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ON THE HYPERGROUP STRUCTURE OF CENTRAL $\Lambda(p)$ SETS

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Let G be a compact group and let Γ be the set of equivalence classes of the continuous irreducible unitary representations of G. For $\gamma \in \Gamma$ denote by χ_γ the character of γ, then for $E \subset \Gamma$ any function of the form $\sum_{j=1}^{n} a_n \chi_{\gamma_n} (\gamma_1, \cdots, \gamma_n \in E$ and $a_1, \cdots, a_n \in C)$ will be called a central E-polynomial, and the set of all such functions will be denoted ${}^z\mathscr{T}_E$. A set $E \subset \Gamma$ is a central Sidon set when the norms $\| \ \|_\infty$ and $\| \ \|_A (\|f\|_A = \sum |a_n|$, where $f = \sum a_n \chi_{\gamma_n})$ are equivalent on ${}^z\mathscr{T}_E$, and it is a central $\Lambda(p)$ set when the norms $\| \ \|_1$ and $\| \ \|_p$ are equivalent on ${}^z\mathscr{T}_E$. When G is abelian the algebraic structure of $\Lambda(p)$ and Sidon set has been studied extensively. In this paper the structure of central $\Lambda(p)$ sets is investigated in terms of the hypergroup structure of Γ. In particular it is shown that central $\Lambda(p) (p > 2)$ sets cannot contain arbitrarily long "arithmetic progressions."

1. Preliminary remarks. Following Helgason [2] we shall say that a set S is hypergroup if to any pair (α, β) of elements from S there corresponds a measure $\mu_{\alpha,\beta}$ on S. For Γ, a hypergroup structure is induced by the decomposition of tensor products. Thus if $\alpha, \beta \in \Gamma \alpha \otimes \beta = \bigoplus_{\gamma \in \Gamma} [\gamma : \alpha \otimes \beta]\gamma$, where $[\gamma : \alpha \otimes \beta]$ is a nonnegative integer which is called the multiplicity of γ in $\alpha \otimes \beta$, and the measure assigned to the pair (α, β) is the discrete measure whose mass at γ is $[\gamma : \alpha \otimes \beta]$. From the elementary properties of characters we write $\chi_{\gamma_1} \cdots \chi_{\gamma_n} = \chi_{\gamma_1 \otimes \cdots \otimes \gamma_n} = \sum_{\gamma \in \Gamma} [\gamma : \gamma_1 \otimes \cdots \otimes \gamma_n] \chi_\gamma$. We shall denote by 1 the class of the trivial one dimensional representation, and by $\tilde{\gamma}$ the class containing the conjugates of representations in γ. All the basic facts about representations needed in this paper may be found in [3].

2. A necessary and sufficient condition for central $\Lambda(2s)$. Although the condition we are about to give is cumbersome, it will allow us to get both necessary conditions and sufficient conditions which are reminiscent of conditions given by Rudin [6, Thm. 4.5] for the case where G is the circle group.

THEOREM. *Let $E \subset \Gamma$ and let s be a natural number, then the following are equivalent.*

(a) *E is a central $\Lambda(2s)$ set.*

(b) *There exists a constant B depending only on E and s such that for every choice of positive real numbers $a_1, \cdots, a_N$ and elements*

$\gamma_1, \cdots, \gamma_N \in E$ *the inequality*

$$\sum_{\gamma \in \Gamma} (\sum a_{k_1} \cdots a_{k_s}[\gamma: \gamma_{k_1} \otimes \cdots \otimes \gamma_{k_s}])^2 \leqq \left(B\left(\sum_{k=1}^{N} a_k^2\right)^{1/2}\right)^{2s}$$

holds, where the inner sum on the left is over all

$$(k_1, \cdots, k_s) \in \{1, \cdots, N\}^s .$$

Proof. The logarithmic convexity of the $\|\ \|_p$ norms shows that for $p > 2$ a set E is central $\Lambda(p)$ if $\|\ \|_2$ and $\|\ \|_p$ are equivalent on ${}^z\mathscr{T}_E$ [6, Thm. 1.4]. Accordingly, we will work with the $\|\ \|_2$ and $\|\ \|_{2s}$ norms.

