = r(A)$ will be denoted by A' . If A is an ideal of a semiprime ring R , then, again, $A' = l(A) = r(A)$. If R is semiprime, the set of annihilator ideals of R is a Boolean algebra [17, p. 110].

A submodule N of the right R -module M is essential in M if $N \cap K \neq 0$ for each nonzero submodule K of M . The set of essential submodules of M_R is a dual ideal in the lattice of submodules of M . The closure of N in M is the submodule $\text{Cl}(N) = \{x \in M \mid xD \subseteq N \text{ for some essential right ideal } D \text{ of } R\}$, and N is closed in M if $N = \text{Cl}(N)$. Since the intersection of a family of closed submodules is closed, the collection of closed submodules of M forms a complete lattice which will be denoted by $C_r(M)$. The singular submodule of M is $Z(M_R) = \{x \in M \mid r(x) \text{ is essential in } R_R\}$ [14]. If $Z(M_R) = 0$, then $\text{Cl}(N)$ is closed and is the largest essential extension of N in M ; and N is closed if and only if N is a complement in M —i.e., there is a

submodule K of M such that N is maximal with respect to $N \cap K = 0$ [8, p. 61].

A ring S is a right quotient ring of its subring R if for all $x, y \in S$ with $y \neq 0$, there exists $r \in R$ with $yr \in R$ and $yr \neq 0$. Each ring R with $l(R) = 0$ has a maximal right quotient ring [21]. Throughout this paper R will denote a ring with $l(R) = 0$ and maximal right quotient ring $Q = Q(R)$. Let E denote the injective hull of the right R -module R_R . A right ideal D of R is dense if $l_E(D) = 0$; equivalently, if $h \in \text{Hom}_R(E, E)$ and $h(D) = 0$, then $h = 0$; equivalently, $Q(D) = Q$ [8, p. 58 and § 8]. For each $q \in Q$ there is a dense right ideal D of R such that $qD \subseteq R$. If $Z(R_R) = 0$, then a right ideal of R is dense if and only if it is essential [21, Theorem 3]; also, in this case, $C_r(R)$ contains each right annihilator of R [8, p. 71, Proposition 5].

The ring R is regular if for each $a \in R$ there exists $x \in R$ with $a = axa$; equivalently, each principal left (right) ideal is generated by an idempotent [8, p. 42]. R is strongly regular if it is regular and has no nonzero nilpotent elements; alternatively, each idempotent is central. The ring R has a regular right quotient ring, and then Q is regular, right injective, and the injective hull of R_R , exactly when $Z(R_R) = 0$ [8, p. 69]. If Q is regular, then the closure in Q_R of each right ideal of R is a summand of Q_Q [8, p. 70, Theorem 4].

LEMMA 1. *Let R be a reduced ring and let $a_1, \dots, a_n, x$ be elements of R . Let P be a product of $a_1, \dots, a_n$ (in any order); let P_i be a product of $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n$; and let P_x be a product of $a_1, \dots, a_n, x$.*

- (1) $P = 0 \leftrightarrow a_1 \cdots a_n = 0$.
- (2) If $a_i = a_j$ for some $i \neq j$, then $P = 0 \leftrightarrow P_i = 0$.
- (3) If $P = 0$, then $P_x = 0$.

Proof. Each of these statements is trivial if R is a domain. Since R is a subdirect product of domains [3] each is true when R is merely reduced.

An f -ring is a lattice-ordered ring that is a subring and a sublattice of a product of totally ordered rings. The equivalence of (1) and (2) in Theorem 1 has essentially been proven by Anderson [2, Theorem 4.3] for the case in which R is a unital f -ring, and the equivalence of (1) through (4) has been proven by Renault [19, Theorem 4.1] for the case in which R is a unital reduced ring.

THEOREM 1. *Let R be a ring with zero right singular ideal and maximal right quotient ring Q . The following statements are equivalent.*

- (1) If $a, b \in R$, then $aR \cap bR = 0$ implies $ab = 0$.
 (2) Q has no nilpotent elements.
 (3) Each element of $C_r(R)$ is the right annihilator of an ideal of R .
 (4) Each element of $C_r(R)$ is an ideal of R .
 If R has no nilpotent elements, then (1) is equivalent to each of the following.
 (5) If I is a right ideal of R , then I is essential in I'' (and I'' is the maximal essential extension of I in R).
 (6) If I is a right ideal of R , then I' is a (unique) right complement of I .

