PACIFIC JOURNAL OF MATHEMATICS
Vol. 51, No. 1, 1974

ON FLOW-INVARIANT SETS

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By employing Lyapunov-like functions and the theory of differential inequalities some sufficient conditions are given for flow-invariant and conditionally flow-invariant sets.

1. In a recent paper Redheffer [6] has generalized a remarkable theorem of Bony [1] and has discussed its relation to a similar result of Brezis [2]. Results of general nature are also given by Ladde and Leela [4] which characterize various kinds of flow-invariant sets from a different point of view. In the present paper, we give theorems of very general character which offer sufficient conditions for flow-invariant sets in terms of Lyapunov-like functions and differential inequalities. These results include as special cases the theorems of Bony [1], Brezis [2], and Redheffer [6].

2. Let E be a domain in real Euclidean space R^n and $F \subset E$ be a closed set. For any set A, let A^0, ∂A denote the interior, boundary of A respectively. As usual, $d(x, A)$ denotes the distance of a point x from the set A. We consider the differential system

$$x' = f(t, x) , \quad x(t_0) = x_0 , \quad t_0 \in R^+ , \tag{1}$$

where $f \in C[R^+ \times E, R^n]$, R^+ being the half real line.

The set F is flow-invariant for f if

$$x_0 \in F \quad \text{implies} \quad x(t, t_0, x_0) \in F \quad \text{for} \quad t_0 \leqq t < T$$

where $[t_0, T)$ is the interval of existence of the solution $x(t, t_0, x_0)$ of (1).

Let $V \in C^1[R^+ \times E, R^+]$ and $y \in F$. Let, for each t, S_k be the closed region around a fixed x generated by $V(t, x - z) \leqq k$, that is, $S_k = [z: V(t, x - z) \leqq k]$. Suppose that $F \cap S_k^0 = \emptyset$ and $y \in \partial S_k$. Then the vector $\gamma(t, y) = V_x(t, x - y)$ is said to be normal to F at y for each t.

The function $V(t, x - y)$ is said to be positive definite with respect to the set F, if $w(t, x) = \inf_{y \in F} V(t, x - y)$ is positive definite with respect to f.

Let $g \in C[R^+ \times R^+, R]$, $g(t, 0) \equiv 0$ and the only solution of

$$u' = g(t, u) , \qquad u(t_1) = 0 , \tag{2}$$

on $t_1 \leqq t < t_1 + \varepsilon$ is identically zero, where $\varepsilon > 0$ is some number, for every $t_1 \in R^+$. Then we shall say that g is a uniqueness function.

We shall now prove the following result which offers sufficient conditions for a closed set F to be flow-invariant relative to f.

THEOREM 1. *Let the following assumptions hold:*
(i) *for* $(t, x, y) \in R^+ \times E \times F$,

$$V'(t, x - y) \equiv V_t(t, x - y) + V_x(t, x - y)\cdot[f(t, x) - f(t, y)] \leqq g(t, V(t, x - y)) ;$$

(ii) $\gamma(t, y)\cdot f(t, y) \leqq 0$ *whenever the vector* $\gamma(t, y)$ *is normal to* F *at* y *for each* t;

(iii) $V(t, x - y)$ *is positive definite with respect to the set* F *and* g *is a uniqueness function. Then the closed set* F *is flow-invariant for* f.

Proof. If the theorem is false, there exists a $t_1 > t_0$ such that $x(t_0) \in F$ but $x(t) \notin F$ for $t_1 < t < t_2$ for some t_2, on which $x(t)$ exists. Set $\varepsilon = t_2 - t_1$. Thus we have $d[x(t), F] > 0$ for $t_1 < t < t_2$ and $d[x(t_1), F] = 0$. Define $m(t) = w(t, x(t))$ for $t_1 \leqq t < t_2$ and observe that $m(t_1) = 0$ by (iii). For a fixed t in $t_1 < t < t_2$, let $x = x(t)$. It follows, by the definition of $w(t, x)$ and the fact that F is closed, that there exists a $y_0 \in F$ such that

$$m(t) = V(t, x - y_0) .$$

Consider the closed region $S_k = [z: V(t, x - z) \leqq k = m(t)]$. It is easily verified that $F \cap S_k^0 = \varnothing$. This shows that by definition that the vector $\gamma(t, y_0) = V_x(t, x - y_0)$ is normal to F at y_0. For small $h \geqq 0$ and for any $y \in F$, we have

$$m(t + h) = w(t + h, x(t + h)) \leqq V(t + h, x(t + h) - y) .$$

Hence

$$m(t + h) - m(t) \leqq V(t + h, x(t + h) - y_0) - V(t, x(t) - y_0) ,$$

which yields

$$D^+m(t) \leqq V_t(t, x(t) - y_0) + V_x(t, x(t) - y_0)\cdot f(t, x(t)) .$$

Consequently, using the assumptions (i) and (ii), we get

$$D^+m(t) \leqq V'(t, x(t) - y_0) \leqq g(t, m(t)) , \qquad t_1 < t < t_2 . \tag{3}$$