Suppose that E is a central $\Lambda(2s)$ set, and choose positive real numbers $a_1, \cdots, a_N$ and $\gamma_1, \cdots, \gamma_N \in E$. Let $\mathscr{N}$ denote the set $\{1, \cdots, N\}^s$, denote elements in $\mathscr{N}$ by $\boldsymbol{k}$, write $a_{k_1}, \cdots, a_{k_s}$ as $a(\boldsymbol{k})$, $(\gamma_{k_1}, \cdots, \gamma_{k_s}) \in E^s$ as $\gamma(\boldsymbol{k})$ and $\gamma_{k_1} \otimes \cdots \otimes \gamma_{k_s}$ as $\gamma^{(\boldsymbol{k})}$ were $(k_1, \cdots, k_s) = \boldsymbol{k}$. Define $f = \sum_{k=1}^N a_k \chi_{\gamma_k}$, then

$$\begin{aligned} f^s &= \sum_{\boldsymbol{k} \in \mathscr{N}} a(\boldsymbol{k}) \chi_{\gamma(\boldsymbol{k})} \\ &= \sum_{\boldsymbol{k} \in \mathscr{N}} a(\boldsymbol{k}) \sum_{\gamma \in \Gamma} [\gamma: \gamma^{(\boldsymbol{k})}] \chi_\gamma \\ &= \sum_{\gamma \in \Gamma} (\sum_{\boldsymbol{k} \in \mathscr{N}} a(\boldsymbol{k})[\gamma: \gamma^{(\boldsymbol{k})}]) \chi_\gamma . \end{aligned}$$

Using the fact that the irreducible characters are an orthonormal family and E is a central $\Lambda(2s)$ set we have

$$\| f \|_{2s}^{2s} = \| f^s \|_2^2 = \sum_{\gamma \in \Gamma} \left(\sum_{\boldsymbol{k} \in \mathscr{N}} a(\boldsymbol{k})[\gamma: \gamma^{(\boldsymbol{k})}] \right)^2 \leqq (B \| f \|_2)^{2s} .$$

To show (b) implies (a), let $g = \sum_{k=1}^N b_k \chi_{\gamma_k}$ be any central E-polynomial. As before

$$\begin{aligned} \| g \|_{2s}^{2s} &= \sum_{\gamma \in \Gamma} \left| \sum_{\boldsymbol{k} \in \mathscr{N}} b(\boldsymbol{k})[\gamma: \gamma^{(\boldsymbol{k})}] \right|^2 \\ &\leqq \sum_{\gamma \in \Gamma} \left(\sum_{\boldsymbol{k} \in \mathscr{N}} | b | (\boldsymbol{k})[\gamma: \gamma^{(\boldsymbol{k})}]\right)^2 \end{aligned}$$

which by hypothesis is $\leqq (B(\sum_{k=1}^N | b_k |^2)^{1/2})^{2s} = (B \| g \|_2)^{2s}$.

COROLLARY. *Let $E \subset \Gamma$ be a central $\Lambda(2s)$ set with constant B so that $\| f \|_{2s} \leqq B \| f \|_2$ for all central E-polynomials f, then for any finite subset $F \subset E$,*

$$\sum_{\gamma \in \Gamma} \left(\sum_{(\gamma) \in F^s} [\gamma: \gamma_1 \otimes \cdots \otimes \gamma_s] \right)^2 \leqq B^{2s} (\text{card } E)^s .$$

Proof. In the theorem set $a_1 = \cdots = a_N = 1$.

REMARK. A case where this criterion is violated in a very simple way is that of $G = SU(2)$. Here Γ can be written as $\{\underline{1}, \underline{2}, \cdots\}$ and the Clebsch-Gordan [3, p. 135] formula shows that $\underline{n} \otimes \underline{n} = \underline{1} \oplus \underline{3} \oplus \cdots \oplus \underline{2n-1}$. So if E is any set in Γ, take $F = \{n\} \subset E$, then

$$\sum_{k=1}^{\infty} [\underline{k}: \underline{n} \otimes \underline{n}]^2 = n$$

and hence Γ cannot contain any infinite central $\Lambda(4)$ sets. This fact has already been observed by Helgason [2, p. 789].