Proof. (1) $\rightarrow$ (2): We first claim that $I + r(I)$ is an essential right ideal of R for each right ideal I of R . For if J is a right ideal of R and $[I + r(I)] \cap J = 0$, then, since $I \cap J = 0$, (1) implies that $IJ = 0$. Thus $J \subseteq r(I)$, and hence $J = 0$; so the claim is proven. Next we show that R is semiprime. Let I be a right ideal such that $I^2 = 0$. Then $I[I + r(I)] = 0$, so $Z(R_R) = 0$ implies that $I = 0$. Next we show that R is reduced. Let $a \in R$ and let $J = r(aR)$. Then $aRJ = 0$ yields $RJa = 0$, and hence $Ja = 0$, since RJ is an ideal of the semiprime ring R . If $a^2 = 0$, then $a(aR + J) = 0$; so $a = 0$ by the essentiality of $aR + J$.

Let $q \in Q$ and let D be a dense right ideal of R for which $qD \subseteq R$. Since D is essential in R , $(qD)' \cap D$ is essential in $(qD)'$. So $[(qD)' \cap D] \oplus qD$ is essential in $(qD)' \oplus qD$, and hence $[(qD)' \cap D] \oplus qD$ is an essential right ideal of R . If $x \in (qD)' \cap D$, then $xqD = 0$; so $xq = 0$. Since $qx \in R$ and $(qx)^2 = 0$, $qx = 0$. Thus $q^2 = 0$ implies $q[(qD)' \cap D] \oplus qD = 0$, and hence $q = 0$. So Q is reduced.

(2) $\rightarrow$ (1): Let $a, b \in R$ with $aR \cap bR = 0$. Let I be the closure of $(aR)_R$ in Q and let J be the closure of $(bR)_R$ in Q . Then $I = eQ$ and $J = fQ$ for some idempotents e and f of Q . Since I and J are closed R -submodules of Q_R , and since $aR \subseteq I$ and $bR \subseteq J$, $a \in I$ and $b \in J$. Since aR is essential in I and bR is essential in J , $aR \cap bR = 0$ implies $I \cap J = 0$. Since I and J are ideals of Q , $ab \in IJ \subseteq I \cap J = 0$.

(2) $\rightarrow$ (3): If $I \in C_r(R)$ and J is a complement of I , then I is a complement of J . Let I_1 be the closure of I in Q and let J_1 be the closure of J in Q . Then, as in the previous argument, $IJ \subseteq I_1J_1 = 0$; i.e., $J \subseteq I'$. Since $I \cap I' = 0$, the maximality of J implies that $J = I'$. Similarly, $I = J'$, so $I = I''$.

That (3) implies (4) is trivial; and that (4) implies (1) follows from the fact that $ab \in \text{Cl}(aR) \text{Cl}(bR) \subseteq \text{Cl}(aR) \cap \text{Cl}(bR) = 0$, if $aR \cap bR = 0$.

(1) $\rightarrow$ (5): If I and J are right ideals of R for which $I \cap J = 0$, then (1) implies that $J \subseteq I'$. Thus $J \subseteq I''$ yields $J = 0$, and hence I

is essential in I'' . Since an annihilator is closed, I'' is the closure of I in R .

(5) $\rightarrow$ (6): If $I \cap J = 0$, then, since I is essential in I'' , $I'' \cap J = 0$. But then $J \subseteq I''' = I'$ since I'' is an ideal. Since $I \cap I' = 0$, I' is the unique right complement of I .

(6) $\rightarrow$ (3): Let $I \in C_r(R)$. By (6) I'' is a right complement of I' which contains I . But I is a right complement of I' , so $I = I''$.

There are reduced rings R for which $Q(R)$ is not reduced. For an example take any domain that is not a right Ore domain.

A right ideal A of the ring R is called an M -right ideal provided $x \in A$ whenever there exists a dense right ideal D for which $xD \subseteq A$. In [21, §§3 and 4] Utumi has shown that every right complement and every right annihilator in R is an M -right ideal, and that Q is regular if and only if each M -right ideal of R is a right complement. The following corollary takes this result one step further.

