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TWO CHARACTERIZATIONS OF COMMUTATIVE BAER RINGS

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A commutative ring A is called a Baer ring if the annihilator of each element in A is the principal ideal generated by an idempotent. It is shown that the following three conditions on a semiprime commutative ring A with identity are equivalent: (1) A is a Baer ring, (2) the mapping $Q \to Q \cap E$ is a homeomorphism of Min Spec A **with the Boolean space of the Boolean algebra E of idempotents in A, (3)** Min Spec A **is a retract of** Spec A.

Introduction. A commutative ring A is called a *Baer ring* if the annihilator of each element in A is the principal ideal generated by an idempotent. Baer rings have been the subject of several recent investigations. (See, e.g., [1], [2], [8], [9], [10], [14], [15], [16] and [17].) The main purpose of this note is to give two new characterizations of these rings.

All rings considered in this paper are assumed to be commutative with identity; the symbol A will always denote such a ring, and $E = E(A)$ will denote the Boolean algebra of idempotents in A. Recall that the operations in E are given by $e \cap f = ef$, $e' = 1 - e$, and hence $e \cup f = (e' \cap f')' = e + f - ef$.

If $\mathscr{V} = \mathscr{V}(A)$ is any family of prime ideals in A, and if a is an element of A, then let $\mathscr{V}_a = \{Q \in \mathscr{V}: a \notin Q\}$. We have $\mathscr{V}_a \cap \mathscr{V}_b = \mathscr{V}_{ab}$, and $\mathscr{V}_1 = \mathscr{V}$; so the family $\{\mathscr{V}_a: a \in A\}$ is a base for a topology on $\mathscr{V}$; this topology is called the *Stone* or *Zariski topology*. It is to be understood that any set of prime ideals carries the Stone-Zariski topology.

The *minimal prime spectrum* of A, denoted by $\mathscr{P}(A)$, or also by Min Spec A, is the space of minimal prime ideals of A. As shown in [6] and [9], $\mathscr{P} = \mathscr{P}(A)$ is a Hausdorff space in which each set $\mathscr{P}_a$ is both open and closed.

The set $\mathscr{P}(E)$ of maximal (= prime) ideals in the Boolean algebra E is topologized by taking the family $\{\mathscr{P}_e(E): e \in E\}$ as a base, where $\mathscr{P}_e = \{P \in \mathscr{P}(E): e \notin P\}$. When so topologized, $\mathscr{P}(E)$ is a compact Hausdorff space in which each set $\mathscr{P}_e(E)$ is both open and closed; moreover, each open and closed subset of $\mathscr{P}(E)$ is of the form $\mathscr{P}_e(E)$ for some e in E.

If Q is a prime ideal in A, then $Q \cap E$ is a prime ideal in E, i.e., it is a member of $\mathscr{P}(E)$. Our first characterization of Baer rings is the following one. (Recall that a semiprime ring is one in which there are no nonzero nilpotents.)

THEOREM 1. *A semiprime commutative ring A with identity is a Baer ring if and only if the mapping* $Q \to Q \cap E$ *is a homeomorphism of* $\mathscr{P}(A)$ *upon* $\mathscr{P}(E)$.

Section 1 of this paper will be devoted to a proof of the above result. In the course of proving it, we show that for any ring A, the mapping $Q \to Q \cap E$ is always a continuous surjection of $\mathscr{P}(A)$ upon $\mathscr{P}(E)$, and we characterize those rings A for which this mapping is a bijection.

REMARKS 1. If X is a nonempty set, and if Z is the ring of integers, then $A = Z^X$ is a Baer ring. In [13], D. Scott showed that there is a bijection of the set of minimal prime ideals of A upon the set of ultrafilters on X. The Boolean algebra $E(A)$ is isomorphic to the Boolean algebra of all subsets of X, and $\mathscr{P}(E)$ is in one-to-one correspondence with the ultrafilters on X. Hence, Theorem 1 is a generalization of Scott's result.

