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O-PRIMITIVE ORDERED PERMUTATION GROUPS II

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This paper is a sequal to *o-primitive ordered permutation groups* **[Pacific J. Math., 40 (1972), 349–372]. There it was shown that if $A(\Omega)$ is the lattice-ordered group of all o-permutations of a chain Ω, and if G is an l-subgroup of $A(\Omega)$ which is periodically o-primitive (transitive and lacking proper convex blocks, but neither o-2-transitive nor regular), then the (convex) orbits of any stabilizer subgroup G_α, $\alpha \in \Omega$, themselves form a chain o-isomorphic to the integers. Let Δ be any non-singleton orbit of G_α. Here it is shown that G_α is faithful on Δ and that $G_\alpha \mid \Delta$ is o-2-transitive and contains an element $\neq 1$ of bounded support. From this it follows that all o-primitive groups (except for certain pathological o-2-transitive groups) are complete l-subgroups of $A(\Omega)$, and hence are completely distributive. When G is "full", $G_\alpha \mid \Delta$ satisfies an important "splice" property, and G_α and G are laterally complete. There is a detailed description of the unique full group G for which Δ is an α-set, and a listing of the other "nice" permutation group representations of G.**

We assume throughout that G is a transitive l-subgroup of $A(\Omega)$ (except that the more general *coherent* subgroups of $A(\Omega)$ are discussed briefly in the last section). Thus the orbits of G_α are convex, and the concepts of "orbital" [6] and "orbit" coincide. Although familiarity with [6] is assumed, we recapitulate most of Main Theorem 40 for l-permutation groups.

THEOREM 1. *Let (G, Ω) be an o-primitive l-permutation group which is neither o-2-transitive nor regular, and let $\alpha \in \Omega$. Then the long orbits of G_α form a chain o-isomorphic to the integers. Let $\Delta_1 = (\Delta_1)_\alpha$ denote the first positive long orbit, and let Δ_{j+1} be the first long orbit greater than Δ_j. Between Δ_j and Δ_{j+1} there lies at most one point of Ω. Either there is a positive integer n such that* $\sup \Delta_j \in \Omega$ *if and only if $j \equiv 0 \pmod n$, and we say that G has* Config (n); *or* $\sup \Delta_j \in \Omega$ *only when $j = 0$, and we say that G has* Config (∞). *There is a unique o-permutation z of $\bar{\Omega}$, $\bar{\Omega}$ the Dedekind completion (without end points) of Ω, such that $\alpha z = \sup(\Delta_1)_\alpha$ for each $\alpha \in \Omega$. z is the period of G in the sense that it generates (as a group) the centralizer $Z_{A(\bar{\Omega})}G$; so that $(\bar{\beta} z)g = (\bar{\beta} g)z$ for all $\bar{\beta} \in \bar{\Omega}$, $g \in G$.*

This periodicity is of paramount importance, and is the key to most of the following results. The action of $g \in G$ on any Δ_j determines its action on all of Ω. $(\Delta_{j+1})_\alpha$ is "one period up" from $(\Delta_j)_\alpha$ in the sense that $(\bar{\Delta}_j)_\alpha z = (\bar{\Delta}_{j+1})_\alpha$.

Full groups are those for which G is the entire centralizer $Z_{A(\bar{\Omega})}z$. It is shown that G is full if and only if G_α is the set of all o-permutations of Δ_1 which preserve the sets $\Delta_j z^{1-j} \subseteqq \bar{\Delta}_1$; and that the full group G is determined by these sets. It is shown also that the *primitive* (in the ordinary permutation group sense) l-permutation groups are precisely those which are o-2-transitive or periodically o-primitive with Config (∞).

2. Representations. A *representation* of an l-group G is an l-isomorphism θ of G into some $A(\Sigma)$. Here all representations will be transitive (meaning that $G\theta$ is a transitive l-subgroup of $A(\Sigma)$). Now let (G, Ω) be an l-permutation group, and let $\bar{\omega} \in \bar{\Omega}$. For $g \in G$, form $\hat{g} \in A(\bar{\omega}G)$ by first extending g to $\bar{\Omega}$ and then restricting to $\bar{\omega}G$. If G is faithful on $\bar{\omega}G$ (e.g., if $\bar{\omega}G$ is dense in $\bar{\Omega}$), the map $g \to \hat{g}$ is a representation of G into $A(\bar{\omega}G)$. We shall identify G with its image and speak of $(G, \bar{\omega}G)$. Of course, $(G, \bar{\omega}G)$ depends only on the orbit $\bar{\omega}G$ and not on the particular $\bar{\omega}$.

Holland [3] defined a transitive l-permutation group (G, Ω) to be *weakly o-primitive* if for every o-block system $\tilde{\Delta}$ of G (except the system of singletons) there exists $1 \neq g \in G$ such that $\Gamma g = \Gamma$ for all $\Gamma \in \Delta$. A representation θ of an l-group G into $A(\Sigma)$ is *complete* if it preserves arbitrary sups that exist in G, or equivalently, if $G\theta$ is a complete l-subgroup of $A(\Sigma)$ in the sense that if $g \in G$ is the sup in G of $\{g_i \mid i \in I\}$, then g is also the sup in $A(\Sigma)$ of $\{g_i\}$. In [8], the present author defined an o-2-transitive group G to be *pathological* if it contains no element $\neq 1$ of bounded support. We collect some facts about representations of o-primitive groups.