3. A sufficient condition for central $\Lambda(2s)$. Let F be any subset of Γ and write as (γ) the s-tuples $(\gamma_1, \cdots, \gamma_s) \in F^s$. Write $\otimes(\gamma)$ for $\gamma_1 \otimes \cdots \otimes \gamma_s$, and for $(\gamma) \in F^s$ let $M((\gamma))$ stand for the set of irreducible components of $\otimes(\gamma)$. Furthermore, define

$$r_s(F, \gamma) = \sum_{(\gamma) \in F^s} [\gamma: \otimes(\gamma)]^2 .$$

Note that when G is abelian $r_s(F, \gamma)$ is the number of ways we can write $\gamma = \gamma_{k_1} \otimes \cdots \otimes \gamma_{k_s}$ where $\gamma_{k_j} \in F$ and where a permutation of the same set of γ_{k_j}'s is counted as a distinct partition of γ. The following corollary generalizes Rudin's result [6, Thm. 4.5(b)].

COROLLARY. *Let $E \subset \Gamma$ and let s be a natural number. If E is the union of sets $E_i (i = 1, \cdots, j)$ for which there exist constants C_i and D_i depending only on E_i and s such that*

(i) *$r_s(E_i, \gamma) \leqq C_i$ for all $\gamma \in \Gamma$ and*

(ii) *card $M((\gamma)) \leqq D_i$ for all $(\gamma) \in E_i^s$*

then

(a) *E is a central $\Lambda(2s)$ set and*

(b) *$\| f \|_{2s} \leqq (\sum_{i=1}^{j} (C_i D_i)^{1/s})^{1/2} \| f \|_2$ for all central E-polynomials f.*

Proof. We show first that the E_i are central $\Lambda(2s)$ sets by applying the theorem of §2. Choose positive numbers $a_1, \cdots, a_N$ and $\gamma_1, \cdots, \gamma_N \in E_i$. Then

$$\sum_{\gamma \in \Gamma} \Big(\sum_{k \in \mathscr{N}} a(k) [\gamma: \gamma^{(k)}] \Big)^2$$
$$\leqq \sum_{k \in \mathscr{N}} \Big(\sum_{k \in \mathscr{N}} [\gamma: \gamma^{(k)}]^2 \Big) \Big(\sum_{k \in \mathscr{N}} a^2(k) \phi(\gamma, k) \Big)$$

where $\phi(\gamma, k) = 1$ if γ appears in the decomposition of $\gamma^{(k)}$ and $\phi = 0$ otherwise. Observe that $\sum_{\gamma \in \Gamma} \phi(\gamma, k) = \text{card } M(\gamma(k))$ and so by hypothesis this sum is

$$\leqq C_i D_i \sum_{k \in \mathscr{N}} a^2(k) = C_i D_i \Big(\sum_{k=1}^{N} a_k^2 \Big)^s .$$

Hence E_i is a central $\Lambda(2s)$ set and $\|f\|_{2s} \leqq (C_i D_i)^{1/2s}\|f\|_2$ for any central f.

Now suppose that the E_i's are disjoint, for if not they may be replaced by $E_i - \bigcup_{l=1}^{i-1} E_l$. If $f = \sum a_\gamma \chi_\gamma$ is a central E-polynomial then $f = f_1 + \cdots + f_j$ where $f_i = \sum_{\gamma \in E_i} a_\gamma \chi_\gamma$ and

$$\begin{aligned}\|f\|_{2s} &\leqq \sum_{i=1}^{j} \|f_i\|_{2s} \leqq \sum_{i=1}^{j} (C_i D_i)^{1/2s}\|f_i\|_2 \\ &\leqq \Big(\sum_{i=1}^{j} (C_i D_i)^{1/s}\Big)^{1/2}\Big(\sum_{i=1}^{j} \|f_i\|_2^2\Big)^{1/2} \\ &= \Big(\sum_{i=1}^{j} (C_i D_i)^{1/s}\Big)^{1/2}\|f\|_2\end{aligned}$$

since the f_i are orthogonal.