2. It was stated without proof in [10] that $\mathscr{P}(A)$ and $\mathscr{P}(E)$ are homeomorphic when A is a Baer ring. However, my original proof of that fact was roundabout, and different from the one given here.

As our second characterization, we show that a semiprime commutative ring A with identity is a Baer ring if and only if $\mathscr{P}(A)$ is a retract of Spec A, the space of all prime ideals in A. This result, Theorem 2, is proved in §2.

Theorems 1 and 2 are applied in §3 to obtain a new proof of the known result that a semiprime commutative ring A with identity is a regular ring if and only if each prime ideal in A is maximal. In §4, Theorem 2 is applied to generalize a result of Henriksen and Jerison.

In his 1972 Tulane thesis [*Baer rings and their structure sheaves*], Howard Evans independently established Theorems 1 and 2, and even for noncommutative Baer rings; his methods of proof are entirely different from the methods we use here.

1. *Proof of Theorem* 1. Recall that $\operatorname{rad} J$, the *radical* of an ideal J in a commutative ring A, is the set of all elements a in A such that some power of a is in J. If I is an ideal in the Boolean algebra E, then we denote by $\bar{I}$ the ideal in A generated by I. It is easy to see that $\bar{I}$ consists of all elements a such that $ae' = 0$ for some element e in I.

We shall prove Theorem 1 by a sequence of lemmas.

LEMMA 1.1. *If I is an ideal in E, then* rad $\bar{I}$ *is the intersection of all minimal prime ideals containing I.*

Proof. Let $a \notin \text{rad}\,\bar{I}$ so that $a^n \notin \bar{I}$ for each natural number n. Thus, $a^n e' \neq 0$ for each nonnegative integer n and each element e of I. The family S of all such elements is a multiplicative semigroup, for $a^m e' a^n f' = a^{m+n}(e \cup f)'$, and $e \cup f$ is in I if both e and f are. Since $0 \notin S$, Krull's lemma [7, p. 1] guarantees the existence of a minimal prime ideal Q which does not meet S. It follows that $a \notin Q$ and that $I \subset Q$.

COROLLARY 1.2. *If P is a prime ideal in E, then there is a minimal prime ideal Q in A such that $P = Q \cap E$.*

Proof. By the lemma, there is a minimal prime ideal Q in A such that $\bar{P} \subset Q$. Thus, $P \subset Q$, and so $P = Q \cap E$.

The previous result asserts that the mapping $Q \rightarrow Q \cap E$ is a surjection of $\mathscr{P}(A)$ upon $\mathscr{P}(E)$. If $e \in E$, then $\{Q \in \mathscr{P}(A): e \notin Q \cap E\} = \{Q \in \mathscr{P}(A): e \notin Q\}$, and the latter set is open (and closed) in $\mathscr{P}(A)$. Hence, for any commutative ring A with identity, the mapping $Q \rightarrow Q \cap E$ is a continuous surjection of $\mathscr{P}(A)$ upon $\mathscr{P}(E)$.

LEMMA 1.3. *Let A be a semiprime commutative ring with identity. If I is an ideal in E, then the ideal $\bar{I}$ coincides with its radical.*

Proof. Let $a^n \in \bar{I}$, so that $a^n e' = 0$ for some $e \in I$. Then $(ae')^n = 0$, and so $ae' = 0$, that is, $a \in \bar{I}$.

An ideal J in A is called regular if $J = \overline{J \cap E}$.

PROPOSITION 1.4. *In a semiprime commutative ring A with identity, the surjection $Q \rightarrow Q \cap E$ of $\mathscr{P}(A)$ upon $\mathscr{P}(E)$ is a bijection if and only if each minimal prime ideal in A is regular.*

Proof. Suppose that each minimal prime ideal in A is regular. If Q_1 and Q_2 are minimal primes such that $Q_1 \cap E = Q_2 \cap E$, then $Q_1 = \overline{Q_1 \cap E} = \overline{Q_2 \cap E} = Q_2$, and thus the mapping $Q \rightarrow Q \cap E$ is an injection.

