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OSCILLATORY PROPERTIES OF A DELAY DIFFERENTIAL EQUATION OF EVEN ORDER

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A classification of nonoscillatory solutions according to the sign properties of their derivatives is introduced for a general nonlinear delay differential equation of order $2n$. It is seen that there are n types of positive solutions of this equation. An intermediate Riccatti transformation is employed to obtain integral criteria for the nonexistence of such solutions and for the oscillation of all solutions. Analysis of the Taylor Remainder gives rise to a summability condition which is used to investigate the asymptotic behavior of a class of solutions. The major results are then shown to be special cases of a more general result based on the direct method of Lyapunov.

The purpose of this paper is to discuss the oscillatory and nonoscillatory behavior of solutions of the nonlinear delay differential equation of order $2n$:

$$D^n[r(t)D^n y(t)] + y_\tau(t) f(t, y_\tau(t)) = 0 \,, \tag{1.1}$$

where $y_\tau(t) = y[t - \tau(t)]$, $0 \leqq \tau(t) \leqq T$ and $0 < m \leqq r(t) \leqq M$. Throughout the first three sections $f(t, u)$ is assumed to satisfy the following three hypotheses:

(i) $f(t, u)$ is a continuous real-valued function on $[0, \infty) \times R$, $R = (-\infty, \infty)$;

(ii) for each fixed $t \in [0, \infty)$, $f(t, u) < f(t, v)$ for $0 < u < v$; and

(iii) for each fixed $t \in [0, \infty)$, $f(t, u) > 0$ and $f(t, u) = f(t, -u)$ for $u \neq 0$.

In section four, these assumptions on $f(t, u)$ will be replaced by others as indicated there.

1. A solution $y(t)$ of (1.1) is said to be oscillatory on $[0, \infty)$ if for each $t_0 > 0$, there exists a $T_0 > t_0$ such that $y(T_0) = 0$; it is called nonoscillatory otherwise. Following Kiguradze [4], we say that a solution $y(t)$ is of type A_j if for sufficiently large t the derivatives $D^k y(t) > 0$, $k = 0, 1, \cdots, 2j + 1$ and $(-1)^{k+1} D^k y(t) > 0$, $k = 2j + 2, \cdots, 2n - 1$. In an analogous manner we say that a solution $y(t)$ is of type B_j if for sufficiently large t, $y_k(t) > 0$ for $k = 0, \cdots, 2j + 1$ and $(-1)^{k+1} y_k(t) > 0$, $k = 2j + 2, \cdots, 2n - 1$ where

$$y_k(t) = D^k y(t) \,, \qquad k = 0, \cdots, n - 1$$

and

$$y_k(t) = D^{k-n}[r(t)D^n y(t)] , \qquad k = n, \cdots, 2n-1.$$

When $r(t) \equiv 1$, these definitions reduce to those of types $A_j (0 \leqq j \leqq n-1)$ respectively. In [4] Kiguradze proved a fundamental lemma which we state here as follows.

LEMMA 1.1. *Let $u(t)$ be a continuous nonnegative function on $(0, \infty)$ having continuous derivatives up to order $2n$ inclusive which do not change sign on this interval. If $D^{2n}u(t) \leqq 0$, then there exists an integer $p = 0, \cdots, n-1$ such that*

$$D^k u(t) \geqq 0 , \qquad k = 0, \cdots, l ,$$

$$(-1)^{k+1} D^k u(t) \geqq 0 , \qquad k = l+1, \cdots, 2n-1 ,$$

where $l = 2p+1$. Furthermore, $0 \leqq D^l u(t) \leqq l!\, t^{-l} u(t)$.

In view of this result, all nonoscillatory solutions of (1.1) with $r(t) \equiv 1$ are of type A_j for some $j = 0, \cdots, n-1$. In the general case we argue as follows. Suppose $y(t)$ is a nonoscillatory solution of (1.1) which we may assume to be nonnegative because of (iii). First, no two successive y_k, $k \geqq 1$ can ultimately be negative. Suppose $k \geqq n+1$ and y_k and y_{k+1} are negative for $t > a$. Then y_k is a negative decreasing function on $[a, \infty)$ and there exists a constant $C_k > 0$ such that $y_k(t) < -C_k$ for $t > a$. Thus