COROLLARY 1. *Let R be a semiprime ring with maximal right quotient ring Q . Then Q is strongly regular if and only if each M -right ideal of R is an (annihilator) ideal.*

Proof. In view of the preceding theorem and Utumi's theorem it suffices to show that each M -right ideal I of R is a right complement, assuming that each such I is an (annihilator) ideal. Let J be a right complement of I and let I_1 be a right complement of J containing I . If $I_1 I' \neq 0$, then, since I is essential in I_1 , $I \cap I_1 I' \neq 0$. This contradicts the assumption that R is semiprime.

Note that this corollary is false without the hypothesis that R is semiprime. For an example take any commutative quasi-Frobenius ring that is not regular.

The next corollary generalizes a well-known fact about integral domains.

COROLLARY 2. *A ring R has a strongly regular right quotient ring if and only if each of its right ideals has a strongly regular right quotient ring.*

Proof. Suppose that R has a strongly regular right quotient ring and let A be a right ideal of R . Let $a, b \in A$ with $aA \cap bA = 0$. By Theorem 1, since $Q(R)$ is strongly regular, $aAbA = 0$. Thus $a^2b^2 = 0$, and hence $ab = 0$ by Lemma 1 (2). By Theorem 1, again, $Q(A)$ is strongly regular.

Corollary 2 is false if "strongly" is deleted; for a right ideal

need not even have a right quotient ring (in the sense of Utumi). To give a concrete example let R be the two-by-two matrix ring over the rationals and let $A = e_{11}R$ be the right ideal of R consisting of those matrices whose second row is zero. Then A is a right injective ring that is clearly not regular [20, Example 5.3].

There is a generalization of Corollary 2, however, which is an immediate consequence of the proof of [8, Theorem 1.2, p. 97]: Let R be a semiprime ring with a regular right quotient ring and let A be a right ideal of R with $l_A(A) = 0$. Then A is semiprime and has a regular right quotient ring; and

$$Q(A) \cong eQ(R)e \cong Q(\text{Hom}_R(A, A)) ,$$

where $eQ(R)$ is the injective hull of A_R in $Q(R)$.

COROLLARY 3. (Utumi [22, Theorem 1.4].) *If a ring R has a strongly regular right quotient ring and a strongly regular left quotient ring, then Q is also the maximal left quotient ring of R .*

Proof. Since Q is a strongly regular right injective ring it is also left injective (see [21, Theorem 4], for example). Thus it suffices to show that ${}_R R$ is essential in ${}_R Q$. Let $0 \neq q \in Q$. Since R_R is essential in Q_R there exists $d \in R$ such that $0 \neq qd \in R$. By the left-sided version of Theorem 1 $Rqd \cap Rd \neq 0$, so there exists $a, b \in R$ with $0 \neq aqd = bd$. Let $y \in Q$ with $dyd = d$, and note that $dy = yd$ is idempotent. Then $daq = dydaq = daqdy = dbdy = dydb = db \neq 0$, so $Rq \cap R \neq 0$.

Next we impose chain conditions on the reduced ring R to obtain a ring theoretic generalization of a theorem about f -rings which is due to Anderson [1, Theorems 6.1 and 6.2] (also see [19, Proposition 5.3]).

LEMMA 2. *Let A be an ideal of the reduced ring R . The following statements are equivalent.*

- (1) A' is a maximal annihilator of R .
- (2) R/A' is a domain.
- (3) A is a domain.

Proof. If A is not a domain there are nonzero elements $a, b \in A$ such that $ab = 0$. If J is the ideal of R generated by $A' \cup \{a\}$, then $A' \subseteq J'' \subseteq R$, since $a \in J'' \setminus A'$ and $b \notin J''$ ($b \in A'' \cap a' = J'$); hence (1) implies (3). That (3) implies (2) is straightforward (and known). If R/A' is a domain, then $a \notin A'$ implies $a' \cap A'' \subseteq A' \cap A'' = 0$. Therefore, $(\{a\} \cup A')'' = (a' \cap A'')' = R$; so A' is a maximal annihilator and (2) implies (1).