Conversely, suppose there is a nonregular minimal prime ideal Q in A. Choose a in Q such that $a \notin \overline{Q \cap E}$. By Lemmas 1.1 and 1.3, there is a minimal prime ideal Q_1 such that $a \notin Q_1$, and $Q \cap E \subset Q_1$. Thus, $Q \cap E = Q_1 \cap E$ but $Q \neq Q_1$.

Let A be a ring in which the mapping $Q \rightarrow Q \cap E$ is a bijection of $\mathscr{P}(A)$ upon $\mathscr{P}(E)$. If $P \in \mathscr{P}(E)$ then there is exactly one element

$Q \in \mathscr{P}(A)$ for which $P = Q \cap E$, and hence rad $\bar{P} = Q$. Thus, if A is semiprime, then $\bar{P} = Q$, and consequently the inverse of the above mapping is $P \to \bar{P}$.

If a is an element of a commutative ring A, then ann a, the *annihilator* of a, is the set of all elements b in A for which $ab = 0$. The following characterization of minimal prime ideals can be found in [9]; see also [7, p. 57].

LEMMA 1.5. *A prime ideal Q in a semiprime commutative ring is a minimal prime ideal if and only if ann $a \not\subset Q$ whenever $a \in Q$.*

LEMMA 1.6. *Each minimal prime ideal in a Baer ring is regular.*

Proof. Let Q be a minimal prime ideal in the Baer ring A, and let a be an element of Q. There is an idempotent e such that *ann* $a = Ae$. A Baer ring is semiprime [9], so Lemma 1.5 insures that $e \notin Q$. Hence, $a \in \overline{Q \cap E}$, and therefore $Q = \overline{Q \cap E}$.

The following result can be found in [6] and [9].

LEMMA 1.7. *If a is an element of a semiprime commutative ring A, then ann $a = \bigcap \{Q : Q \in \mathscr{P}_a\}$.*

We now have at hand all the tools with which to prove Theorem 1.

Proof of necessity. If A is a Baer ring, then by Lemma 1.6 and Proposition 1.4, the mapping $Q \to Q \cap E$ is a bijection of $\mathscr{P}(A)$ upon $\mathscr{P}(E)$. By the remark following that proposition, the inverse of this bijection is $P \to \bar{P}$. The space $\mathscr{P}(E)$ is compact, and the space $\mathscr{P}(A)$ is Hausdorff, so to complete the proof, we need only show that the mapping $P \to \bar{P}$ is continuous.

Hence, let a be an element of A. We must show that $\{P \in \mathscr{P}(E): a \notin \bar{P}\}$ is open in $\mathscr{P}(E)$. Let e be the idempotent for which *ann* $a = Ae$. By Lemma 1.5, $a \notin \bar{P}$ if and only if *ann* $a \subset \bar{P}$, and so $a \notin \bar{P}$ if and only if $e \in \bar{P}$, i.e., if and only if $e \in P$. We have shown that $\{P: a \notin \bar{P}\} = \{P: e \in P\}$. The latter set is both open and closed in $\mathscr{P}(E)$, and so the mapping $P \to \bar{P}$ is continuous.

Proof of sufficiency. For $a \in A$, the set $\mathscr{P}_a$ is both open and closed in $\mathscr{P}(A)$. Since the mapping $Q \to Q \cap E$ is a homeomorphism, the set $\{Q \cap E: a \notin Q\}$ is both open and closed in $\mathscr{P}(E)$. Hence, there is an idempotent e in E such that $\{Q \cap E: a \notin Q\} = \{Q \cap E: e \notin Q \cap E\}$. The latter set is the same as $\{Q \cap E: e \notin Q\}$, and consequently, $\{Q \in \mathscr{P}(A): a \notin Q\} = \{Q \in \mathscr{P}(A): e \notin Q\}$. This equality and Lemma 1.7 imply that

$ann\ a = ann\ e$. But $ann\ e = Ae'$, and hence A is a Baer ring.