$$y_{k-1}(t) - y_{k-1}(a) = \int_a^t y_k(s)ds < -C_k(t-a) ,$$

which implies that y_{k-1} is eventually negative. Next, if y_n and y_{n+1} are eventually negative, say for $t > a$, then

$$[My_{n-1}(s)]_a^t = M\int_a^t D^n y(s)ds < \int_a^t y_n(s)ds < -C_n(t-a) ,$$

which implies that y_{n-1} is eventually negative. Since $r(t) > 0$ and $y_n(t) < 0$, $D^n y(t) < 0$. Using the negativity of $D^n y$ and $D^{n-1}y$ we may show as in the first part that y_{n-k} is eventually negative for any $k = 2, \cdots, n$ which contradicts the positivity of $y(t)$.

Using the same technique as above, it follows that if any two consecutive y_k, $k \geqq 1$ are ultimately positive, then all the preceding y_k are eventually positive. We conclude that a positive solution of (1.1) is necessarily of type B_j for some $j = 0, \cdots, n-1$. Thus the nonexistence of nonoscillatory solutions of type $B_j (0 \leqq j \leqq n-1)$ will imply that all solutions are necessarily oscillatory.

In section two integral criteria are given for the nonexistence of solutions of type B_j as well as for the oscillation of all solutions of (1.1). Section three provides a necessary and sufficient condition for the existence of a nonscillatory solution of (1.1) with $r(t) \equiv 1$ having prescribed asymptotic form. Recently Yoshizawa [11, 12] has applied the direct method of Lyapunov to study the oscillatory behavior of solutions of certain nonlinear second order ordinary differential equations. In section four his technique is employed to investigate the nonoscillatory behavior of (1.1). For recent related studies see the papers of Burkowski [2], Gollwitzer [3], Staikos and Petsoulas [7], and Wong [9].

2. In this section integral criteria for the nonexistence of solutions of types $B_j (j = 0, \cdots, n-1)$ are derived. To obtain these results we shall first prove several lemmas.

LEMMA 2.1. *Let $y(t)$ be a solution of* (1.1) *of type B_j where either* (i) *n is even and $j \leqq (n-2)/2$ or* (ii) *n is odd and $j \leqq (n-3)/2$. Then for all sufficiently large t,*

$$ty_k(t) \leqq 2(2j + 2 - k)y_{k-1}(t)\,, \qquad k = 1, \cdots, 2j + 1\,.$$

LEMMA 2.2. *Let $y(t)$ be a B_j-solution of* (1.1) *where n is odd and $j = (n-1)/2$. Then for all sufficiently large t,*

$$ty_n(t) \leqq 2My_{n-1}(t)$$

and

$$ty_k(t) \leqq 2[Mm^{-1} + (n - k)]y_{k-1}(t)\,, \qquad k=1, \cdots, n-1\,.$$

LEMMA 2.3. *Let $y(t)$ be a B_j-solution of* (1.1) *where either* (i) *n is even and $j \geqq n/2$ or* (ii) *n is odd and $j \geqq (n+1)/2$. Then for all sufficiently large t,*

(a) $ty_k(t) \leqq 2(2j + 2 - k)y_{k-1}(t), \qquad k = n + 1, \cdots, 2j + 1.$

(b) $ty_n(t) \leqq 2M(2j - n + 2)y_{n-1}(t)$, *and*

(c) $ty_k(t) \leqq 2[M(2j - n + 2)m^{-1} + (n - k)]y_{k-1}(t), \qquad k = 1, \cdots, n - 1.$

The proof of each of the three lemmas is elementary using only integration by parts and the definition of a B_j-solution. For brevity, we will prove only Lemma 2.3.

Proof. Suppose $y(t)$ is a solution of type B_j. Then there is a $T_0 > 0$ such that $y_k(t) > 0$, $(0 \leqq k \leqq 2j + 1)$ and $y_{2j+2}(t) < 0$ for $t \geqq T_0$. Hence, $y_\tau(t) > 0$ for $t - \tau(t) \geqq T_0$, i.e., for $t \geqq T_0 + T = T_1$. Thus, $y_{2j+1}(t)$ is a decreasing function of t for $t \geqq T_1$. Consequently,

$$y_{2j}(t) \geqq [y_{2j}(s)]_{T_1}^{t} = \int_{T_1}^{t} y_{2j+1}(s)ds \geqq (t - T_1)y_{2j+1}(t)\,.$$

Since $(t - T_1) \geqq t/2$ for $t \geqq 2T_1$, we have $y_{2j}(t) \geqq ty_{2j+1}(t)/2$ which proves (a) for the case $k = 2j + 1$.