THEOREM 2. *Let (G, Ω) be o-primitive, but not pathologically o-2-transitive. Let θ be a weakly o-primitive (alternately, a complete transitive) representation of G as an l-subgroup of some $A(\Sigma)$. Then there is an o-isomorphism ψ from some $\bar{\omega}G$ onto Σ which, together with θ, furnishes an o-isomorphism from $(G, \bar{\omega}G)$ onto $(G\theta, \Sigma)$. The collection of $(G, \bar{\omega}G)$'s constitute (up to o-isomorphism) all weakly o-primitive (alternately, all complete transitive) representations of G.*

Proof. At present we shall treat only weakly o-primitive representations; after Theorem 6, we shall return to complete representations. If (G, Ω) is o-2-transitive (and not pathological), the first statement is a special case of [3, Theorem 7]; and by the proof of that theorem it suffices in general to show that a prime subgroup of G which moves every $\bar{\beta} \in \bar{\Omega}$ must in fact be all of G. If (G, Ω) is regular, this is obvious. Now suppose (G, Ω) is periodically o-primitive, that P is a prime subgroup of G moving every $\bar{\beta} \in \bar{\Omega}$, and that $1 < g \in G$. Then (as in [2, p. 329]) given any bounded interval Π of Γ, we may take the sup of an appropriate finite collection of elements of P and raise it to an appropriate power to obtain $1 < f \in L$ such that f exceeds g on Π. We take Π to be of the form $(\alpha, \alpha z)$, and periodicity guarantees that f exceeds g on Ω. Therefore $P = G$, concluding the periodically o-primitive case. Finally, anticipating Theorem 3, we find that every $(G, \bar{\omega}G)$ is in fact weakly o-primitive (indeed, primitive).

An l-group G is *laterally complete* if every pairwise disjoint set of elements has a sup in G. For an l-permutation group (G, Ω), we formulate a much stronger property: Let $\mathscr{D}_1 = \{\Delta_{i,1} \mid i \in I\}$ be a collection of pairwise disjoint nondegenerate segments of Ω such that $\bigcup \mathscr{D}_1$ is dense in Ω, with $\mathscr{D}_1$ totally ordered in the natural way; and similarly for $\mathscr{D}_2 = \{\Delta_{i,2} \mid i \in I\}$. Let f be an o-isomorphism from $\mathscr{D}_1$ onto $\mathscr{D}_2$ such that whenever $\bar{\mu}$ is a proper Dedekind cut in $\mathscr{D}_1$ (and thus may be considered as an element of $\bar{\Omega}$), $\bar{\mu}$ and $\bar{\mu}f$ lie in the same orbit of G. Suppose that for each $i \in I$, there exists $g_i \in G$ such that $\Delta_{i,1}g_i = \Delta_{i,2}$. We shall say that (G, Ω) has the *splice property* if whenever these circumstances occur, there exists $g \in G$ such that $g \mid \Delta_{i,1} = g_i$ for each i. It is easily checked that it suffices to consider the special case in which each g_i is positive ($\omega \leqq \omega g_i$ for each $\omega \in \Delta_{i,1}$).

If (G, Ω) is *entire* (i.e., if $G = A(\Omega)$), then G has the splice property. Actually, $A(\Omega)$ satisfies a stronger property, whereby $\bar{\mu}$ and $\bar{\mu}f$ are required only to be both in Ω or both in $\bar{\Omega}\backslash\Omega$; but there are examples in which this stronger property does not carry over to $(A(\Omega), \bar{\omega}A(\Omega))$.

(G, Ω) is *depressible* if given any $g \in G$ and $\gamma \in \Omega$ for which $\gamma g \neq \gamma$, there exists $h \in G$ such that $\omega h = \omega g$ if ω lies in the interval of support $\mathrm{Conv}\{\gamma g^n \mid n \text{ an integer}\}$, and $\omega g = \omega$ otherwise. Clearly groups having the splice property are depressible as well as laterally complete. On the other hand, if Ω is the real numbers, and G is the set of all o-permutations g of Ω for which there exists no monotone sequence $\omega_n \to \omega$ such that $(\omega_n g - \omega)/(\omega_n - \omega) \to 0$ or ∞, then G is a depressible, laterally complete, o-2-transitive l-permutation group, but it does not have the splice property. (When showing that G is closed under product, use the fact that if $u_n v_n \to 0$ or ∞ $(u_n, v_n > 0)$, then some subsequence of $\{u_n\}$ or of $\{v_n\}$ approaches 0 or ∞, respectively.)

Holland [3] defined a transitive l-permutation group (G, Ω) to be locally *o-primitive* if in the totally ordered set of o-block systems (excluding the system of singletons), there is a smallest system $\tilde{\Delta}$. The o-blocks in $\tilde{\Delta}$ are called the *primitive segments* of G. If Γ is a primitive segment, let $G \mid \Gamma$ denote the restriction of G to Γ, i.e., $\{g \mid \Gamma : g \in G \text{ and } \Gamma g = \Gamma\}$. All $(G \mid \Gamma, \Gamma)$'s are isomorphic as o-permutation groups, and they are o-primitive. A property preserved by isomorphism and enjoyed by $(G \mid \Gamma, \Gamma)$ for one (hence every) primitive segment Γ will be said to be enjoyed locally by (G, Ω). In view of Theorem 1, every locally o-primitive group must be locally o-2-transitive, locally regular, or locally periodically o-primitive.