As the following discussion will show, there are rings A for which the mapping $Q \rightarrow Q \cap E$ is a bijection of $\mathscr{P}(A)$ upon $\mathscr{P}(E)$, but which are not Baer rings.

A semiprime ring A is called *complementedly normal* [2, p. 196] if whenever a, b are elements of A such that $ab = 0$ then there is an idempotent e in A such that $ae = 0 = be'$.

Now the ring $C(X)$ of real-valued continuous functions on a completely regular Hausdorff space X is complementedly normal if and only if X is a U-space [2, p. 218], and it is a Baer ring if and only if X is basically disconnected, i.e., if and only if the lattice $C(X)$ is conditionally σ-complete [9, p. 45]. There are U-spaces which are not basically disconnected spaces [4, p. 390], so in virtue of Proposition 1.4 and the following result, there are semiprime rings A which are not Baer rings, but for which the mapping $Q \rightarrow Q \cap E$ is a bijection of $\mathscr{P}(A)$ upon $\mathscr{P}(E)$.

LEMMA 1.8. *If the semiprime ring A is complementedly normal, then each minimal prime ideal in A is regular.*

Proof. Let Q be a minimal prime ideal in the complementedly normal ring A. If a is an element of Q, then by Lemma 1.5, there is an element $b \notin Q$ such that $ab = 0$. Hence, there is an idempotent e such that $ae' = 0 = be$. We must have e in Q, and so a is in $\bar{Q}$. Thus, each minimal prime ideal is regular.

2. Retracts. Recall that Spec A, the *prime spectrum* of a commutative ring A, is the space of all prime ideals of A. This section will be devoted to a proof of the following result.

THEOREM 2. *A semiprime commutative ring A with identity is a Baer ring if and only if the minimal prime spectrum of A is a retract of the prime spectrum of A.*

The above result was suggested by a paper of DeMarco and Orsatti [3], and some of the arguments used in its proof are similar to arguments used by those authors.

For P in Spec A, let O_P be the intersection of all minimal prime ideals contained in P. A proof of the following result can be found in [2, p. 105] and [3, p. 460].

LEMMA 2.1. *In a semiprime commutative ring A, $O_P = \{a \in A$: $ann\ a \not\subset P\}$ for each prime ideal P.*

For a subset S of A, let $h(S)$ be the set of all prime ideals which contain S, and for a subset S of Spec A, let $k(S)$ be the intersection of all members of S. It is well-known and easy to prove that closed subsets of Spec A are exactly those of the form $h(S)$ for some subset S of A, and that the closure of a subset S of Spec A is $h(k(S))$. In particular then, the closure of a point P of Spec A consists of all prime ideals which contain P.

The next result is essentially Lemma 9.2 of [9].

LEMMA 2.2. *An ideal I in a commutative ring A with identity is a direct summand of A if and only if $h(I)$ is an open—as well as closed—subset of* Spec A.

We now prove the necessity of Theorem 2. If a is an element of any commutative ring A, then $ann\, a \cap ann\, ann\, a$ is contained in each prime ideal of A. This remark and Lemma 2.1 imply that $O_P \subset \{a \in A: ann\, ann\, a \subset P\}$ for each prime ideal P in a semiprime ring A. A Baer ring A is semiprime, and $ann\, a + ann\, ann\, a = A$ for each element a in such a ring, so $O_P = \{a \in A: ann\, ann\, a \subset P\}$. In a semiprime ring, $ann\, ann\, ab = ann\, ann\, a \cap ann\, ann\, b$ for each pair a, b. Therefore, O_P is a prime ideal for each prime ideal P in a Baer ring.