THEOREM 2. *Let R be a reduced ring with the maximum condition on annihilators. Then $Q = Q(R)$ is reduced if and only if R is a right Ore ring. If this is the case Q is the classical right quotient ring of R and it is a direct sum of division rings.*

Proof. Since the Boolean algebra $\mathcal{A}$ of annihilators of R has the maximum condition, it is finite. Let $A_1, \dots, A_n$ be the maximal elements of $\mathcal{A}$. Then $\bigcap A_i = 0$. If $B_i = A'_i$, then $B_i \cap \sum_{j \neq i} B_j \subseteq B_i \wedge (\bigvee_{j \neq i} B_j) = \bigvee_{j \neq i} B_i \wedge B_j = 0$, where the lattice operations are taken in $\mathcal{A}$. Thus $B = \sum \bigoplus B_i \subseteq R$. Since $B' = 0$ B is essential in R . Since each B_i is a domain B has regular elements, and since $B' = 0$ the regular elements of B are still regular in R .

If R is right Ore, then so is B , and R and B have the same classical right quotient ring T . Since T is a direct sum of division rings it is equal to Q .

Suppose, conversely, that Q is reduced. Let a and x be elements of R with x regular. If $aB_i = 0$ let b_i be any nonzero element of B_i . If $aB_i \neq 0$, then the regularity of x implies that $aB_i xR \neq 0$. By Theorem 1, then, $aB_i \cap xR \neq 0$. Let $b_i \in B_i$ with $0 \neq ab_i \in xR$. Then $b = b_1 + \dots + b_n$ is a regular element of R and $ab \in xR$. Thus R is a right Ore ring.

In [20, Theorem 3.5] we have shown that if R is an f -ring in which each right ideal that has a finite number of positive generators has a single positive generator, then Q is an f -ring extension of R . This has motivated the next result.

THEOREM 3. *If R is a reduced ring in which all finitely generated right ideals are principal, then Q is strongly regular.*

Proof. It is well-known that a domain S can be embedded in a unital domain. (Let S_1 be the ring obtained from S by freely adjoining the integers to S . Then S_1 is semiprime and S_1/S' is a domain.) Since R is a subdirect product of domains it can be embedded in a unital reduced ring T . Let R_1 be the subring of T generated by R and 1.

We now show that Q is reduced. Suppose that $q \in Q$ with $q^2 = 0$, and let $d \in R$ with $qd \in R$. Then $qdR_1 + dR_1 = cR_1$ for some $c \in R$. Hence there exist elements x_1, y_1, a_1 , and b_1 in R_1 such that

$$\begin{aligned}(qd)x_1 + dy_1 &= c \\ d &= ca_1 \\ qd &= cb_1.\end{aligned}$$

Hence

$$cb_1y_1b_1 = qdy_1b_1 = q[(c - qdx_1)b_1] = qcb_1 = q^2d = 0.$$

Since R_1 is reduced, Lemma 1 (2) implies that $cb_1y_1 = 0$. Thus $qdy_1 = 0$ and hence $qc = 0$. Consequently $qd = 0$, and, since $D = \{d \in R \mid qd \in R\}$ is a dense right ideal of R , necessarily $q = 0$.

COROLLARY 4. (Amitsur [1, p. 62], Cohn [7, Theorems 3.3, 4.1, and 4.2].) *An integral domain in which each finitely generated right ideal is principal is a right Ore domain.*

The following corollary is an immediate consequence of Theorem 3.

COROLLARY 5. *If R is a reduced ring in which each finitely generated one-sided ideal is principal, then Q is also the maximal left quotient ring of R .*

If Q can be made into an f -ring extension of the f -ring R , then R is called a qf -ring.

COROLLARY 6. *A semiprime f -ring R in which each finitely generated right ideal is principal is a qf -ring.*

Proof. Since R is reduced [4, p. 63, Theorem 16], Q is strongly regular. Thus Theorem 1 and the nonunital version of [2, Theorem 4.3] imply that R is a qf -ring.

We close this section with a short proof of the following result which has been proven by J. Georgoudis for the case in which R is an f -ring [11, Proposition 7]. (It is valid even if S is not a faithful extension of R .) The largest subring of Q that is a left quotient ring of R is the maximal two-sided quotient ring of R [25, Theorem 8].