For any o-primitive (G, Ω), and any $\bar{\omega} \in \bar{\Omega}$, $\bar{\omega}G$ is dense in $\bar{\Omega}$ ([3, Theorem 2]). Thus Theorem 2 leads us to

THEOREM 3. *Let (G, Ω) be a transitive l-permutation group and let $\bar{\omega} \in \bar{\Omega}$ with $\bar{\omega}G$ dense in $\bar{\Omega}$. Then*

(1) *Extension to $\bar{\Omega}$ followed by restriction to $\bar{\omega}G$ provides a canonical one-to-one correspondence between the collection of non-singleton o-blocks of (G, Ω) and that of $(G, \bar{\omega}G)$.*

(2) *The image under this correspondence of an o-block system of (G, Ω) is an o-block system of $(G, \bar{\omega}G)$.*

(3) *A canonical o-isomorphism from the tower of o-block systems of (G, Ω) onto that of $(G, \bar{\omega}G)$ is given by (2).*

(4) *Corresponding o-primitive components of (G, Ω) and $(G, \bar{\omega}G)$ are o-isomorphic, with the following exception: In the locally o-primitive case, if Γ is the primitive segment of (G, Ω) for which $\bar{\omega} \in \bar{\Gamma}$, $\Gamma_{\bar{\omega}} = \Gamma \cap \bar{\omega}G$ is the primitive segment of $\bar{\omega}G$ containing $\bar{\omega}$. But at least $(G \mid \Gamma, \Gamma)$ and $(G \mid \Gamma_{\bar{\omega}}, \Gamma_{\bar{\omega}})$ are both pathologically (or both nonpathologically) o-2-transitive, both regular, or both periodically o-primitive (in which case both have the same period z).*

Moreover, the following properties are enjoyed either by both (G, Ω) and $(G, \bar{\omega}G)$, or by neither:

(5) *Weak o-primitivity.*

(6) *The splice property.*

(7) *Depressibility.*

(8) *Entirety on some dense orbit of G.*

(9) *Completeness in the entire group $A(\Omega)$, respectively $A(\bar{\omega}G)$. (See the comments preceding Theorem 8.)*

All parts of the theorem are routine except for (9), which is contained in [5, Theorem 1]. In connection with (8), we mention that examples in which entirety does not carry over to $(A(\Omega), \bar{\omega}A(\Omega))$ are to be found in [7, Theorem 15].

3. **Periodic o-primitivity.** Let (G, Ω) be periodically o-primitive with Config (n), and let I_n be $\{1, \cdots, n\}$ if n is finite, and be the integers if $n = \infty$. Let $\Omega_k = \Omega z^{1-k} \subseteq \bar{\Omega}(k \in I_n)$. Since z centralizes G, $\Omega_k G = \Omega_k$ for each k. Let $\alpha \in \Omega$, and let the long orbits $\{\Delta_j\}$ of G_α be denoted as in the introduction. $\Delta_j z^{1-j} = \bar{\Delta}_1 \cap \Omega_k$, where $k \equiv j \pmod{n}$. By the *signature* of G we shall mean the collection $\{\bar{\Delta}_1 \cap \Omega_k (= \Delta_k z^{1-k}) \mid k \in I_n\}$. [6, Theorem 54] gives three conditions which the signature must satisfy.

By an abstract n-signature ($n = 1, 2, \cdots, \infty$), we shall mean a collection $\{\Sigma_k \mid k \in I_n\}$, with each Σ_k a specified subset of $\bar{\Sigma}_1$, satisfying the conditions (for the particular n) of [6, Theorem 54]. By an o-isomorphism of one such signature $\{\Sigma_k\}$ onto another $\{\Pi_k\}$ we shall mean an o-isomorphism φ from $\bar{\Sigma}_1$ onto $\bar{\Pi}_1$ such that $\Sigma_k\varphi = \Pi_k$ for each $k \in I_n$. The signature defined above for G is of course an n-signature, and by the transitivity of G, it is independent (up to o-isomorphism) of the choice of α.

Recall that the full groups are universal in the sense that every periodically o-primitive group G is contained in a full group having the same period z, namely $Z_{A(\bar{\Omega})}z$.

THEOREM 4. *For each $n = 1, 2, \cdots, \infty$, a one-to-one correspondence between the collection of (o-isomorphism classes of) full periodically o-primitive groups (G, Ω) of* Config (n) *and the collection of (o-isomorphism classes of) n-signatures is given by mapping each group to its signature. When $n = 1$, the chain Ω determines the (full) group G and the signature of (G, Ω).*

Proof. In [6, Theorem 54] we constructed from an arbitrary n-signature $\{\Sigma_k\}$ a full group (G, Ω) such that for an appropriate α, $\Delta_k z^{1-k} = \Sigma_k$ for all $k \in I_n$. Hence our mapping is onto. It is also one-to-one, for the signature of (G, Ω) determines Ω and determines the period z on Ω and hence on $\bar{\Omega}$; and then since G is full, $G = Z_{A(\bar{\Omega})}z$. If $n = 1$, $\Sigma_1 = \Delta_1$ is a closed interval of Ω. Since G is o-primitive, $A(\Omega)$ must be o-primitive and thus o-2-transitive, so that all nondegenerate closed intervals of Ω are o-isomorphic. Hence Ω determines Δ_1 and thus determines G.

LEMMA 5. *Let (G, Ω) be a periodically o-primitive l-permutation group. Let $\alpha \in \Omega$ and let Δ be any long orbit of G_α. Then G_α is faithful on Δ, and (G_α, Δ) is a nonpathologically o-2-transitive l-permutation group.*

Proof. Periodicity guarantees that G_α is faithful on Δ and that, in view of part (4) of Theorem 3, we may assume that Δ is the first positive orbit of G_α. Let $\Gamma = \Delta'$, the last negative orbit of G_α. Since (G_α, Δ) is transitive, it will be o-2-transitive if for $\beta \in \Delta$, $(G_\alpha)_\beta = G_\alpha \cap G_\beta$ is transitive on $\Pi = \{\delta \in \Delta \mid \beta < \delta\}$. Pick any $\gamma, \delta \in \Pi$, with $\gamma < \delta$. Next pick $1 \leqq h \in G_\alpha$ so that $\gamma h = \delta$. Then $\beta \leqq \beta h < \delta$, so $\beta h \in \Delta$. Now pick $1 \geqq r \in G_\delta$ so that $(\beta h)r = \beta$. Since $\alpha r \leqq \alpha < \beta < \delta$, and since both αr and α lie in the last negative orbit Γ_δ of G_δ, we have $\alpha r, \alpha \in \Gamma_\beta$. Hence we may pick $1 \leqq s \in G_\beta$ so that $(\alpha r)s = \alpha$. Now $hrs \in (G_\alpha)_\beta$, and $\gamma hrs = \delta rs = \delta s \geqq \delta$, so that γ and δ lie in the same (convex) orbit of $(G_\alpha)_\beta$. Therefore, $(G_\alpha)_\beta$ has only one positive orbit in Δ, so that (G_α, Δ) is o-2-transitive.