Now $\{P \in \text{Spec}\, A: a \notin O_P\} = h(ann\, a)$, and, by Lemma 2.2, in a Baer ring A, the latter set is open in Spec A, so $P \to O_P$ is a continuous mapping of Spec A into Min Spec A.

In any ring, obviously $a \in ann\, ann\, a$, and thus $O_P \subset P$. Hence, in a Baer ring, $O_P = P$ for each minimal prime ideal P.

We have shown that the mapping $P \to O_P$ is a retraction of Spec A upon Min Spec A when A is a Baer ring.

The sufficiency of the condition in Theorem 2 for a ring to be a Baer ring is included in the following result. In proving this result, we use the easily verified fact that a semiprime commutative ring A with identity is a Baer ring if $ann\, ann\, a$ direct summand of A for each element a.

PROPOSITION 2.3. *Let A be a semiprime commutative ring with identity. If τ is a retraction of* Spec A *upon* Min Spec A, *then $\tau(P) = O_P$ for each P in* Spec A, *and A is a Baer ring.*

Proof. If $Q \in \mathscr{P}(A)$, then $Q = \tau(Q)$, i.e., $Q \in \tau^{\leftarrow}(Q)$. Since τ is a continuous mapping, and since $\mathscr{P}(A)$ is a Hausdorff space, $\tau^{\leftarrow}(Q)$ is closed in Spec A, so cl $\{Q\} \subseteq \tau^{\leftarrow}(Q)$. Thus, if P is in Spec A, and $P \supset Q$, then $\tau(P) = Q$. Consequently, each prime ideal in A contains a unique minimal prime ideal, and so $\tau(P) = O_P$ for each P in Spec A.

Since A is semiprime, Lemma 2.1 insures that $\{P \in \text{Spec } A: a \notin O_P\} = h(ann\, a)$ for each element a in A. Since the mapping $P \to O_P$ is continuous, $h(ann\, a)$ is open as well as closed for each a. By Lemma 2.2, $ann\, a$ is a direct summand of A for each a, and thus A is a Baer ring.

3. von Neumann regular rings.

In this section, we shall apply Theorems 1 and 2 to obtain a new proof of the following result.

THEOREM 3.1. *Let A be a semiprime commutative ring with identity. If each prime ideal in A is maximal, then A is a von Neumann regular ring.*

REMARK. As is well-known, the above theorem has a valid converse: if A is a commutative von Neumann regular ring, then A is semiprime—in fact, even semisimple—and each prime ideal in A is maximal.

Proofs of Theorem 3.1 have been given by Cornish [2] and Peercy [11]. In his book on commutative rings [7], Kaplansky leaves the proof as an exercise, but with hints for doing it; Cornish's proof is essentially the one outlined by Kaplansky. Peercy's proof is a sheaf-theoretic one, while the other two are not. Although we also use some results from sheaf theory to prove Theorem 3.1, our proof is different from Peercy's.

We begin with a summary of Pierce's [12] representation of a commutative ring with identity.

Recall that a sheaf $(\mathscr{B}, Y)$ of commutative rings is reduced if (i) Y is a Boolean space, i.e., a compact Hausdorff space with a base of open-and-closed sets, and (ii) for each $y \in Y$ the only idempotents in the stalk $\mathscr{B}_y$ are 0_y and 1_y.

THEOREM 3.2. (Pierce) *Let A be a commutative ring with identity, and for each $P \in \mathscr{P}(E)$, let $\mathscr{A}_P = (A/\bar{P}, P)$. Then $\mathscr{A} = \dot{\bigcup} \{\mathscr{A}_P: P \in \mathscr{P}(E)\}$ is the sheaf space of a sheaf of reduced commutative rings with base space $\mathscr{P}(E)$, and the mapping $a \to \hat{a}$, where $\hat{a}(P) = a/\bar{P}$ is an isomorphism of A upon the ring $\Gamma(\mathscr{P}(E), \mathscr{A})$ of global sections of the sheaf $(\mathscr{A}, \mathscr{P}(E))$.*

For the remainder of this section, $(\mathscr{A}, \mathscr{P}(E))$ will denote the sheaf of reduced commutative rings defined above.