We proceed inductively and suppose that for some integer k, $n + 1 < k \leqq 2j + 1$,

$$(t - T_1)y_k(t) \leqq (2j + 2 - k)y_{k-1}(t) . \tag{2.1}$$

An integration of (2.1) yields

$$[(s - T_1)y_{k-1}(s)]_{T_1}^t - \int_{T_1}^t y_{k-1}(s)ds = \int_{T_1}^t (s - T_1)y_k(s)ds$$

so that

$$\begin{aligned}(t - T_1)y_{k-1}(t) &\leqq [2j + 2 - (k - 1)] \int_{T_1}^t y_{k-1}(s)ds \\ &\leqq [2j + 2 - (k - 1)]y_{k-2}(t) ,\end{aligned}$$

which proves (a). Specifically, for $k = n + 1$, (2.1) becomes

$$(t - T_1)y_{n+1}(t) \leqq (2j - n + 1)y_n(t) .$$

Integrating this by parts results in

$$\begin{aligned}(t - T_1)y_n(t) - \int_{T_1}^t y_n(s)ds &= \int_{T_1}^t (s - T_1)y_{n+1}(s)ds \\ &\leqq (2j - n + 1) \int_{T_1}^t y_n(s)ds .\end{aligned}$$

Thus, for $t \geqq T_1$ we obtain

$$\begin{aligned}(t - T_1)y_n(t) &\leqq (2j - n + 2) \int_{T_1}^t y_n(s)ds \\ &\leqq (2j - n + 2)My_{n-1}(t) .\end{aligned} \tag{2.2}$$

As in (a) the estimate for $t \geqq 2T_1$ becomes

$$ty_n(t) \leqq 2(2j - n + 2)My_{n-1}(t) ,$$

which proves (b).

Now integration of (2.2) by parts establishes the anchor for an inductive proof of (c).

$$\begin{aligned}[m(s - T_1)y_{n-1}(s)]_{T_1}^t - m \int_{T_1}^t y_{n-1}(s)ds &= m \int_{T_1}^t (s - T_1)D^n y(s) \\ &\leqq \int_{T_1}^t (s - T_1)y_n(s)ds \\ &\leqq (2j - n + 2)M \int_{T_1}^t y_{n-1}(s)ds .\end{aligned}$$

Thus, for $t \geqq 2T_1$, we have

$$mty_{n-1}(t)/2 \leqq m(t - T_1)y_{n-1}(t) \leqq [(2j - n + 2)M + m] \int_{T_1}^{t} y_{n-1}(s)ds$$
$$\leqq [(2j - n + 2)M + m]y_{n-2}(t) \,,$$

which proves (c) in the case $k = n - 1$. Now assume inductively that

$$(t - T_1)y_k(t) \leqq [M(2j - n + 2)m^{-1} + (n - k)]y_{k-1}(t)$$

for some k, $1 < k \leqq n - 1$. Then

$$(t - T_1)y_{k-1}(t) - \int_{T_1}^{t} y_{k-1}(s)ds$$
$$= \int_{T_1}^{t} (s - T_1)y_k(s)ds$$
$$\leqq [M(2j - n + 2)m^{-1} + (n - k)] \int_{T_1}^{t} y_{k-1}(s)ds$$

so that for $t \geqq T_1$

$$(t - T_1)y_{k-1}(t) \leqq [M(2j - n + 2)m^{-1} + (n - k + 1)]y_{k-2}(t)$$

and for $t \leqq 2T_1$

$$ty_{k-1}(t) \leqq 2[M(2j - n + 2)m^{-1} + (n - k + 1)]y_{k-2}(t)$$

which proves (c).

We remark that if $y(t)$ is a B_j-solution of (1.1) on $[0, \infty)$ with $r(t) \equiv 1$ and $\tau(t) \equiv 0$, we may take $T_1 = 0$ and $m = M = 1$. The above proof will yield

$$ty_{2j+2-k} \leqq ky_{2j+1-k}$$

for $k = 1, \cdots, 2j + 1$, which is another form of Kiguradze's lemma [4].