We can pick $g \in G$ such that $\alpha g < \alpha$ and $\beta g > \beta$. Then $1 \neq g \vee 1$ fixes each point in some segment Λ of Ω which meets both Δ and Γ, so that $1 \neq (g \vee 1) \mid \Delta$ has support which is certainly bounded below, and by periodicity, is also bounded above. Therefore, (G_α, Δ) is not pathological.

The author showed in [4, Theorem 7] that for a transitive l-subgroup G of $A(\Omega)$, the following are equivalent:

(1) G_α is *closed* subgroup of G for one (hence every) $\alpha \in \Omega$, i.e., if $g = \bigvee_{i \in I} g_i$ with each $g_i \in G_\alpha$, then $g \in G_\alpha$.

(2) G is a *complete* subgroup of $A(\Omega)$.

(3) Sups *in G are pointwise*, i.e., if $g = \bigvee_{i \in I} g_i$ with each $g_i \in G$, then for each $\beta \in \Omega$, βg is the sup in Ω of $\{\beta g_i \mid i \in I\}$.

Moreover, it was shown in [4, Corollary 15] that in the presence of these conditions, we have

(4) G is a *completely distributive* l-group, i.e., $\bigwedge_{i \in I} \bigvee_{k \in K} g_{ik} = \bigvee_{f \in K^I} \bigwedge_{i \in I} g_{if(i)}$ for any collection $\{g_{ik} \mid i \in I, k \in K\}$ of G for which the indicated sups and infs exist.

THEOREM 6. *Let (G, Ω) be an o-primitive l-permutation group. Then Conditions* (1), (2), (3), *and* (4) *are all equivalent, and they fail if and only if G is pathologically o-2-transitive.*

Proof. The o-2-transitive case is precisely the content of [8, Theorem 1], and the conditions hold automatically in the regular case. In the periodically o-primitive case, the proof parallels the proof in [8] for the nonpathologically o-2-transitive case: Suppose $g = \bigvee_{i \in I} g_i$, with $g \in G$ and each $g_i \in G_\alpha$, but with $\alpha < \alpha g$. By Lemma 5, we can pick $1 > h \in G_\alpha$ such that $h \mid \Delta_0$ (Δ_0 the last negative orbit of G_α) has support contained in $(\alpha g^{-1}, \alpha)$. Then for each $i \in I$, $g_i \leqq hg < g$ on Δ_0, and hence on all of Ω by periodicity, giving a contradiction. Therefore G_α is closed, and the other conditions follow.

Now we prove Theorem 2 for complete representations. Theorem 6 states that every $(G, \bar{\omega}G)$ is complete. For the rest, it suffices to show that every proper closed prime subgroup of G is $G_{\bar{\omega}}$ for some $\bar{\omega} \in \bar{\Omega}$; and the proof of Theorem 11 of [5] shows that if this were not the case, G would have a proper o-block, violating o-primitivity.

LEMMA 7. *Let (G, Ω) be periodically o-primitive, and let $\Gamma \neq \Omega$ be a* (*not necessarily convex*) *block of G. Then $\Gamma \subseteqq FxG_\alpha$ for every $\alpha \in \Gamma$.*

Proof. If $\delta \in \Gamma \backslash FxG_\alpha$, then Γ would contain the segment δG_α, so that $\Gamma = \Omega$ by periodicity.

THEOREM 8. *Let G be a transitive l-permutation group. Then G is primitive if and only if G is o-2-transitive or G is periodically* *o-primitive and has* Config (∞).

Proof. If G is primitive, then *a fortiori*, G is o-primitive. If G is o-2-transitive, it is clearly primitive. If G is periodically o-primitive and has Config (∞), $FxG_\alpha = \{\alpha\}$, so G is primitive by the lemma. However, if G has Config (n) for some finite n, the block FxG_α violates primitivity; and if (G, Ω) is regular, it is the regular representation of some subgroup of the reals ([6, Proposition 24]) and thus is not primitive.

We now borrow some terminology from [1, pp. 142–4] and [7], assuming for convenience that Ω is homogeneous and dense in itself (which will necessarily be the case if (G, Ω) is o-primitive, unless Ω is the integers). The point or hole (i.e., proper Dedekind cut) $\bar{\omega}$ of Ω is said to have *character* $c_{\beta\gamma}$ if ω_β is the unique regular ordinal number which is o-isomorphic to a cofinal subset of $\{\sigma \in \Omega \mid \sigma < \bar{\omega}\}$ and dually for ω_γ. All elements of any one orbit $\bar{\omega}G$ have the same character. Ω is said to have *final character* c_β if ω_β is the unique regular ordinal o-isomorphic to a cofinal subset of Ω; and dually for *initial character*. Alternately, any of these characters can be determined by using subsets not of Ω, but of any dense subset of $\bar{\Omega}$.