PROPOSITION 1. *A two-sided quotient ring S (in particular, the maximal two-sided quotient ring) of a reduced ring R is reduced.*

Proof. Let $s \in S$ with $s^2 = 0$, and let $D_1(D_2)$ be a dense left (right) ideal of R with $D_1s \subseteq R$ ($sD_2 \subseteq R$). If $d_1 \in D_1$ and $d_2 \in D_2$, then $d_1s^2d_2 = 0$ and Lemma 1 (3) imply that $(d_1sd_2)^2 = 0$. Thus $D_1sD_2 = 0$, so $s = 0$.

2. Strongly regular rings. The next theorem, which has also been obtained by Herstein, by Snider [10], and by Wong [24], characterizes regular rings among the class of reduced rings. Its motivation is the commutative version given in [16, p. 61]. Our proof is of interest because it is quite similar to that of the commutative version given in [16]. R. Snider has given a counterexample to show that the theorem is not true for regular rings (see [10]), and the theorem

has been obtained by Fisher and Snider [10] as a corollary of a characterization of regular rings.

If P is a prime ideal of a ring R , then the intersection of a maximal chain of prime ideals contained in P is a minimal prime ideal. Recall that a minimal prime ideal of a reduced ring is completely prime; i.e., the quotient is a domain [3]. (A prime ideal in a reduced ring need not be completely prime. For example, let S be a right primitive domain that is not a division ring. Then there exists a division ring D and a subring R of S such that the two-by-two matrix ring over D is a homomorphic image of R [11, Theorem 2.14, p. 43].)

LEMMA 3. *Let P be a prime ideal of the reduced ring R . Then P is a minimal prime if and only if $x' \not\subseteq P$ whenever $x \in P$.*

Proof. Suppose the condition is satisfied and let N be a minimal prime ideal contained in P . If $x \in P \setminus N$ there exists $y \notin P$ such that $xy = 0$. This contradicts the fact that N is completely prime, so $N = P$.

Suppose conversely that P is a minimal prime and let $x \in P$. Let $M = R \setminus P$ and let

$$S = M \cup \{a_1 x a_2 x \cdots a_n x a_{n+1} \mid a_i \in M \text{ and } n \geq 1\}.$$

Then M is a multiplicatively closed subset of R maximal with respect to excluding 0, and S is a multiplicative subset properly containing M . Thus $0 = a_1 x a_2 \cdots a_n x a_{n+1}$ for some $a_i \in M$. By Lemma 1(1) $a_1 a_2 \cdots a_n x = 0$, so $x' \not\subseteq P$.

THEOREM 4. *A reduced ring is regular if and only if R/P is regular for each prime ideal P of R .*

Proof. Suppose that for each prime ideal P of R R/P is regular. Since a regular domain is a division ring, each minimal prime is a maximal left ideal, and therefore each prime is minimal. Let $0 \neq a \in R$ and let $J = a'$. Then $\bar{a}$ is a regular element in the reduced ring $\bar{R} = R/J$. Suppose that $\bar{R}$ has an identity element. If $\overline{Ra} \subseteq \bar{R}$ let $\bar{A}$ be a maximal left ideal containing $\overline{Ra}$, and let $\bar{P}$ be a primitive ideal contained in $\bar{A}$. Then $\bar{P} = \bar{A}$ and hence $\bar{a} \in \bar{P}$. This contradicts Lemma 3 which implies that $\bar{P}$ consists of zero divisors.

If $\bar{R}$ does not have an identity, then, by a change of notation, we have a is a regular element of R and $1 \notin R$. Let

$$S = \{a^{n_1}(1 - x_1)a^{n_2}(1 - x_2) \cdots a^{n_k}(1 - x_k) \mid k \geq 1, n_i \geq 1 \text{ and } x_i \in R\}.$$

Then S is a multiplicatively closed subset of R . If $0 \in S$, then, since

R is embeddable in a unital reduced ring, Lemma 1 (2) implies that $a(1 - x) = 0$ for some $x \in R$; hence x is the identity of R . If $0 \notin S$, and P is an ideal maximal with respect to being disjoint from S , then P is prime, so R/P is a division ring. If $e + P$ is the identity of R/P , then $a(1 - e) \in P$ gives the contradiction.