REMARKS. (1) The condition (ii) of the previous corollary is also necessary. Take $F = \{\gamma_1, \cdots, \gamma_s\} \in E$ and apply the corollary in §2. Then we have

$$\begin{aligned}B^{2s}s^s &\geqq \sum_{\gamma \in \Gamma}\Big(\sum_{(\gamma) \in F^s} [\gamma : \otimes (\gamma)]\Big)^2 \\ &\geqq s! \sum_{\gamma \in \Gamma} [\gamma : \otimes (\gamma)] = s! \operatorname{card} M((\gamma))\end{aligned}$$

where (γ) in the last two expressions is the s-tuple whose components are the elements of F.

(2) The condition (ii) is always satisfied when $\sup\{\deg \gamma \mid \gamma \in E\} = P < \infty$. For if $(\gamma) \in E^s$, then the degree of $\otimes(\gamma)$ is not larger than P^s and hence there can be at most P^s elements in $M((\gamma))$.

4. The relationship between central Sidon and central $\Lambda(p)$ sets. A set $E \subset \Gamma$ will be called a central Λ set if there exists a constant C depending only on E such that $\|f\|_p \leqq Cp^{1/2}\|f\|_2$ for all $2 < p < \infty$ and all central E-polynomials f. In the case of abelian groups, Rudin [7, p. 128] shows that every Sidon set is a central Λ set. Using essentially the same technique Parker [5, p. 43] extends this result to central Sidon sets which have a uniform bound on the degrees of the representations in the set. Moreover, Parker [5, p. 73] shows by an example that some sort of condition is required; he gives an example of a central Sidon set which is not even central $\Lambda(4)$. Using essentially the same technique as Rudin and Parker we will characterize those central Sidon sets which are also central $\Lambda(2s)$ or central Λ. An interesting consequence of this result is that a central Sidon set which is also central $\Lambda(p)$ for all p must be a central Λ set. It should be noted that sets which are central $\Lambda(p)$ for all $p < \infty$ need not in general be central Λ sets, in fact such sets exist in every infinite abelian group [1, p. 788].

THEOREM. *Let $E \subset \Gamma$ be a central Sidon set.*

(i) *E is central $\Lambda(2s)$ if and only if there exists a constant B depending on E and s, so that $\|\chi_\gamma\|_{2s} \leqq B$ for all $\gamma \in E$.*

(ii) *E is central Λ if and only if there exists a constant B depending only on E such that $\|\chi_\gamma\|_{2s} \leqq B$ for all $\gamma \in E$ and $s = 1, 2, \cdots$.*

Proof. Since $\|\chi_\gamma\|_2 = 1$ for all $\gamma \in \Gamma$ we clearly have the "only if" parts of (i) and (ii).

Suppose E is a central Sidon set and we have a constant B as in (i). Let $f = \sum_{n=1}^{N} a_n \chi_{\gamma_n}$ be a central E-polynomial. Let

$$\Omega = \Pi_1^N\{-1, 1\}$$

with the operation of coordinatewise multiplication and let $\varepsilon_n\colon \Omega \to \{-1, 1\}$ be projection onto the nth coordinate. Since E is a central Sidon set, for every $\omega \in \Omega$ there exists a central measure μ_ω on G such that $\hat{\mu}_\omega(\gamma_n) = \varepsilon_n(\omega) I_{d_{\gamma_n}} (n = 1, \cdots, N)$ and $\|\mu_\omega\|_1 \leqq C$ where C depends only on E [5, p. 27]. We have

$$\begin{aligned} \|f\|_{2s}^{2s} &= \|\mu_\omega * \mu_\omega * f\|_{2s}^{2s} \leqq \|\mu_\omega\|_1^{2s} \|\mu_\omega * f\|_{2s}^{2s} \\ &\leqq C^{2s} \int_G \left| \sum_{n=1}^{N} a_n \chi_{\gamma_n}(x) \varepsilon_n(\omega) \right|^{2s} dx \,. \end{aligned}$$