LEMMA 9. *Let (G, Ω) be periodically o-primitive, and suppose that the points of Ω have character $c_{\beta\gamma}$. Then if Δ is any long orbit of G_α, $\alpha \in \Omega$, the initial character of Δ is c_γ and the final character is c_β.*

Proof. $\Delta = \Delta_j$ for some j. Δ has the same initial character as Δz^{1-j}; since the latter is dense in $\bar{\Delta}_1$, its initial character is that of Δ_1, which is c_γ. A similar argument works for final characters.

PROPOSITION 10. *Suppose that (G, Ω) is periodically o-primitive, and that in its order topology, Ω satisfies the first countability axiom* (i.e., *the points of Ω have character c_{00}*). *Then all long orbits of G_α, $\alpha \in \Omega$, are o-isomorphic.*

Proof. Let Γ and Δ be long orbits of G_α. Picking $g \in G$ such that Γg meets Δ, we obtain an o-isomorphism between some interval of Γ and some interval of Δ. Since $G_\alpha \mid \Gamma$ and $G_\alpha \mid \Delta$ are o-2-transitive, all nondegenerate closed intervals of Γ and of Δ are o-isomorphic to each other. Since the points of Ω have character c_{00}, the lemma guarantees that Γ and Δ both have c_0 as initial and final characters. The proposition follows.

4. Extracts of periodically o-primitive groups. The results about (G_α, Δ) mentioned in the introduction will be needed also for $G_{\bar{\beta}}$, $\bar{\beta} \in \bar{\Omega}$.

PROPOSITION 11. *Let (G, Ω) be periodically o-primitive, and let*

$\bar\beta \in \bar\Omega$. Then the long orbits in Ω of $G_{\bar\beta}$ are the sets $\Delta_j = \{\omega \in \Omega \mid \bar\beta z^{j-1} < \omega < \bar\beta z^j\}$, j an integer; and $G_{\bar\beta}$ is faithful on each Δ_j.

Proof. The statement about the long orbit structure of $G_{\bar\beta}$ is equivalent to the statement that the fixed points in $\bar\Omega$ of $G_{\bar\beta}$ are precisely those of the form $\bar\beta z^j$. Periodicity guarantees that $G_{\bar\beta}$ fixes these points and that to show it fixes no others, it suffices to consider $\bar\beta < \bar\gamma < \bar\beta z$. Now pick $\alpha \in \Omega$ such that $\alpha < \bar\beta < \bar\gamma < \alpha z$. Lemma 5 guarantees that G_α is o-2-transitive on its first positive orbit, so there exists $h \in G_\alpha$ with $\bar\beta h \leqq \bar\beta$ and $\bar\gamma h > \bar\gamma$. Now $h \vee 1$ fixes $\bar\beta$ and moves $\bar\gamma$, as desired. Therefore, the long orbits of $G_{\bar\beta}$ are as described, and by periodicity, G is faithful on each of them.

We now define the $\bar\beta$-*extract* of (G, Ω), where G is periodically o-primitive and $\bar\beta \in \bar\Omega$, to be $(G_{\bar\beta}, \Delta_1)$. (Warning: $\Delta_1 \not\subseteq \bar\beta G$ unless $\bar\beta \in \Omega$.) Of course if $\bar\beta$ and $\bar\gamma$ lie in the same orbit of G, the $\bar\beta$- and $\bar\gamma$-extracts are isomorphic as o-permutation groups.

LEMMA 12. *Let $\mathscr{P}$ be an o-permutation group property which carries over from (H, Σ) to $(H, \bar\sigma H)$ when $\bar\sigma H$ is dense in $\bar\Sigma$* (cf. Theorem 2). *Suppose that $\mathscr{P}$ holds for every α-extract, $\alpha \in \Omega$, of every periodically o-primitive group (G, Ω). Then for every periodically o-primitive (G, Ω), $\mathscr{P}$ holds for every $\bar\beta$-extract, $\bar\beta \in \bar\Omega$, and hence for $(G_{\bar\beta}, \Delta)$, where Δ is any long orbit of $G_{\bar\beta}$.*

Proof. $\mathscr{P}$ holds for the $\bar\beta$-extract of (G, Ω) because it carries over from the $\bar\beta$-extract of $(G, \bar\beta G)$. ($(G, \bar\beta G)$ is periodically o-primitive by Theorem 3.) $(G_{\bar\beta}, \Delta)$ is merely the $\bar\gamma$-extract of G, where $\bar\gamma = \inf \Delta$.

THEOREM 13. *Every extract of a periodically o-primitive l-permutation group is a nonpathologically o-2-transitive l-permutation group.*

Proof. Use Lemmas 5 and 12.

LEMMA 14. *Let (G, Ω) be periodically o-primitive, let $\bar\beta \in \bar\Omega$, and let $g \in G$ be such that $\bar\beta g \notin \bar{F}xG_{\bar\beta}$* (i.e., $\{\bar\omega \in \bar\Omega \mid \bar\omega G_{\bar\beta} = \bar\omega\}$). *Then G is generated as a group by $G_{\bar\beta}$ and g.*

Proof. We may assume that $\bar\beta \in \Omega$. (If not, replace (G, Ω) by $(G, \bar\beta G)$.) Now let C be the subgroup of G generated by $G_{\bar\beta}$ and g. Then $\bar\beta C$ is a block of G (by [9, Theorem 7.5]), contradicting Lemma 7.

LEMMA 15. *Let (G, Ω) be full and let H and K be periodically o-primitive l-subgroups of G having the same period z as G. If there exists $\bar\beta \in \bar\Omega$ such that $H_{\bar\beta} = K_{\bar\beta}$ and $\bar\beta g \notin \bar{F}xG_{\bar\beta}$ for some $g \in H \cap K$, then $H = K$.*

Proof. $\bar{F}xG_{\bar{\beta}} = \bar{F}xH_{\bar{\beta}} = \bar{F}xK_{\bar{\beta}}$, so that this lemma follows from the previous one.

