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ON EXTREMELY REGULAR FUNCTION SPACES

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In this paper extremely regular function spaces are defined and discussed. A necessary and sufficient condition for the existence of proper extremely regular linear subspaces of $C_0(X)$ is obtained.

If X is a locally compact space[1], we denote by $C_0(X)$ the Banach space of continuous, complex-valued functions vanishing at infinity on X, provided with the usual supremum norm.

We call a closed linear subspace A of $C_0(X)$ *extremely regular* (over X), if for each $x_0 \in X$ and each neighborhood V of x_0 and each $0 < \varepsilon < 1$ there is a function f in A such that

$$(1) \qquad 1 = \|f\| = f(x_0) > \varepsilon > |f(x)| \quad \text{for every } x \in X \backslash V .$$

If ε can be replaced by zero in the above definition, that is,

$$(2) \qquad 1 = \|f\| = f(x_0) > f(x) = 0 \quad \text{for every } x \in X \backslash V ,$$

we call A *extremely regular of type zero*.

If for each $x_0 \in X$, each open neighborhood V of x_0 there is a function $f \in A$ such that

$$1 = \|f\| = f(x_0) > \sup\{|f(x)| : x \in X \backslash V\}$$

then we call A *completely regular*.

Myers [3] has proved that a sufficient condition for compact spaces X and Y to be homeomorphic is that a completely regular linear subspace of $C^r(X)$ and such a subspace of $C^r(Y)$ be isometrically isomorphic, where $C^r(X)$ (resp. $C^r(Y)$) denotes the real-valued continuous functions on X (resp. Y).

Cambern [1] has shown that if ϕ is a linear isomorphism of $C_0(X)$ onto $C_0(Y)$, for any locally compact spaces X and Y, with $\|\phi\| \cdot \|\phi^{-1}\| < 2$, then X and Y are homeomorphic.

In [2], we have shown that in certain special cases the above mentioned generalizations of the well-known Banach-Stone theorem can be combined. More precisely, if ϕ is a linear isomorphism of an extremely regular subspace of $C_0(X)$ onto such a subspace of $C_0(Y)$ with $\|\phi\| \cdot \|\phi^{-1}\| < 2$, then X and Y are homeomorphic.

This result suggests that we should know more about the extremely regular function spaces. It is clear that every proper extremely regular function space is contained in a maximal one.

[1] Throughout this paper all topological spaces will be Hausdorff.

The main purpose of this article is to prove the following:

THEOREM. *Let X be a locally compact space and let $M_c(X)$ denote the set of all nonzero, continuous, complex-valued finite regular Borel measures on X. For each $\mu \in M_c(X)$, let $K(\mu)$ denote the kernel of μ in $C_0(X)$. Then,*

(a) *For each $\mu \in M_c(X)$, $K(\mu)$ is a maximal extremely regular linear subspace of $C_0(X)$ of type zero.*

(b) *If A is a maximal extremely regular linear subspace of $C_0(X)$, then $A = K(\mu)$ for some $\mu \in M_c(X)$.*

(c) *$C_0(X)$ has no proper extremely regular linear subspace if, and only if, X is dispersed* (i.e. *the* αth *derived set of X is void for some ordinal number α*).

Before beginning the proof of the theorem, we wish to establish some conventions concerning notation. For a finite regular Borel measure μ on a locally compact space X we denote $\int_X f d\mu$ by $\mu(f)$, $f \in C_0(X)$, and $|\mu|(X)$ by $\|\mu\|$, where $|\mu|$ denotes the total variation of μ. For a point $x \in X$, μ_x denotes the unit point mass at x.

PROPOSITION. *Let μ be a finite regular Borel measure on a locally compact space X such that $K(\mu)$ contains an extremely regular linear subspace of $C_0(X)$. Then, μ is continuous, that is, the atomic part of μ is zero.*

Proof. Let A be an extremely regular linear subspace of $C_0(X)$ contained in $K(\mu)$, and suppose that μ is not continuous. Then, there is a finite (with at least two points) or countably infinite subset $F = \{x_1, x_2, \cdots\}$ of X such that

$$\mu = \sum_i \alpha_i \mu_{x_i} + \nu ,$$

where α_i are nonzero constants, and where ν is the continuous part of μ.

Let ε be any number with $0 < \varepsilon < 1$, and let m be a positive integer which is either the number of points in F or such that

$$\sum_{i=m+1}^{\infty} |\alpha_i| < \varepsilon$$

according as F is finite or infinite.

Let V be any open neighborhood of x_1 that contains none of the points $x_2, \cdots, x_m$ and that $|\nu|(V) < \varepsilon$. Now, choose an element f of A with

$$1 = \|f\| = f(x_1) > \varepsilon \geqq |f(x)| \quad \text{for all } x \text{ in } X \backslash V .$$

From

$$0 = \mu(f) = \alpha_1 + \sum_{i \geqq 2} \alpha_i f(x_i) + \nu(f)$$

we obtain

$$|\alpha_1| < \varepsilon\,(\|\mu\| + 2).$$

From this inequality it follows that $\alpha_1 = 0$, and this contradiction completes the proof.

Proof of the theorem. (b) follows from the Hahn-Banach and Riesz representation theorems and the above proposition. (c) follows from the above proposition and the fact that X is dispersed if, and only if, $M_c(X) = \varnothing$. (Cf. Pelczynski-Semadeni [4].)

For Part (a), we shall first show that for each $\mu \in M_c(X)$, $x_0 \in X$ there exists $f \in C_0(X)$ with $1 = \|f\| = f(x_0)$ and $\mu(f) = 0$.

Let V be a compact neighborhood of x_0. We may assume that the restriction of μ to $X \backslash V$ is not zero. Then, it follows that there exists a function g in $C_0(X)$ such that $1 = \|g\|$, $g(x) = 0$ for all $x \in V$ and that $\mu(g) \neq 0$. Now choose a function h in $C_0(X)$ with $1 = \|h\| = h(x_0)$, $h(x) = 0$ for each $x \in X \backslash V$ and $|\mu(h)| \leqq |\mu(g)|$. Clearly, the function $f = h + \alpha g$, where $\alpha = -\mu(h)/\mu(g)$, satisfies the above requirements.

Now, to complete the proof of Part (a) (thus, that of the theorem) take any open neighborhood U of x_0. Then by the above result (X and μ replaced by U and the restriction of μ to U respectively) there exists a function f in $C_0(X)$ such that $\mu(f) = 0$ and that $1 = \|f\| = f(x_0) > f(x) = 0$ for all $x \in X \backslash U$.

REMARK 1. A proper extremely regular linear subspace of $C_0(X)$ of type zero need not be maximal. (Let X denote the closed unit interval [0, 1] and m denote the Lebesgue measure on X and let m' be such that $m'(B) = m(B \cap [0, 1/2])$, for every Borel set B in X. Consider $K(m) \cap K(m')$.)

REMARK 2. An extremely regular linear subspace of $C_0(X)$ need not be of type zero. An example of this kind can be obtained by restricting the functions which are continuous on the closed unit disc and analytic inside to the unit circle in the Euclidean plane.

REMARK 3. Every extremely regular function space is completely regular. But the converse is not true, since it can be shown that if X is a locally compact space with at least three points, $C_0(X)$ has closed completely regular proper linear subspaces. (This fact is clear if X is non-dispersed. If X is dispersed, it has at least three isolated points, x, y, and z, say. Consider $K(\mu)$, where $\mu = \mu_x + \mu_y + \mu_z$.)

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