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ON FINDING THE DISTRIBUTION FUNCTION FOR AN ORTHOGONAL POLYNOMIAL SET

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Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be real sequences with $b_n > 0$ and $\{b_n\}_{n=0}^{\infty}$ bounded. Let $\{P_n(x)\}_{n=0}^{\infty}$ be a sequence of polynomials satisfying the recurrence formula

$$(1.1) \qquad \begin{cases} xP_n(x) = b_{n-1}P_{n-1}(x) + a_nP_n(x) + b_nP_{n+1}(x) & (n \geqq 0) \\ P_{-1}(x) = 0 \qquad P_0(x) = 1. \end{cases}$$

Then there is a substantially unique distribution function $\psi(t)$ with respect to which the $P_n(x)$ are orthogonal. That is,

$$\int_{-\infty}^{\infty} P_n(x)P_m(x)d\psi(x) = K_n\delta_{n,m} \qquad (n, m \geqq 0),$$

where $K_n \neq 0$ and $\delta_{n,m}$ is the kronecker delta. This paper gives a method of constructing $\psi(x)$ for the case $\lim_{n\to\infty} b_{2n} = 0$, $\lim_{n\to\infty} b_{2n+1} = b < \infty$, the set of limit points of $\{a_n\}_{n=1}^{\infty}$ equals $\{-\alpha, \alpha\}$ and $\lim_{n\to\infty}\{a_{2n} + a_{2n+1}\} = 0$. The same method can be used in the case $\lim_{n\to\infty} b_n = 0$ and the set of limit points of $\{a_n\}_{n=0}^{\infty}$ is bounded and finite in number.

This continues the investigation started by Dickinson, Pollak, and Wannier [3] in which they studied the distribution function under the assumption $a_n = 0$ and $\Sigma\, b_n < \infty$. Goldberg [4] extended their results by considering the case $a_n = 0$ and $\lim_{n\to\infty} b_n = 0$. Finally, Maki [5] showed how to construct the distribution function when $\lim_{n\to\infty} b_n = 0$ and the set of limit points of $\{a_n\}_{n=0}^{\infty}$ are bounded and finite in number. In all these cases their approach was to study the continued fraction

$$(1.2) \qquad K(z) = \frac{1\ |}{|\ z - a_0} - \frac{b_0^2\ |}{|\ z - a_1} - \frac{b_1^2\ |}{|\ z - a_2} \cdots,$$

where $\{b_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ consist of the same numbers as given in (1.1).

Our approach is different from that of the above mentioned authors. If $S(\psi)$ denotes the spectrum of ψ, i.e., the set $\{\lambda \mid \psi(\lambda+\varepsilon) - \psi(\lambda-\varepsilon) > 0$ for all $\varepsilon > 0\}$, then, in our case, we will show from the properties of the sequences $\{a_n\}$ and $\{b_n\}$ how to find the derived set of $S(\psi)$ and that the $S(\psi)$ consists of a denumerable set of points.

To prove our results we make use of the following theorem due to M. Krein ([1], p. 230-231).

THEOREM 1.1. *The polynomial set defined by* (1.1) *is associated with a determined Hamburger moment problem with solution ψ, such that $S(\psi)$ is bounded and the set of limit points of $S(\psi)$ is contained* *in* $\{\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_p\}$ (α_i *real*) *if and only if the numbers* a_i *and* b_i $(i=0, 1, 2\cdots)$ *form a bounded set and* $\lim_{i,k\to\infty} g_{i,k} = 0$ *where* $g_{i,j}$ *is the element in the* ith *row and* jth *column of the matrix*

$$\prod_{i=1}^{p} (A - \alpha_i I) ,$$

where

$$A = \begin{Vmatrix} a_0 & b_0 & 0 & \cdots \\ b_0 & a_1 & b_1 & \cdots \\ 0 & b_1 & a_2 & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{Vmatrix} .$$