LEMMA 16. *Let* (G, Ω) *be full and let* $H \neq G$ *be a periodically o-primitive l-subgroup of* G. *Then* $H_{\bar{\beta}} \neq G_{\bar{\beta}}$ *for every* $\bar{\beta} \in \bar{\Omega}$.

Proof. If H has the same period z as G, the previous lemma suffices. But if z is not the period of H, $\bar{F}xH_{\bar{\beta}} \neq \bar{F}xG_{\bar{\beta}}$, so $H_{\bar{\beta}} \neq G_{\bar{\beta}}$.

THEOREM 17. *Let* $(G_{\bar{\beta}}, \Delta)$ *be any extract of a periodically o-primitive group* (G, Ω). *Then* $G_{\bar{\beta}}$ *preserves the subsets* $\bar{\Delta} \cap \Omega_k (k \in I_n)$ *of* $\bar{\Delta}$. *Moreover,* G *is full if and only if* $G_{\bar{\beta}}$ *consists of all o-permutations of* Δ *which preserve these sets.*

Proof. $G_{\bar{\beta}}$ preserves $\bar{\Delta}$ and Ω_k, and thus preserves $\bar{\Delta} \cap \Omega_k$. Suppose G is full. If $h \in A(\Delta)$ preserves the sets $\bar{\Delta} \cap \Omega_k$, the unique o-permutation of $\bar{\Omega}$ which extends h and commutes with z will preserve Ω, so that $G_{\bar{\beta}}$ will be as described. Conversely, if $G_{\bar{\beta}}$ fits the description, and if K is the full periodically o-primitive group containing G and having the same period z as G ([6, Proposition 53]), then $G_{\bar{\beta}} = K_{\bar{\beta}}$, so that $G = K$ (Lemma 16) and G is full.

THEOREM 18. *Let* (G, Ω) *be periodically o-primitive. If one of its extracts is entire, so are they all; and similarly for the splice property. If the extracts are entire,* G *is full* (*and conversely if* G *has* Config (1)). *If* G *is full, the extracts have the splice property.*

Proof. Suppose that the extract $(G_{\bar{\beta}}, \Pi)$ has the splice property. In the extract $(G_{\bar{\gamma}}, \Lambda)$, let $\mathscr{D}_j = \{\Delta_{i,j} \mid i \in I\}$, f, and $\{g_i \mid i \in I\}$ satisfy the conditions of the splice property. With no loss of generality, we may suppose first (since $\bar{\beta}G$ is dense in $\bar{\Omega}$) that $\bar{\beta} \in \bar{\Lambda}$, and next (by multiplying by an appropriate element of $G_{\bar{\gamma}}$) that $\bar{\beta}$ is fixed by the permutation of $\bar{\Lambda}$ obtained by splicing the g_i's. Now since $(G_{\bar{\beta}}, \Pi)$ has the splice property, when we splice together the g_i's for which $\Delta_{i,1} \subseteqq \Pi \cap \Lambda$ and $\hat{g}$ (the identity on $\Pi \backslash \Lambda$), we obtain an element h_1 of $G_{\bar{\beta}}$ which acts as desired on $\Pi \cap \Lambda$ and (by periodicity) is the identity on $\Lambda \backslash \Pi$. Similarly, for an appropriate $\bar{\eta}$ ($\bar{\beta}z^{-1} < \bar{\eta} < \bar{\gamma}$), there exists $h_2 \in G_{\bar{\eta}}$ which acts as desired on $\Lambda \backslash \Pi$ and is the identity on $\Pi \cap \Lambda$. The product $h_1 h_2$ satisfies the conclusion of the splice property.

For entirety of extracts, we proceed similarly, letting $g \in A(\Lambda)$. We may not assume that $\bar{\beta}$ is fixed, but we do obtain an h_1 which agrees with g on $\Pi \cap \Lambda$ and an h_2 which agrees with g on $\Lambda \backslash \Pi$. Splicing these together, we find that $g \in G_{\bar{\gamma}}$. (Since one extract is entire, it, and hence every extract, satisfies the splice property.)

If the extracts are entire, then by Theorem 17, G is certainly full; and if G has Config (1), there is only one $\bar{\Delta} \cap \Omega_k$, namely Δ, so the converse holds. If G is full, Theorem 17 guarantees that its extracts have the splice property.

THEOREM 19. *Let (G, Ω) be periodically o-primitive, and let $\bar{\beta} \in \bar{\Omega}$. Then $G_{\bar{\beta}}$ is laterally complete if and only if G is laterally complete; and these conditions obtain for all full groups.*

Proof. For the "iff" statement, we can assume that $\bar{\beta} = \beta \in \Omega$. (If not, consider $(G, \bar{\beta}G)$.) Certainly if G is laterally complete, so is G_β, for G_β is closed under arbitrary sups in G (Theorem 6). Now suppose that G_β is laterally complete, and let $\{h_i \mid i \in I\}$ be a set of pairwise disjoint elements of G. At most one of the h_i's moves β, so we may suppose with no loss of generality that $\{h_i \mid i \in I\} \subseteqq G_\beta$. Let h be the sup in G_β of $\{h_i\}$. If this sup is pointwise, h will also be the sup in G of $\{h_i\}$, and we shall be finished. Thus let Δ be a long orbit of G_β. G_β is faithful on Δ, so in $G_\beta \mid \Delta$, $h \mid \Delta = \sup \{h_i \mid \Delta\}$. Since $G_\beta \mid \Delta$ is nonpathologically o-2-transitive, this sup is pointwise on Δ (Theorem 6); by periodicity, the sup is then pointwise on Ω, as required. If G is full, the $\bar{\beta}$-extract has the splice property by Theorem 18, so $G_{\bar{\beta}}$ is laterally complete.