We finish up by using rings of quotients to generalize some theorems of R. Wiegand [23] on commutative regular rings to strongly regular rings. If $B(S)$ is the set of central idempotents of the ring S , then $B(S)$ is a generalized Boolean algebra with lattice operations $e \wedge f = ef$ and $e \vee f = e + f - ef$. (It is a Boolean algebra if $1 \in S$.) The following lemma, though undoubtedly known, is included for completeness.

LEMMA 4. *Let R be a semiprime subring of S , and suppose that S_R is an essential extension of R_R . If A is an ideal of R , then B , the closure of A_R in S_R , is an ideal of S .*

Proof. Let $b \in B$ and $s \in S$. Since R_R is essential in S_R , there exists an essential right ideal D of R such that $bD \subseteq A$, $sD \subseteq D$, and $sbD \subseteq R$. So $bsD \subseteq bD \subseteq A$; hence B is a right ideal of S . Also $sbDA' = 0$, so $sbD \subseteq A'$. Since A_R is essential in A'' , $D_1 = \{d \in D \mid sbd \in A\}$ is an essential right ideal of R . Thus $sb \in B$ and B is a left ideal of S .

The next result appears in [9, p. 90, Theorem 11.9] for the case that R is a commutative semisimple ring with unity.

PROPOSITION 2. *Let R be a regular ring with maximal right quotient ring Q . Then $B(Q)$ is the completion of $B(R)$ if and only if each nonzero annihilator ideal of R contains a nonzero central element.*

Proof. Since the center of R is contained in the center of Q [21, 1.3], $B(R) \subseteq B(Q)$. Since Q is right injective each annihilator ideal A of Q , being a closed right ideal, is a summand of Q_q , and hence is a ring summand of Q . Thus $B(Q)$ is isomorphic to the Boolean algebra of annihilator ideals of Q , and, in particular, $B(Q)$ is complete. In [5] it is shown that the maximal quotient ring of a Boolean ring B coincides with the completion of B considered as a generalized Boolean algebra. So $B(Q)$ is the completion of $B(R)$ if and only if it is a quotient ring of $B(R)$.

Suppose that $B(Q)$ is the completion of $B(R)$. Let A be a nonzero annihilator ideal of R and let B be the closure of A in Q_R . By Lemma 4 B is an ideal of Q , so $B = eQ$ for some $e \in B(Q)$. If $f \in B(R)$ with

$0 \neq ef \in B(R)$, then $ef \in eQ \cap R = A$ [8, p. 70, Theorem 4].

Assume, on the other hand, the condition of the proposition. Let $0 \neq e \in B(Q)$ and let $A = R \cap eQ$. Then $A = [R \cap (1 - e)Q]'$, so A contains a nonzero central element a . Since R is regular there exists $b \in R$ with $a = aba$. Then $ab = ba = f \in B(R)$ and $aR = fR$. So $0 \neq f = ef$ and hence $B(Q)$ is a quotient ring of $B(R)$.

A ring R is said to satisfy a polynomial identity if there exists a (homogeneous multilinear) polynomial $f(x_1, x_2, \dots, x_n) = \sum_i w_i x_{i_1} x_{i_2} \cdots x_{i_n}$ in noncommuting indeterminates $\{x_j\}$ (where i ranges over the symmetric group S_n and w_i lies in the centroid of R) such that $f(r_1, r_2, \dots, r_n) = 0$ for all $r_1, r_2, \dots, r_n \in R$ and the kernel of $w_1 = 0$. Such a ring is called a P.I. ring.

COROLLARY 7. *If R is strongly regular or a regular P.I. ring, then $B(Q)$ is the completion of $B(R)$.*

Proof. If R is strongly regular this is obvious. In the other case, the conclusion is a consequence of the following theorem of L. Rowen [18, Theorem A]: Each nonzero ideal of a semiprime P.I. ring contains a nonzero central element.