2. Our main results.

THEOREM 2.1. *Let* $\lim_{n\to\infty} b_{2n} = 0$ *and* $\lim_{n\to\infty} b_{2n+1} = b < \infty$, *where* $b > 0$. *The set of limit points of* $\{a_n\}_{n=0}^{\infty}$ *is* $\{-\alpha, \alpha\}$ *and* $\lim_{n\to\infty}\{a_{2n-1} + a_{2n}\} = 0$ *if and only if the derived set of* $S(\psi)$ *equals*

$$\{-(\alpha^2 + b^2)^{1/2}, (\alpha^2 + b^2)^{1/2}\} .$$

Proof. By using the notation of Theorem 1.1, it is easy to show that the element in the ith row and jth column of the matrix $A^2 - (\alpha^2 + b^2)I$ is given by

$$g_{n,n+j} = \begin{cases} 0 & \text{if } |j| > 2 , \\ b_{n-1}\, b_n & \text{if } j = 2 , \\ b_{n-1}(a_{n-1} + a_n) & \text{if } j = 1 , \\ b_{n-2}^2 + a_{n-1}^2 + b_{n-1}^2 - \alpha^2 - b^2 & \text{if } j = 0 , \\ b_{n-2}(a_{n-2} + a_{n-1}) & \text{if } j = -1 , \\ b_{n-2}\, b_{n-3} & \text{if } j = -2 . \end{cases}$$

Let $\{-\{\alpha^2 + b^2)^{1/2}, (\alpha^2 + b^2)^{1/2}\}$ constitute the derived set of $S(\psi)$. Because $\{b_n\}_{n=0}^{\infty}$ is bounded, then the Hamburger moment problem associated with (1.1) is determined (see [7], p. 59). Thus by Theorem 1.1 $\lim_{i,j\to\infty} g_{i,j} = 0$. Therefore, $\lim_{n\to\infty}(a_{2n-1} + a_{2n}) = 0$ and $\lim_{n\to\infty}(a_n^2 - \alpha^2) = 0$. But this implies that the set of limit points of $\{a_n\}_{n=0}^{\infty}$ is $\{-\alpha, \alpha\}$.

Conversely if the limit points of $\{a_n\}_{n=0}^{\infty}$ is $\{-\alpha, \alpha\}$ and

$$\lim_{n\to\infty}(a_{2n-1} + a_{2n}) = 0 ,$$

then $\lim_{i,j\to\infty} g_{i,j} = 0$. Thus by Theorem 1.1 this implies that the derived set of $S(\psi)$ has $-(\alpha^2+b^2)^{1/2}$ and $(\alpha^2+b^2)^{1/2}$ as its only two points. This completes the proof of the theorem.

Let k be a positive integer and $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ be a set of real numbers. If $g_{i,j,k}$ is the element in the ith row and jth column of the matrix

$$\prod_{i=1}^{k}(A-\alpha_i I)$$

then it is easy to show by mathematical induction on k that

$$(2.1) \quad g_{n,n-i,k} = \begin{cases} h_{i,n,k}\prod_{l=1}^{i} b_{n-l-1} & \text{if } 1 \leqq i \leqq k\,, \\ s_{n,k}\, b_{n-1}^2 + q_{n,k}\, b_{n-2}^2 + \prod_{i=1}^{k}(a_{n-1}-\alpha_i) & \text{if } i = 0\,, \\ r_{i,n,k}\prod_{l=0}^{-i-1} b_{n+l-1} & \text{if } -k \leqq i \leqq -1\,, \\ 0 & \text{if } |i| > k\,, \end{cases}$$

where $\{h_{i,n,k}\}$, $\{r_{i,n,k}\}$, $\{s_{n,k}\}$, and $\{q_{n,k}\}$ are bounded sequences in n for fixed k and i.

By using Equation (2.1) and the same technique as that used in the proof of Theorem 2.1 we have

THEOREM 2.2. *Let* $\lim_{n\to\infty} b_n = 0$ *and* $\{a_n\}_{n=0}^{\infty}$ *be a bounded sequence. The derived set of* $S(\psi)$ *equals* $\{\alpha_1, \alpha_2, \cdots, \alpha_p\}$ *if and only if the set of limit points of* $\{a_n\}_{n=0}^{\infty}$ *is* $\{\alpha_1, \alpha_2, \cdots, \alpha_p\}$.