LEMMA 2.4. *Let $y(t)$ be a solution of* (1.1) *of type B_j. Then* $y_{2j}(t - \tau(t)) \sim y_{2j}(t)$.

Proof. Let $y(t)$ be a B_j-solution of (1.1). For $j \neq (n - 1)/2$ and $t \geqq T_1$, $y_\tau(t)$ and $y_k(t)$, $k = 0, \cdots, 2j + 1$ are all positive while $y_{2j+2}(t) < 0$. Since $\tau(t) \geqq 0$ and $y_{2j}(t)$ is an increasing function on (T_1, ∞), $y_{2j}(t - \tau(t)) < y_{2j}(t)$, so that with the help of Lemma 2.1 or 2.3, we have

$$\left| \frac{y_{2j}(t - \tau(t))}{y_{2j}(t)} - 1 \right| = \tau(t)\frac{y_{2j+1}(s)}{y_{2j}(t)} < \frac{2T}{s}\,\frac{y_{2j}(s)}{y_{2j}(t)} < \frac{2T}{s} \,,$$

where $t - \tau(t) \leqq s \leqq t$. Since s tends to infinity with t, the lemma follows in this case. The case $j = (n - 1)/2$ follows in a similar manner using Lemma 2.2 and the estimate $mDy_{2j}(s) \leqq y_{2j+1}(s)$.

We remark that this lemma is analogous to one proved by Bradley [1] for the linear equation

$$y''(t) + p(t)y_\tau(t) = 0 .$$

His proof can be modified to yield Lemma 2.4. We note that Lemmas 2.1, 2.2, and 2.3 are valid for unbounded t, provided $\lim_{t\to\infty} (t - \tau(t)) = +\infty$. A weaker version of Lemma 2.4 is also true for unbounded $\tau(t)$, provided $0 \leqq \tau(t) \leqq \mu t$, where μ can be specified. In general, if $y(t)$ is a B_j-solution of (1.1) described by Lemmas 2.1 or 2.3, then we may take $\mu < 1/2$; otherwise the stronger estimate $\mu < m/(M + m)$ is required. The conclusion of Lemma 2.4 is changed to read: There are constants $k_j > 0$ and $t_j > 0$ such that $y_{2j}(t - \tau(t)) > k_j y_{2j}(t)$ for $t > t_j$. With these lemmas we can now give criteria for the non-existence of solutions of type $B_j (0 \leqq j \leqq n - 1)$.

THEOREM 2.5. *Suppose that for all constants $C > 0$ and some $j = 0, \cdots, (n - 1)$,*

$$\int^\infty t^{2j} f(t, Ct^{2j})dt = +\infty . \tag{2.3}$$

Then (1.1) *has no solutions of type B_j.*

Proof. Let $y(t)$ be a positive solution of type B_j and let $w(t) = y_{2n-1}(t)/y_{2j}(t)$. Then (1.1) shows that

$$w'(t) + y_{2n-1}(t)Dy_{2j}(t)y_{2j}^{-2}(t) + y_\tau(t)f(t, y_\tau(t))y_{2j}^{-1}(t) = 0 .$$

For $j \neq (n - 1)/2$, $Dy_{2j}(t) = y_{2j+1}(t) > 0$; if n is odd and $j = (n - 1)/2$, $Dy_{2j}(t) = y_{2j+1}(t)/r(t) > 0$ since $r(t) > 0$. Since $y_{2n-1}(t)$ and $y_{2j}(t)$ are positive for $t > T_1$, this reduces to

$$w'(t) + y_\tau(t)f(t, y_\tau(t))y_{2j}^{-1}(t) < 0, t \geqq T_1 . \tag{2.4}$$

There are three cases to consider

(i) n is even and $j \leqq (n - 2)/2$ or n is odd and $j \leqq (n - 3)/2$;

(ii) n is odd and $j = (n - 1)/2$; and

(iii) n is even and $j \geqq n/2$ or n is odd and $j \geqq (n + 1)/2$.