Integrating both sides of the inequality over Ω and using Fubini's theorem and the inequality

$$\left(\int_\Omega \left| \sum_{n=1}^{N} b_n \varepsilon_n(\omega) \right|^{2s} d\omega \right)^{1/2s} \leqq 2\sqrt{s} \left(\sum_{n=1}^{N} |b_n|^2 \right)^{1/2}$$

whose proof is the same as that of [8, 8.4, p. 213], we have

$$\begin{aligned} \|f\|_{2s}^{2s} &\leqq C^{2s} 2^{2s} s^s \int_G \left(\sum_{n=1}^{N} |a_n|^2 |\chi_{\gamma_n}(x)|^2 \right)^s dx \\ &= (2\sqrt{s}\,C)^{2s} \sum |a_{n_1}{}^2| \cdots |a_{n_s}|^2 \int_G |\chi_{\gamma_{n_1}}|^2 \cdots |\chi_{\gamma_{n_s}}|^2 dx \end{aligned}$$

where the sum is over all $(n_1, \cdots, n_s) \in \{1, \cdots, N\}^s$. By Hölder's inequality this expression is

$$\begin{aligned} &\leqq (2\sqrt{s}\,C)^{2s} \sum |a_{n_1}|^2 \cdots |a_{n_s}|^2 \Pi_{j=1}^{s} \left(\int_G |\chi_{\gamma_{n_j}}|^{2s} dx \right)^{2/2s} \\ &\leqq (2\sqrt{s}\,C)^{2s} \left(\sum_{n=1}^{N} |a_n|^2 \right)^s B^{2s} \end{aligned}$$

that is, $\|f\|_{2s} \leqq (CB\sqrt{2})\sqrt{2s}\, \|f\|_2$.

REMARKS. (1) Since $\deg \gamma = \|\chi_\gamma\|_\infty = \lim_{s\to\infty} \|\chi_\gamma\|_{2s}$, (ii) is a restatement of Parker's result.

(2) The following are equivalent.

(a) There exists a constant B depending only on E and s so that $\|\chi_\gamma\|_{2s} \leqq B$ for all $\gamma \in E$.

(b) There exist constants C and D depending only on E and s so that

(i) $[\sigma : \otimes(\gamma)] \leqq C$ for all $\sigma \in \Gamma$ and $\gamma \in E$ where (γ) is the s-tuple whose components are γ, and

(ii) card $M((\gamma)) \leqq D$ for all $\gamma \in E$.

The orthogonality of the characters gives

$$\begin{aligned}
\|\chi_\gamma\|_{2s}^{2s} &= \int_G \chi_\gamma^s \bar{\chi}_\gamma^s dx \\
&= \int_G \Big(\sum_{\sigma \in \Gamma} [\sigma : \otimes (\gamma)]\chi_\sigma\Big)\Big(\sum_{\nu \in \Gamma} [\nu : \otimes (\gamma)]\bar{\chi}_\nu\Big) dx \\
&= \sum_{(\sigma,\nu) \in \Gamma \times \Gamma} [\sigma : \otimes (\gamma)][\nu : \otimes (\gamma)] \int_G \chi_\gamma \bar{\chi}_\nu dx \\
&= \sum_{\sigma \in M((\gamma))} [\sigma : \otimes (\gamma)]^2 .
\end{aligned}$$

Since the terms in this last sum are positive we have the equivalence of (a) and (b).