5. Periodically o-primitive groups constructed from α-sets. Let ω_α be a regular ordinal number. An *α-set* is a chain Ω of cardinality $\aleph_\alpha$ in which for any two (possibly empty) subsets $\Gamma < \Delta$ of cardinality less than $\aleph_\alpha$, there exists $\omega \in \Omega$ such that $\Gamma < \omega < \Delta$. If we consider only nonempty Γ and Δ (though still requiring that Ω has neither a first nor a last point), so that the terminal characters need not be c_α, we obtain a generalization which we shall call a *truncated α-set*. We shall need some information from [7] about truncated α-sets Ω. For any regular ω_α, and any regular ω_β, ω_γ less than or equal to ω_α, there exists (assuming the generalized continuum hypothesis) a truncated α-set Ω having initial character ω_β and final character ω_γ; and it is unique up to o-isomorphism. The points of Ω have character $c_{\alpha\alpha}$, and the holes have character $c_{\alpha\alpha}$, $c_{\alpha\beta}$, or $c_{\beta\alpha}$ (with ω_β regular and $\omega_\beta < \omega_\alpha$), with all of these characters actually occurring. Conversely, these conditions on characters (including the terminal characters), together with the requirement that card $(\Omega) \leqq \aleph_\alpha$, force Ω to be the truncated α-set above. Hence any segment (without end points) of an α-set is a truncated α-set, and every truncated α-set arises in this way. If both terminal characters are c_0, the set will be called *countably truncated.* The set of all holes in a truncated α-set Ω of a given character $c_{\gamma\delta}$ form an orbit $\Omega_{\gamma\delta}$ of $(A(\Omega), \bar{\Omega})$, and these orbits are dense in $\bar{\Omega}$. All have cardinality $\aleph_\alpha$ except for $\Omega_{\alpha\alpha}$, which has cardinality $2^{\aleph_\alpha}$.

LEMMA 20. *Suppose that (G, Ω) is a periodically o-primitive group having an α-set Δ as the first positive orbit of a stabilizer subgroup G_ξ. Then all long orbits of G_ξ are o-isomorphic to the α-set Δ, and Ω is o-isomorphic to the countably truncated α-set.*

Proof. Use characters. The terminal characters of any long orbit are c_α by Lemma 9; and those of Ω are c_0 because of the configuration of G.

LEMMA 21. *Let Φ_j, $j = 1, 2$, be truncated α-sets having the same initial character and same final character. Let $\{\Psi_{j,i} \mid i \in I\}$, I the positive integers, be a collection of dense pairwise disjoint subsets of $\bar{\Phi}_j$ for which $\Psi_{j,1} = \Phi_j$ and each $\Psi_{j,i}$ is o-isomorphic to Φ_j. Then there exists an o-isomorphism g from Φ_1 onto Φ_2 such that $\Psi_{1,i}g = \Psi_{2,i}$ for each i.*

Proof. First, the lemma holds for nontruncated α-sets; for we may apply the standard proof of the uniqueness of α-sets [1, p. 182], noting that if $\Gamma < \Delta$ in an α-set Ω, and if both sets have cardinality less than $\aleph_\alpha$, then $\{\omega \in \Omega \mid \Gamma < \omega < \Delta\}$ contains more than one point of Ω. Now for truncated α-sets, we may proceed exactly as in the proof of [7, Theorem 5].

THEOREM 22. *Let $n = 1, 2, \cdots$, or ∞, let ω_α be a regular ordinal number, and let Δ be an α-set. Then there exists a unique (up to o-isomorphism) full periodically o-primitive group (G, Ω) having Δ as the first positive orbit of a stabilizer subgroup G_ξ and having* Config (n). *Its extracts are entire if and only if $n = 1$. Let $\hat{\Omega}_{\alpha\alpha} = \Omega_{\alpha\alpha} \backslash \bigcup \{\Omega_k \mid k \in I_n\}$, which is o-isomorphic to $\Omega_{\alpha\alpha}$. (G, Ω) itself, $(G, \hat{\Omega}_{\alpha\alpha})$, and the $(G, \Omega_{\alpha\beta})$'s and $(G, \Omega_{\beta\alpha})$'s constitute (up to o-isomorphism) all weakly o-primitive (alternately, all complete transitive) representations of the l-group G, and distinct representations in the list are non-o-isomorphic. All except (G, Ω) have* Config (1). *The $(G, \Omega_{\alpha\beta})$'s and $(G, \Omega_{\beta\alpha})$'s are never full; and $(G, \hat{\Omega}_{\alpha\alpha})$ is full if and only if $n = 1$.*

Proof. First we show that Δ satisfies Conditions (a), (b), and (c) of [6, Theorem 54]. Let $\Sigma_1 = \Delta$. $\Delta_{\alpha\alpha}$ is dense in $\bar{\Delta}$. From each open interval of Δ, pick an element of $\Delta_{\alpha\alpha}$; and let Σ_2 be the ordered set of holes thus obtained. Now Δ is a dense subset of $\bar{\Sigma}_2 = \bar{\Delta}$, so we may use subsets of Δ to determine characters for Σ_2. Hence each point in Σ_2 has character $c_{\alpha\alpha}$; each hole in Σ_2 has character $c_{\alpha\alpha}$, $c_{\alpha\beta}$, or