If R is a regular ring, then $B(Q)$ need not be the completion of $B(R)$. For instance, if R is the ring of linear transformations of finite rank of an infinite dimensional vector space (or any regular prime ring without an identity element), then $B(R) = 0$ while $B(Q) = \{0, 1\}$. On the other hand, it is clear from the proof of Proposition 2 that if R is any semiprime ring with zero right singular ideal in which each nonzero annihilator ideal contains a nonzero central idempotent, then $B(Q)$ is the completion of $B(R)$. Included in this class of rings are the biregular rings and the semisimple weakly reducible rings (see [21, p. 13]). (A ring R is biregular if each principal ideal is generated by a central idempotent. An ideal of a ring is called a matrix ideal if it is a matrix ring of finite degree over a unital reduced ring in which each nonzero one-sided ideal contains a nonzero central idempotent. A ring is semisimple weakly reducible if every nonzero ideal contains a nonzero matrix ideal.)

PROPOSITION 3. *The following statements are equivalent for a regular ring R for which $B(Q)$ is the completion of $B(R)$.*

- (1) $B(R)$ is complete.
- (2) Each ideal of R is an essential submodule of a principal right (left) ideal.
- (3) R is unital and each annihilator ideal is a summand.

Proof. To see that (1) implies (2) let I be an ideal of R and let eQ be the closure of I in Q_R . By Lemma 4 $e \in B(Q) = B(R)$, so I is essential in eR . Trivially (2) implies that R is unital and, since an annihilator ideal is closed, it is a summand. Finally, (3) implies that $B(R)$ is isomorphic to the Boolean algebra of annihilator ideals of R , and hence it is complete.

For the sequel, let I be an ideal of the regular ring R , $\hat{I} = eQ$ the closure of I in Q_R , $S = \text{Hom}_R(I_R, I_R)$ and $T = \text{Hom}_R(\hat{I}_R, \hat{I}_R)$. Then, since $Z(I_R) = 0$, S is embedded naturally in T ; and $T \cong eQe$, $S \cong S_1 = \{eqe \mid eqeI \subseteq I\}$ [8, p. 97, Theorem 1.2].

THEOREM 5. *Let I be an ideal of the strongly regular ring R . Then $S = \text{Hom}_R(I_R, I_R) \cong \text{Hom}_R({}_R I, {}_R I)$ is strongly regular.*

Proof. Let $a \in Q$ and let $b \in Q$ with $aba = a$. Let $a_* = ab = ba$ and let $\tilde{a} = bab$. Then a_* is the unique idempotent of Q with $Qa = Qa_*$, and $\tilde{a}$ is the inverse of a in Qa . If $c \in Q$, then, since each one-sided ideal of Q is an ideal,

$$Qac = QaQc = Qa_*Qc_* = Qa_*c_*.$$

Thus $*$ ($\sim$) is a multiplicative (anti-multiplicative) mapping of Q which leaves each idempotent fixed. If $a \in I$, then, since R is regular, a_* , $\tilde{a} \in I$. We claim that for $a \in Q$, $aI \subseteq I$ if and only if $a_*I \subseteq I$. For if $c \in I$, then $ac \in I$ if and only if $a_*c_* = (ac)_* \in I$; if and only if $a_*c_*R \subseteq I$; if and only if $a_*c \in I$. Note, also, that since $Qa = Q\tilde{a}$, $aI \subseteq I$ if and only if $\tilde{a}I \subseteq I$.

We will now show that S is regular. Take $ae \in S_1$ and let $g = a_*e$. Then $g \in S_1$ since $aeI \subseteq I$. We claim that $S_1g = S_1ae$. Since $ae = aea_*e = aeg$, $S_1ae \subseteq S_1g$. However, $g = a_*e = \tilde{a}eae \in S_1ae$ since $\tilde{a}e \in S_1$; so $S_1g \subseteq S_1ae$. Thus S_1 , and hence S , is regular.

Since $aI \subseteq I$ if and only if $Ia \subseteq I$, $S \cong \text{Hom}_R({}_R I, {}_R I)$.

With the same notation as above let $\mu_I: R \rightarrow S$ be the homomorphism given by $\mu_I(x)$ is left multiplication by x . Note that S is a right R -module: If $s \in S$, $r \in R$, and $a \in I$, then $(sr)a = s(ra)$.