5. Product groups and lacunary projections. Let G_α, $\alpha \in I$ be a collection of compact groups with dual objects Γ_α. Let $G = \prod_{\alpha \in I} G_\alpha$ be the complete direct product and $\Gamma = \prod_{\alpha \in I}^* \Gamma_\alpha$ be the incomplete direct product. Then Γ is the dual object of G and the operations are all the obvious coordinatewise ones [3, p. 27]. Let $\sigma_\alpha \in \Gamma_\alpha$ and let π_α: $G \to G_\alpha$ be the projection onto the α'th coordinate, then $\sigma_\alpha \circ \pi_\alpha \in \Gamma$. Write σ_α^j for the j-fold tensor product of σ_α in Γ_α and let $M(\sigma_\alpha^j)$ be the set of irreducible components of σ_α^j in Γ_α.

THEOREM. *Let G and Γ be as above and consider $E = \{\gamma_\alpha = \pi_\alpha \circ \sigma_\alpha \mid \alpha \in I\}$. A necessary and sufficient condition that E be a central $\Lambda(2s)$ set is that there exist constants K and L both depending only on s and the set $\{\sigma_\alpha \mid \alpha \in I\}$ so that*

(a) $[\tau_\alpha : \sigma_\alpha^s] \leqq L$ *for all* $\tau_\alpha \in \Gamma_\alpha$ *and* $\alpha \in I$, *and*

(b) card $M(\sigma_\alpha^s) \leqq K$ *for all* $\alpha \in I$.

Proof. Parker [5, p. 70] shows that E is a central Sidon set, hence by the theorem in §4 we need a uniform bound on $\|\chi_{\gamma_\alpha}\|_{2s}$ as α ranges over I. Since Haar measure on G is just the product of the Haar measures on the G_α, we have $\|\chi_{\gamma_\alpha}\|_{2s} = \|\chi_{\sigma_\alpha}\|_{2s}$ but by remark (2) of §4 this is equivalent to the conditions (a) and (b).

REMARK. If $\sup\{\deg \sigma_\alpha \mid \alpha \in I\} = P < \infty$, then E is a central $\Lambda(2s)$ set.

6. Intersections with arithmetic progressions. Let $\sigma \in \Gamma$ and let N be a natural number, we define the arithmetic progression of length N generated by σ to be

$$A(\sigma, N) = \bigcup_{j=1}^{N} M(\sigma^j)$$

where σ^j is the j-fold tensor product of σ.

THEOREM. *Let E be a central $\Lambda(p)$ set $(p > 2)$ with constant B so that $\|f\|_p \leqq B\|f\|_2$ for all central E-polynomials f. Let $\sigma \in \Gamma$, then*

$$\operatorname{card}(A(\sigma, N) \cap E) = 0\,(N^{4(\deg\sigma)^2/p}) \quad as \quad N \longrightarrow \infty .$$

Proof. Choose ε and let $D^\sigma_{2N} = \sum_{\gamma \in A(\sigma,2N)} d_\gamma \chi_\gamma$ and

$$F^\sigma_{2N} = |D^\sigma_{2N}|^2/(\sum_{\gamma \in A(\sigma,2N)} d^2_\gamma)$$

so

$$\begin{aligned} F^\sigma_{2N} &= (\sum d_\gamma \chi_\gamma)(\sum d_\nu \bar{\chi}_\nu)/(\sum d^2_\gamma) \\ &= (\sum_{\zeta \in \Gamma} (\sum d_\gamma d_\nu [\zeta : \gamma \otimes \bar{\nu}])\chi_\zeta)/(\sum d^2_\gamma) \end{aligned}$$

where the inner sum is over all $(\gamma, \nu) \in A(\sigma, 2N) \times A(\sigma, 2N)$. If we write $F^\sigma_{2N} = \sum_{\zeta \in \Gamma} d_\zeta \alpha(F^\sigma_{2N}, \zeta)\chi_\zeta$ then Mayer [4, p. 688] shows that for all N sufficiently large and $\zeta \in A(\sigma, N)$

$$\alpha(F^\sigma_{2N}, \zeta) \geqq r_\sigma(N)/d_\zeta r_\sigma(2N)$$

where r_σ is a polynomial of degree $\leqq d^2_\sigma$. Choose $\eta > 0$ small enough so that $(2^{-(\deg\sigma)^2} - \eta)^{-1} \leqq 2^{(\deg\sigma)^2} + \varepsilon$. Then for this η and $\zeta \in A(\sigma, N)$ we have for N sufficiently large that

$$(1) \qquad \alpha(F^\sigma_{2N}, \zeta) \geqq (2^{-(\deg\sigma)^2} - \eta)/d_\zeta\,.$$