Proof. Let $L = \{\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_p\}$ be the set of limit points of $\{a_n\}_{n=0}^{\infty}$. From Equation (2.1) and Theorem 1.1 we have that D, the derived set of $S(\psi)$, is contained in L. Assume D is a proper subset of L. That is, $D = \{\beta_1, \beta_2, \beta_3, \cdots, \beta_k\}$ where $k < p$. Thus, if $g_{i,j,k}$ is the element in the ith row and jth column of the matrix $\prod_{i=1}^{k}(A-\beta_i I)$, then by Theorem 1.1 and Equation (2.1)

$$\lim_{n\to\infty}\prod_{i=1}^{k}(a_{n-1}-\beta_i) = 0\,.$$

That is, L is a proper subset of D. But this is a contradiction. Thus $D = L$.

The converse may be proved in a similar manner.

Maki [6] conjectured, that in the case $\lim_{n\to\infty} b_n = 0$, the set of limit points of $S(\psi)$ equals the set of limit points of $\{a_n\}_{n=0}^{\infty}$. Theorem 2.2 shows that this conjecture is true for the case when $\{a_n\}_{n=0}^{\infty}$ is bounded and has a finite set of limit points. Chihara [2] has shown by using the theory of continued fractions that Maki's conjecture is true in general.

3. Construction of the distribution function. Because the sequence $\{b_n\}_{n=1}^{\infty}$ is bounded we are dealing with the determined Hamburger moment problem, so the continued fraction given in Equation (1.2) converges uniformly on every closed half plane,

$$\operatorname{Im}(z) \geqq s > 0 , \tag{3.1}$$

to an analytic function $F(z)$ which is not a rational function. $F(z)$ has the form

$$F(z) = \int_{-\infty}^{\infty} (z-t)^{-1} d\psi(t) , \tag{3.2}$$

where z satisfies (3.1). The polynomial set $\{P_n(x)\}_{n=0}^{\infty}$ given in (1.1) is orthogonal on $(-\infty, \infty)$ with respect to the distribution $\psi(x)$.

Let us define,

$$A(x) = \psi(x+0) - \psi(x-0) .$$

LEMMA 3.1. *Let T be a bounded countable set of real numbers such that the derived set of T is $B = \{\beta_1, \beta_2, \cdots, \beta_n\}$. Also let*

$$\begin{aligned} H &= T \backslash B \\ &= \{h_i \mid i = 1, 2, 3, \cdots\} . \end{aligned}$$

(i) *$S(\psi) = H \cup B$ $A(h_j) = M_j$ $(j = 1, 2, 3, \cdots)$, and $A(\beta_k) = N_k$ $(k = 1, 2, 3 \cdots n)$, if and only if*

(ii) *$M_j > 0$ $(j = 1, 2, 3, \cdots)$, $N_k \geqq 0$ $(k = 1, 2, 3, \cdots n)$,*

$$\sum_{j=1}^{\infty} M_j + \sum_{k=1}^{n} N_k < \infty ,$$

and

$$F(z) = \sum_{j=1}^{\infty} (z - h_j)^{-1} M_j + \sum_{k=1}^{n} (z - \beta_k)^{-1} N_k .$$

Proof. It is easy to show that $S(\psi)$ is closed. From this and by the definition of the Lebesgue-Stieltjes Integral, (i) implies (ii). Also from the fact that $S(\psi)$ is closed and from the Stieltjes inversion formula, (ii) implies (i). This completes the proof of the lemma.

Let $\mathscr{C}$ represent the field of complex numbers.