The following result is contained in Pierce's memoir [12]; it should be noted that the proof of its necessity is essentially a proof of the assertion that every prime ideal in a commutative von Neumann regular ring is maximal.

THEOREM 3.3. *A commutative ring A with identity is a von Neumann regular ring if and only if each stalk $(A/\bar{P}, P)$ of the sheaf $(\mathscr{A}, \mathscr{P}(E))$ is a field.*

We are now ready to prove Theorem 3.1. To do so, let each prime ideal in the semiprime ring A be maximal. Thus, $\mathscr{P}(A) = \operatorname{Spec} A$, and so, trivially, the first of these spaces is a retract of the second. By Theorem 2, A is a Baer ring. Theorem 1 can obviously be recast as follows: A semiprime commutative ring A with identity is a Baer ring if and only if the mapping $P \to \bar{P}$ is a homeomorphism of $\mathscr{P}(E)$ with $\mathscr{P}(A)$. Therefore, $\bar{P}$ is a prime ideal in A for each $P \in \mathscr{P}(E)$, and thus it is maximal. By Theorem 3.3, the ring A is von Neumann regular.

4. **Another application of Theorem 2.** It is known [5] that every prime ideal in the ring $C(X)$ of all real-valued continuous functions on a completely regular space X is contained in a unique maximal ideal. For each prime ideal P in $C(X)$, let $\mu(P)$ be the unique maximal ideal containing it, and let ι be the restriction of μ to $\mathscr{P}(C(X))$, the space of minimal prime ideals in $C(X)$. The following result was obtained by Henriksen and Jerison [6].

THEOREM 4.1. (a) *ι is a continuous mapping of $\mathscr{P}(C(X))$ onto βX, the Stone-Čech compactification of X.*

(b) *ι maps no proper closed subset of $\mathscr{P}$ onto βX.*

(c) *ι is one-to-one if and only if each prime ideal contains a unique minimal prime ideal, i.e., X is an F-space.*

(d) *ι is a homeomorphism if and only if X is basically disconnected.*

(e) *If X is an F-space, then $\mathscr{P}(C(X))$ is compact if and only if X is basically connected.*

Now let A be a commutative ring with identity in which each prime ideal is contained in a unique maximal ideal. For P in $\operatorname{Spec} A$, let $\mu(P)$ be the unique maximal ideal containing P, and let ι be the restriction of μ to $\mathscr{P}(A)$. Let $\mathscr{M}(A)$ be the space of maximal ideals of A. In case $A = C(X)$, $\mathscr{M}(A)$ is homeomorphic with βX, so the following theorem is a generalization of the above result of Henriksen and Jerison.

THEOREM 4.2. (a) *ι is a continuous mapping of $\mathscr{P}(A)$ upon $\mathscr{M}(A)$.*

(b) *If A is semisimple, that is, if 0 is the only element common to all maximal ideals in A, then ι maps no proper closed subset of $\mathscr{P}$ upon $\mathscr{M}$.*

(c) *ι is injective if and only if each prime ideal in A contains a unique minimal prime ideal.*

(d) *If A is semiprime, then ι is a homeomorphism if and only if A is a Baer ring.*

(e) *If each prime ideal in the semiprime ring A contains a unique minimal prime ideal, then $\mathscr{P}(A)$ is compact if and only if A is a Baer ring.*

Proof. (a) This is a consequence of the fact, established by DeMarco and Orsatti [3], that μ is a continuous mapping of Spec A upon $\mathscr{M}(A)$.