We also have $\|F^\sigma_{2N}\|_2 \leqq \|F^\sigma_{2N}\|_\infty = (D^\sigma_{2N}(e))^2/(\sum d^2_\gamma)$, and since $\chi_\gamma(e) = d_\gamma$ we have

$$\|F^\sigma_{2N}\|_2 \leqq \sum_{\gamma \in A(\sigma,2N)} d^2_\gamma = r_\sigma(N)$$

for all N sufficiently large as shown in [4, p. 687]. Hence

$$(2) \qquad \|F^\sigma_{2N}\|_2 \leqq KN^{(\deg\sigma)^2}$$

for all N sufficiently large. Let $f = \sum_{\zeta \in E \cap A(\sigma,N)} \chi_\zeta$, define $\alpha(f, \zeta)$ so that $f = \sum_{\zeta \in \Gamma} d_\zeta \alpha(f, \zeta)\chi_\zeta$, and suppose N is large enough to satisfy (1)

and (2).
Then

$$\begin{aligned}\operatorname{card}(E \cap A(\sigma, N)) &= \sum_{\zeta \in \Gamma} d_\zeta \alpha(f, \zeta) \\ &= \frac{1}{(2^{-(\deg\sigma)^2} - \eta)} \sum_{\zeta \in \Gamma} d_\zeta \alpha(f, \xi)(2^{-(\deg\sigma)^2} - \eta) \\ &\leqq \frac{1}{(2^{-(\deg\sigma)^2} - \eta)} \sum_{\zeta \in \Gamma} d_\zeta \alpha(f, \zeta) d_\zeta \alpha(F_{2N}^\sigma, \zeta) \\ &= (2^{-(\deg\sigma)^2} - \eta)^{-1} \int_G f(x) F_{2N}^\sigma(x) dx \\ &\leqq (2^{-(\deg\sigma)^2} - \eta)^{-1} \| f \|_p \| F_{2N}^\sigma \|_q .\end{aligned}$$

The logarithmic convexity of the $\| \ \|_p$ norms gives $\| \ \|_q \leqq \| \ \|_1^{(2-q)/q} \| \ \|_2^{2/p}$. Using this fact and the hypothesis that E was a central $\Lambda(p)$ set, the last expression is

$$\leqq B(2^{-(\deg\sigma)^2} + \varepsilon) \| f \|_2 \| F_{2N}^\sigma \|_1^{(2-q)/q} \| F_{2N}^\sigma \|_2^{2/p} .$$

Note that $\| f \|_2 = (\operatorname{card}(A(\sigma, N) \cap E))^{1/2}$ and $\| F_{2N}^\sigma \|_1 = \hat{F}_{2N}^\sigma(1) = 1$, so that by (2) we have

$$(\operatorname{card}(A(\sigma, N) \cap E))^{1/2} \leqq B(2^{(\deg\sigma)^2} + \varepsilon)(KN^{(\deg\sigma)^2})^{2/p}$$

for all N sufficiently large, the size of N depending only on σ and ε.

COROLLARY. *Let E be a central Λ set, and let $\sigma \in \Gamma$. Then*

$$\operatorname{card}(A(\sigma, N) \cap E) = 0(\log N) .$$

Proof. For a central Λ set we may take $B = Cp^{1/2}$ where C depends only on E. In the last inequality of the previous proof, set $p = 4 \log (KN^{(\deg\sigma)^2})$, then

$$\operatorname{card}(A(\sigma, N) \cap E) \leqq (2^{(\deg\sigma)^2} + \varepsilon)^2 C^2 e 4 \log (KN^{(\deg\sigma)^2})$$

for all N sufficiently large.

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