Applying Lemmas 2.1, 2.2, and 2.3 to cases (i), (ii), and (iii) respectively, we obtain

$$t^{2j}y_{2j}(t) \leqq M_j y(t) ,$$

where

$$M_j = \begin{cases} 2^{2j}(2j + 1)! , & \text{case (i)}, \\ 2^{2j} \prod_{k=1}^{2j} [Mm^{-1} + (n - k)] , & \text{case (ii)}, \end{cases}$$

$$2^{2j}(2j-n+2)!\, M\prod_{k=1}^{n-1}[(2j-n+2)Mm^{-1}+(n-k)]\,, \quad \text{case (iii)}.$$

Letting $N_j = M_j^{-1}$, we have $y(t) \geqq N_j t^{2j} y_{2j}(t)$ for $t \geqq 2T_1$, so for $t \geqq 2T_1 + T = T_2$ we have

$$y_\tau(t) \geqq N_j(t-\tau(t))^{2j} y_{2j}(t-\tau(t)) \geqq N_j(t-T)^{2j} y_{2j}(t-\tau(t))$$

because $0 \leqq \tau(t) \leqq T$. Since $Dy_{2j}(t) > 0$ for $t \geqq T_1$, there is a constant C_0 such that $y_{2j}(t) \geqq C_0$ for $t \geqq T_1$. Hence, $y_{2j}(t-\tau(t)) \geqq C_0$ for $t \geqq T_1 + T$. Moreover, $(t-T) \geqq t/2$ for $t \geqq 2T$. Because $T_2 = 2T_1 + T > \max(T_1 + T, 2T)$, both estimates hold for $t \geqq T_2$. Combining the above estimates with (2.4) via hypothesis (ii) and using Lemma 2.4, we arrive at

$$w'(t) + NKt^{2j} f(t, Ct^{2j}) \leqq 0,\ t \geqq T_3$$

for some $T_3 \geqq T_2$, where $N = N_j 2^{-2j}$, $C = 2^{-2j} N_j C_0$ and $K = K_j > 0$ is the constant from Lemma 2.4. An integration of this together with (2.3) shows that $w(t)$ is eventually negative, which is absurd.

Since the divergence of $t^{2j} f(t, Ct^{2j})$ implies that of $t^{2(j+1)} f(t, Ct^{2(j+1)})$ the conclusion of Theorem 2.5 may be strengthened to exclude solutions of type B_k where $j \leqq k \leqq n-1$. The theorem may also be restated as follows.

THEOREM 2.6. *Suppose* (2.3) *holds for all constants* $C > 0$ *and for some* $j = 0, \cdots, n-1$. *Then either* (1.1) *is oscillatory or else* $y(t)y_{2j}(t) < 0$ *for* t *sufficiently large.*

For $j = n-1$ and $r(t) \equiv 1$, Theorem 2.6 reduces to the alternative that either (1.1) is oscillatory or else $y(t)D^{2n-2}y(t) < 0$ for t sufficiently large, which is essentially Theorem 3.1 of Ladas [5].

Moreover, in view of the fact that all positive solutions are of types B_j for some $j (0 \leqq j \leqq n-1)$, we can immediately restate Theorem 2.5 as a criterion of oscillation.

COROLLARY 2.7. *If for all constants* $C > 0$

$$\int^\infty f(t, C)dt = +\infty\,,$$

then all solutions of (1.1) *are oscillatory.*

COROLLARY 2.8. *Suppose* $p(t)$ *is continuous and eventually positive and that*

$$\int^\infty p(t)dt = +\infty\,.$$

Then all solutions of the equation

$$D^n[r(t)D^n y(t)] + p(t)y_\tau^{2\lambda+1}(t) = 0,\ \lambda > 0 \tag{2.5}$$

are oscillatory.

The conclusion of Corollary 2.8 is true in the case $\lambda = 0$ in (2.5). In this instance, (2.5) is not a special case of (1.1) since $f(t, u) = p(t)$ does not satisfy hypothesis (ii) of section one. To permit this extension, we may suppose that $y(t)$ is a B_j-solution of (2.5) and let $w(t) = y_{2n-1}(t)/y_{2j}(t)$. Equation (2.4) becomes

$$w'(t) + p(t)y_\tau(t)y_{2j}^{-1}(t) < 0,\ t \geqq T_1\ .$$

Applying Lemmas 2.1–2.4, which are independent of hypothesis (ii), produces the same contradiction as in the proof of Theorem 2.5. Thus, if $\lambda = 0$ and

$$\int^\infty t^{2j} p(t)dt = \infty$$

for some $j = 0, \cdots, n-1$, (2.5) has no B_k-solutions for $k = j, \cdots, n-1$. Corollary 2.8 then follows by specifying $j = 0$.