(b) This can be proved in the same way that Henriksen and Jerison proved (b) of Theorem 4.1. We repeat their argument. Every proper closed set in $\mathscr{P}$ is contained in a set of the form $h(a)$ for some nonzero element a in A, because such sets form a base for the closed sets. If M is a maximal ideal such that $a \notin M$, then $M \notin \iota(h(a))$.

(c) It is easy to see that the following three statements are equivalent:

(i) ι is one-to-one;

(ii) each maximal ideal contains a unique minimal prime ideal;

(iii) O_M is a minimal prime ideal for each maximal ideal M.

For each prime ideal P in A, we have $O_{\mu(P)} \subset P \subset \mu(P)$. Thus, if (iii) holds, then each prime ideal contains a unique minimal prime ideal. Conversely, if each prime ideal contains a unique minimal prime ideal, then, in particular, $O_M \in \mathscr{P}(A)$ for each M in $\mathscr{M}(A)$, so ι is one-to-one with inverse $M \to O_M$.

(d) If A is a Baer ring, then by Theorem 2, the mapping $P \to O_P$ is a retraction of Spec A upon $\mathscr{P}(A)$. In particular then, the continuous mapping $M \to O_M$ of $\mathscr{M}(A)$ upon $\mathscr{P}(A)$ is the inverse of ι, so the latter mapping is a homeomorphism. Conversely, if ι is a homeomorphism, then the composition $P \to \mu(P) \xrightarrow{\iota^{-1}} O_{\mu(P)} = O_P$ is a retraction of Spec A upon $\mathscr{P}(A)$, so by Theorem 2, the semiprime ring A is a Baer ring.

(e) is a consequence of (c), (d), and the fact, again established by DeMarco and Orsatti [loc. cit.], that $\mathscr{M}(A)$ is a Hausdorff space.

REFERENCES

1. W. H. Cornish, *The variety of commutative Rickart rings*, to appear in Nanta Mathematica.

2. ———, *Abelian Rickart semirings*, Thesis, School of Mathematical Sciences, The Flinders University of South Australia, 1970.

3. Giuseppe DeMarco and Adelberto Orsatti, *Commutative rings in which every prime ideal is contained in a unique maximal ideal*, Proc. Amer. Math. Soc., **30** (1971), 459–466.

4. L. Gillman and M. Henriksen, *Rings of continuous functions in which every finitely generated ideal is principal*, Trans. Amer. Math. Soc., **82** (1956), 366–391.

5. L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, Princeton, N. J., 1960.

6. M. Henriksen and M. Jerison, *The space of minimal prime ideals of a commutative ring*, Trans. Amer. Math. Soc., **115** (1965), 110–130.

7. Irving Kaplansky, *Commutative Rings*, Allyn and Bacon, Inc., Boston, 1970.

8. Klaus Keimel, *Baer extensions of rings and Stone extensions of semigroups*, Semigroup Forum, **2** (1971), 55–63.

9. J. Kist, *Minimal prime ideals in commutative semigroups*, Proc. London Math. Soc., (3) **13** (1963), 31–50.

10. ———, *Compact spaces of minimal prime ideals*, Math. Z., **111** (1969), 151–158.

11. D. E. Peercy, *The complete Baer extension of a commutative ring*, Thesis, New Mexico State University, 1970.

12. R. S. Pierce, *Modules over commutative regular rings*, Memoirs Amer. Math. Soc., No. 70 (1967).

13. D. Scott, *On Constructing Models for Arithmetic, Infinitistic Methods*, Proceedings of the symposium on Foundations of Mathematics, 235–255, Pergamon Press, 1961.

14. T. P. Speed and M. W. Evans, *A note on commutative Baer rings*, J. Austral. Math. Soc., **13** (1972), 1–6.

15. ———, *A note on commutative Baer rings II*, to appear in J. Australian Math. Soc.

16. ———, *A note on commutative Baer rings III*, to appear in J. Australian Math. Soc.

17. ———, *On commutative Baer rings*, to appear.

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