By Theorem 1.4.1 in [5], we then obtain

$$m(t) \leqq r(t, t_1, m(t_1)) , \qquad t_1 \leqq t < t_2 ,$$

where $r(t, t_1, u_0)$ is the maximal solution of

$$u' = g(t, u)\,, \qquad u(t_1) = u_0\,.$$

Since $m(t_1) = 0$, the hypothesis (iii) implies that

$$m(t) \equiv 0 \quad \text{for} \quad t_1 \leqq t < t_2\,,$$

which shows that $x(t) \in F$ for $t_1 < t < t_2$. This contradiction proves the theorem.

The special case $V(t, x) = \|x\|^2$ and $g(t, u) = \rho(u)$, where ρ is a uniqueness function of Theorem 1 includes the important result of Bony [1] as generalized by Redheffer in [6].

For an example of f which does not satisfy either a Lipschitz or a monotonicity condition but for which there does exist a Lyapunov function satisfying the uniqueness hypothesis of Theorem 1, see [3, p. 137].

3. In this section we shall generalize Theorem 1 a little further. Let $H \subset E$ be an open set such that $F \subset H$. Then the set H is said to be *conditionally flow-invariant* for f with respect to the set F, if

$$x_0 \in F \quad \text{implies} \quad x(t, t_0, x_0) \in H \quad \text{for} \quad t_0 \leqq t < T\,,$$

where $[t_0, T)$ is the interval of existence of $x(t, t_0, x_0)$ of (1). See [5] for the notion of conditionally invariant sets and their stability properties. We then have

THEOREM 2. *Let the assumptions* (i) *and* (ii) *of Theorem* 1 *hold. Suppose further that*

(a) $w(t, x) = \inf_{y \in F} V(t, x - y)$, $a \in C[R^+, R]$, $w(t, x) \geqq a(t)$, *if* $x \in \partial H$ *and* $w(t, x) < a(t)$, *if* $x \in \partial F$;

(b) *any solution* $u(t, t_1, u_0)$ *of* $u' = g(t, u)$, $u(t_1) = u_0$, *satisfies* $u(t, t_1, u_0) < a(t)$, *for* $t_1 \leqq t < t_1 + \varepsilon$ *where* $\varepsilon > 0$ *is some number, provided* $u_0 < a(t_1)$ *for every* $t_1 \in R^+$. *Then the set* H *is conditionally flow-invariant for* f *with respect to the set* F.

Proof. The proof is almost the same as that of Theorem 1. We shall only indicate the required changes.

Suppose that the theorem is false. Then there exists a $t^* > t_0$ such that $x_0 \in F$, $x(t) = x(t, t_0, x_0) \in H$ for $t_0 \leqq t < t^*$ and $x(t^*) \in \partial H$. This implies that there is a t_1, $t_0 \leqq t_1 < t^*$ such that $x(t_1) \in \partial F$ and $x(t) \in H \backslash F$ for $t_1 < t < t^*$. By (a), we then have

$$w(t^*, x(t^*)) \geqq a(t^*) \quad \text{and} \quad w(t_1, x(t_1)) > a(t_1)\,. \tag{4}$$

Defining $m(t) = w(t, x(t))$, we proceed as in the proof of Theorem 1 till we arrive at the differential inequality (3). We now choose $u_0 = a(t_1)$ so that by Theorem 1.4.1 in [5], we get, as before,

$$m(t) \leqq r(t, t_1, a(t_1)), \qquad t_1 \leqq t < t^* .$$

By the continuity of the functions involved, the assumption (b) and the relations (4), we arrive at the contradiction

$$a(t^*) \leqq m(t^*) \leqq r(t^*, t_1, a(t_1)) < a(t^*) .$$

Hence the proof is complete.

Notice that Theorem 2 enlarges the class of useful Lyapunov-like functions V and offers more flexibility. To see this we give the following application.