$c_{\beta\alpha}$; the initial and final characters of Σ_2 are c_α; and card $(\Sigma_2) \leqq \aleph_\alpha$. Therefore Σ_2 is an α-set, and of course $\Sigma_2 \subseteqq \bar{\Delta}$ and $\Sigma_2 \cap \Sigma_1 = \Box$. $\Sigma_1 \cup \Sigma_2$ is also an α-set, so we may continue this process, obtaining a collection $\{\Sigma_i \mid i \in I_n\}$ of dense pairwise disjoint subsets of $\bar{\Omega}$, all of them α-sets. Thus Condition (a) is satisfied. If $\bar{\eta} \in \bar{\Delta}$ has character $c_{\alpha\alpha}$, then for any Σ_i, $\{\lambda \in \Omega \mid \lambda < \bar{\eta}\}$ and $\{\lambda \in \Omega \mid \lambda > \bar{\eta}\}$ are α-sets, so by applying Lemma 21 we get Conditions (b) and (c). This proves the existence of (G, Ω); and by the proof of [6, Theorem 54], $\Sigma_k = \bar{\Delta} \cap \Omega_k$ for each $k \in I_n$. But Lemmas 20 and 21 show that for a given n, there is (up to o-isomorphism) only one signature with Σ_1 an α-set, so that the uniqueness of (G, Ω) follows from Theorem 4. That the extracts of G are entire if and only if $n = 1$ follows from Theorem 17 and that fact that $\Delta_{\alpha\alpha}$ is an orbit of $A(\Delta)$.

By Theorem 2, every weakly o-primitive or complete transitive representation of G is o-isomorphic to some $(G, \bar{\omega}G)$, $\bar{\omega} \in \bar{\Omega}$, and hence to some $(G, \bar{\delta}G)$, $\bar{\delta} \in \bar{\Delta} \cup \{\xi\}$. These representations (all periodically o-primitive by Theorem 3) are of three kinds:

(1) $\bar{\delta} \in \bigcup \Sigma_k$, so that $G_{\bar{\delta}} = G_\omega$ for some $\omega \in \Omega$ and thus $(G, \bar{\delta}G)$ is o-isomorphic to (G, Ω).

(2) $\bar{\delta}$ has character $c_{\alpha\alpha}$, but $\bar{\delta} \notin \bigcup \Sigma_k$. $G_\xi \mid \Delta$ is the set of all o-permutations of Δ which preserve the Σ_k's (by Theorem 17), so by Lemma 21, $\bar{\delta}G_\xi = \Delta_{\alpha\alpha} \backslash \bigcup \Sigma_k = (\Omega_{\alpha\alpha} \backslash \Pi) \cap \bar{\Delta}$, where $\Pi = \bigcup \Omega_k$. Hence $\bar{\delta}G \supseteqq (\Delta_{\alpha\alpha}G) \backslash (\Pi G) = \Omega_{\alpha\alpha} \backslash \Pi$. Since $\bar{\delta} \notin \Pi$ and $\Pi G = \Pi$, $\bar{\delta}G = \Omega_{\alpha\alpha} \backslash \Pi = \hat{\Omega}_{\alpha\alpha}$. Π is a countably truncated α-set, and $\hat{\Omega}_{\alpha\alpha} = \Pi_{\alpha\alpha}$, which is o-isomorphic to $\Omega_{\alpha\alpha}$. Now card $(\hat{\Omega}_{\alpha\alpha}) = 2^{\aleph_\alpha}$, whereas $\hat{\Omega}_{\alpha\alpha} = \Pi_{\alpha\alpha}$ and card $(\bar{\Pi}_{\alpha\alpha} \backslash \Pi_{\alpha\alpha}) = \aleph_\alpha$, forcing $(G, \hat{\Omega}_{\alpha\alpha})$ to have Config (1), for otherwise $\tilde{\Omega}_{\alpha\alpha}z$ would be disjoint from $\hat{\Omega}_{\alpha\alpha}$. Next, $(G, \hat{\Omega}_{\alpha\alpha})$ is full if and only if $G_\xi \mid (\bar{\Delta} \cap \hat{\Omega}_{\alpha\alpha})$ is entire (by Theorem 17); and since $\bar{\Delta} \cap \hat{\Omega}_{\alpha\alpha} = (\bar{\Delta} \cap \Pi)_{\alpha\alpha}$, entirety holds if and only if $G_\xi \mid (\bar{\Delta} \cap \Pi)$ is entire (by [7, Theorem 15]); which is the case if and only if $n = 1$.

(3) $\bar{\delta}$ has character $c_{\alpha\beta}$, $\beta < \alpha$. The argument used in (2) shows that $\bar{\delta}G = \Omega_{\alpha\beta}$. $(G, \Omega_{\alpha\beta})$ has Config (1), for $\Omega_{\alpha\beta}$ has no holes of character $c_{\alpha\beta}$; and the argument in (2) then shows that $(G, \Omega_{\alpha\beta})$ is not full.

(4) $\bar{\delta}$ has character $c_{\beta\alpha}$, $\beta < \alpha$. This case is dual to (3).

Finally, the chains for any two distinct representations in our list differ either in cardinality or in point character, and hence are not o-isomorphic.

PROPOSITION 23. *Let Δ be an α-set and let $\Lambda = \Delta_{\alpha\alpha}$ or $\Lambda = \Delta \cup \Delta_{\alpha\alpha}$. (When $\alpha = 0$, Λ is the irrationals or the reals.) Then there is a full periodically o-primitive group of* Config (1) *having Λ as the first positive orbit of a stabilizer subgroup.*

Proof. Λ is homogeneous ([7, Corollary 16]) and for any $\lambda \in \Lambda$,

$\{\beta \in \Lambda \mid \beta < \lambda\}$ and $\{\beta \in \Lambda \mid \beta > \lambda\}$ are both o-isomorphic to Λ. [6, Theorem 54] guarantees the existence of the desired group of Config (1). The proof of (2) in the previous theorem shows that any group having Λ as the first positive orbit of a stabilizer subgroup must necessarily have Config (1).

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