THEOREM 3.1. *Let $\lim_{n\to\infty} b_{2n} = 0$ and $\lim_{n\to\infty} b_{2n+1} = b < \infty$, where $b > 0$. Also let the set of limit points of $\{a_n\}_{n=0}^{\infty}$ be $\{-\alpha, \alpha\}$ and $\lim_{n\to\infty} \{a_{2n-1} + a_{2n}\} = 0$.*

(i) *$K(z)$ as defined by Equation (1.2) is a meromorphic function in $\mathscr{C} \backslash \{-(\alpha^2 + b^2)^{1/2}, (\alpha^2 + b^2)^{1/2}\}$ and it has a representation of the form*

$$K(z) = \frac{A(-(\alpha^2 + b^2)^{1/2})}{z + (\alpha^2 + b^2)^{1/2}} + \frac{A((\alpha^2 + b^2)^{1/2})}{z - (\alpha^2 + b^2)^{1/2}} + \sum_{i=0}^{\infty} \frac{A(t_i)}{z - t_i} \tag{3.4}$$

where $A(\pm(\alpha^2 + b^2)^{1/2}) \geqq 0$ *and* $A(t_i) > 0$.

(ii) *If* $T = \{t_i \mid i = 1, 2, 3 \cdots\}$, *where* t_i *is as given in Equation* (3.4), *then* $S(\psi) = T \cup \{-(\alpha^2 + b^2)^{1/2}, (\alpha^2 + b^2)^{1/2}\}$.

(iii) *The limit points of* $S(\psi)$ *are* $-(\alpha^2 + b^2)^{1/2}$ *and* $(\alpha^2 + b^2)^{1/2}$.

Proof. We know from Theorem 2.1 that $S(\psi)$ is countable and its derived set consists only of the points $-(\alpha^2 + b^2)^{1/2}$ and $(\alpha^2 + b^2)^{1/2}$. Thus by Lemma 3.1

$$F(z) = \frac{A(-(\alpha^2 + b^2)^{1/2})}{z + (\alpha^2 + b^2)^{1/2}} + \frac{A((\alpha^2 + b^2)^{1/2})}{z - (\alpha^2 + b^2)^{1/2}} + \sum_{i=1}^{\infty} \frac{A(t_i)}{z - t_i}$$

where $T \cup \{-(\alpha^2 + b^2)^{1/2}, (\alpha^2 + b^2)^{1/2}\} = S(\psi)$. Because ψ is monotonically non-decreasing and $-(\alpha^2 + b^2)^{1/2}$, $(\alpha^2 + b^2)^{1/2}$ are the only limit points of its spectrum we obtain, $A(t_i) > 0$ for $t_i \in T$ and

$$A(\pm(\alpha^2 + b^2)^{1/2}) \geqq 0 .$$

But the continued fraction given in Equation 1.2 converges uniformly to $F(z)$ on any closed bounded set that doesn't contain $S(\psi)$. Thus $K(z) = F(z)$, for $z \notin S(\psi)$. This completes the proof of the theorem.

By working directly with $K(z)$ Maki ([5] Theorem (5.4)) proves that if $\lim_{n\to\infty} b_n = 0$ and the set of limit points of $\{\alpha_n\}_{n=0}^{\infty}$ is $\{\alpha_1, \alpha_2 \cdots \alpha_p\}$ with $|\alpha_i| < \infty$ $i = 1, 2 \cdots p$, then

(i) $K(z)$ is a meromorphic function in $\mathscr{C} \setminus \{\alpha_1, \alpha_2, \cdots, \alpha_p\}$ and has a representation of the form

$$K(z) = \sum_{i=1}^{p} (z - \alpha_i)^{-1} A(\alpha_i) + \sum_{i=0}^{\infty} (z - t_i)^{-1} A(t_i) , \tag{3.5}$$

where $A(\alpha_i) \geqq 0$ and $A(t_i) > 0$,

(ii) if $T = \{t_i \mid i = 1, 2, 3 \cdots\}$ where t_i is as given in Equation (3.5), then $S(\psi) = \{\alpha_1, \alpha_2, \cdots, \alpha_p\} \cup T$, and

(iii) the derived set of $S(\psi)$ is $\{\alpha_1, \alpha_2, \cdots, \alpha_p\}$.

By using Theorem 2.2 and a technique similar to the one used in our proof of Theorem 3.1 it is easy to see how to give a short proof of Maki's theorem.

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