THEOREM 3. *Suppose that the following conditions hold:*

(i) $F \subset R^n$ *is a closed set,* $V \in C^1[R^+ \times S(F, \rho)\backslash F, R]$ *and* $V'(t, x-y) \leqq g(t, V(t, x-y))$ *for* $(t, x, y) \in R^+ \times S(F, \rho)\backslash F \times F$;

(ii) $w(t, x) = \inf_{y \in F} V(t, x-y)$, $b \in C[R^+ \times (0, \rho], (-1, \infty)]$, $b(t, d(x, F)) \leqq w(t, x)$ *for* $(t, x) \in R^+ \times S(F, \rho)\backslash F$, *and* $w(t, x) \to -1$ *as* $d(x, F) \to 0$ *uniformity in* t;

(iii) $\gamma(t, y)\cdot f(t, y) \leqq 0$ *whenever the vector* $\gamma(t, y)$ *is normal to* F *at* y *for each* t;

(iv) $g \in C[R^+ \times R, R]$ *and any solution* $u(t, t_1, u_0)$ *of*

$$u' = g(t, u), \qquad u(t_1) = u_0 ,$$

satisfies $u(t, t_1, u_0) < b(t, \eta)$, $t_1 \leqq t < t_1 + \varepsilon$, ε *is some positive number, provided* $u_0 < b(t_1, \eta)$ *for every* $t_1 \in R^+$ *and a fixed* $\eta \in (0, \rho]$. *Then the set* $S(F, \eta)$ *is conditionally flow-invariant for* f *with respect to* F.

Proof. Since $w(t, x) \to -1$ as $d(x, F) \to 0$ uniformly in t,

$$w(t, x) < b(t, \eta)$$

for each $t \in R^+$, whenever $x \in \partial F$. Setting

$$E = S(F, \rho)\backslash F, \quad H = S(F, \eta), \quad a(t) = b(t, \eta),$$

we see that all the hypotheses of Theorem 2 are verified. Hence the conclusion follows.

If hypothesis (iv) holds for every $\eta \in (0, \rho]$, instead of a fixed η, Theorem 3 shows that F is flow-invariant for f, because $\lim_{\eta \to 0} S(F, \eta) = F$.

4. We can formulate Theorem 1 in such a way as to include the result of Brezis [2] as generalized by Redheffer in [6].

THEOREM 4. *Assume that all the hypotheses of Theorem 1 are satisfied except that the assumption* (ii) *is replaced by* (ii*) $$\liminf_{h\to 0^+} \frac{1}{h} w(t, y + hf(t, y)) = 0 \quad \textit{for each} \quad y \in F .$$

Then the conclusion of Theorem 1 remains valid. Provided that $w(t, x)$ possesses the property

$$w(t, x_1) - w(t, x_2) \leqq V(t, x_1 - x_2) . \tag{5}$$

The condition (ii*) is needed only at each $y \in F$ that has a normal in the sense defined earlier. One can show that the hypothesis (ii*) together with (5) imply the assumption (ii) of Theorem 1. Indeed, let $\gamma(t, y)$ be normal to F at y for each t and let S_k be the closed region around a fixed x generated by $V(t, x - z) \leqq k$ so that $\gamma(t, y) = V_x(t, x - y)$. Since $S_k^0 \cap F = \varnothing$, we have

$$V(t, x - y) = w(t, x) .$$

In view of the condition (5), this implies, for small $h > 0$,

$$V(t, x - y) \leqq V(t, x - y - hf(t, y)) + w(t, y + hf(t, y)) .$$

Hence

$$0 \leqq V(t, x - y - hf(t, y)) - V(t, x - y) + \varepsilon(h) , \tag{6}$$

where $\varepsilon(h) = w(t, y + hf(t, y))$. By (iv*), we see that

$$\liminf_{h\to 0^+} \frac{\varepsilon(h)}{h} = 0 .$$

Consequently, the inequality (6) assures

$$0 \leqq - V_x(t, x - y) \circ f(t, y)$$

which is condition (iv) of Theorem 1.

We could have, following the proof of Theorem 1, directly proved Theorem 4. The proof crucially depends on the inequality (5). This we leave to the reader.

As before, the choice $V(t, x) = \|x\|$ or $\|x\|^2$ and $g(t, u) = \rho(u)$ where ρ is a uniqueness function includes the result of Brezis [2] as given in [6]. Unfortunately, the restrictive condition (5) seems to be unavoidable which makes Theorem 4 less flexible compared to Theorem 1.

It is possible to formulate results analogous to Theorems 2 and 3 in the spirit of Theorem 4. This we do not undertake to avoid monotony.

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Received December 21, 1972.

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