THEOREM 6. *Let I be an ideal of the unital regular ring R . Let $S = \text{Hom}_R(I_R, I_R)$ and let $\mu_I: R \rightarrow S$ be the left multiplication map. Then S_R is projective if and only if I is a summand of R and μ_I is onto.*

Proof. Let $\mu = \mu_I$ and let L_e denote left multiplication by e ($L_e = \mu(1)$). The actions of R on S and of $L_e \cdot R$ on S correspond

in the natural way. In particular, the R -submodules and the $L_e \cdot R$ -submodules of S are the same. By the proof of [8, Theorem 1.2, p. 97] T is the maximal right quotient ring of $\mu(I)$. Since $\mu(I) \subseteq L_e \cdot R \subseteq S \subseteq T$, $L_e \cdot R$ is an essential submodule of S_R .

Consider the right R -module sequence

$$(x) \quad 0 \longrightarrow I' \longrightarrow R \longrightarrow L_e \cdot R \longrightarrow 0.$$

If S_R is projective, then by [15, Lemma 4] $L_e \cdot R$ is a summand of S . Thus $L_e \cdot R = S$ and (x) splits; i.e., I' is a summand of R . Conversely, if I' is a summand and $L_e \cdot R = S$, then (x) splits and hence S_R is projective.

COROLLARY 8. *Let R be either a unital strongly regular ring or a unital regular P.I. ring. Then the following conditions are equivalent.*

- (1) R is self-injective.
- (2) $\mu_I: R \rightarrow \text{Hom}_R(I_R, I_R)$ is epic for each ideal I of R .
- (3) $B(R)$ is complete and $I'' \cong \text{Hom}_R(I_R, I_R)$ for each ideal I of R .
- (4) $\text{Hom}_R(I_R, I_R)$ is a projective R -module for each ideal I of R .

Proof. We first give a proof assuming that R is strongly regular.

(1) $\leftrightarrow$ (2): If I is an ideal of R and $\phi \in \text{Hom}_R(I_R, R_R)$, then, since R is regular, $\phi(I) \subseteq I$. But R is injective if and only if every such ϕ is the restriction of a left multiplication map of R [6, Theorem I.3.2].

(1) $\rightarrow$ (3): By Corollary 7 $B(R)$ is complete. Since $R = I' \oplus I''$,

$$I'' \cong \mu_I(I'') = \mu_I(R) = \text{Hom}_R(I_R, I_R).$$

(3) $\rightarrow$ (4): By Proposition 3 I' is a summand of R . Since $I'' \cong \text{Hom}_R(I_R, I_R)$, μ_I is epic. By Theorem 6 $\text{Hom}_R(I_R, I_R)$ is a projective R -module.

(4) $\rightarrow$ (2): This implication follows from Theorem 7.

Now if R is a regular P.I. ring, then the only part of the preceding proof that needs modification is the implication (2) $\rightarrow$ (1). Let J be a right ideal of R and let $\phi \in \text{Hom}_R(J, R)$. If K is a complement of J , then $J \oplus K$ is an essential right ideal of R and ϕ can be extended to $J \oplus K$. Thus we may assume that J is an essential right ideal of R . By [18, Theorem 6], J contains an essential ideal I of R . Then the restriction of ϕ to I is given by left multiplication by some element x of R . If $a \in J$, then, since I_R is essential in J , there is an essential right ideal D of R such that $aD \subseteq I$. So $\phi(a)d = \phi(ad) = xad$ for each $d \in D$. Hence $\phi(a) = xa$, and so R is right injective. But then R is also left injective since $Q(T)$ is the maximal left quotient ring of any semiprime P.I. ring T [18, Theorem 5].

Added in proof. The following example shows that the analogue of Theorem 5 is false for regular P.I. rings; i.e., the endomorphism ring of an ideal of a regular P.I. ring need not be regular (but by Martindale's theorem [18] it is P.I., since it is a subring of Q). Let R be the ring of all sequences of 2×2 matrices over the field F which are eventually in $\begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}$, and let I be the ideal of all sequences that are eventually in $\begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}$. Then Q is the ring of all sequences of 2×2 matrices over F [21, 2.1], and $S \cong \{q \in Q \mid qI \subseteq I\} = \{q \in Q \mid q \text{ is eventually in } \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}\}$.

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