NOTE. When $n = 2$ and $j = 1$, Lemma 2.3 reduces to Lemma 2.1 of Terry and Wong [8]. Similarly, letting $n = 2$ and $j = 0, 1$ in Lemma 2.4, we obtain Lemma 2.2 (a), (b) of [8]. Moreover, Theorem 2.5, Corollary 2.7, and Corollary 2.8 here are, respectively, the analogues of Theorem 2.8, Theorem 2.4, and Corollary 2.5 of [8].

3. In this section an asymptotic result is established for the equation

$$D^{2n}y(t) + f(t, y_\tau(t))y_\tau(t) = 0\ , \tag{3.1}$$

where $f(t, u)$ satisfies the three conditions of section one.

LEMMA 3.1. *Let $y(t)$ be a solution of* (3.1) *which is eventually positive. Then*

$$D^{2n-1}y(t) \sim (2n-1)!\, t^{1-2n}y(t)\ .$$

Proof. Suppose $y(t)$ is a solution of (3.1) such that $y(t) > 0$ for $t \geqq T_1$. Then $y_\tau(t) > 0$ for $t - \tau(t) \geqq T_1$, i.e., for $t \geqq T_1 + T = T^*$. By Taylor' theorem with remainder, for $t \geqq T^*$

$$(2n-1)!\, R(t) = (2n-1)!\, y(t) + \int_{T^*}^t (t-s)^{2n-1}y_\tau(s)f(s, y_\tau(s))ds\ , \tag{3.2}$$

where

$$R(t) = \sum_{k=0}^{2n-1} \frac{1}{k!} D^k y(T^*)(t - T^*)^k .$$

Since $y_\tau(s)$, and hence $-D^{2n}y(s)$, is positive for $s > T^*$, condition (iii) together with $(t - T^*) > (t - s) > 0$ imply that

$$\begin{aligned}(2n-1)!\, R(t)\\ &\leqq (2n-1)!\, y(t) + (t - T^*)^{2n-1}[D^{2n-1}y(T^*) - D^{2n-1}y(t)] .\end{aligned}$$

Dividing this by $(t - T^*)^{2n-1}$ and noting that

$$\lim_{t\to\infty} (2\mathrm{n} - 1)!\, (t - T^*)^{1-2n} R(t) = D^{2n-1}y(T^*) ,$$

it follows upon passage to the limit that

$$\begin{aligned}D^{2n-1}&y(T^*)\\ &\leqq \varliminf_{t\to\infty} (2n-1)!\, (t - T^*)^{1-2n} y(t) + D^{2n-1}y(T^*) - \lim_{t\to\infty} D^{2n-1}y(t) .\end{aligned}$$

Hence,

$$\lim_{t\to\infty} D^{2n-1}y(t) \leqq \varliminf_{t\to\infty} (2n-1)!\, (t - T^*)^{1-2n} y(t) . \tag{3.3}$$

To prove the reverse inequality let σ be chosen such that $T^* < \sigma < t$. By restricting s to lie in the interval $[T^*, \sigma]$, we have $(t - s)^{2n-1} \geqq (t - \sigma)^{2n-1}$ and

$$\begin{aligned}(2n-1)!\, R(t) &\geqq (2n-1)!\, y(t) + (t - \sigma)^{2n-1} \int_{T^*}^{\sigma} y_\tau(s) f(s, y_\tau(s)) ds\\ &= (2n-1)!\, y(t) + (t - \sigma)^{2n-1}[D^{2n-1}y(T^*) - D^{2n-1}y(\sigma)] .\end{aligned}$$

Multiplying this by $(t - \sigma)^{1-2n}$, keeping σ fixed and letting $t \to \infty$ through a sequence of points for which $(t - \sigma)^{1-2n}y(t)$ tends to its limit superior, we obtain

$$\begin{aligned}D^{2n-1}&y(T^*)\\ &\geqq \varlimsup_{t\to\infty} (2n-1)!\, (t - T^*)^{1-2n} y(t) + D^{2n-1}y(T^*) - D^{2n-1}y(\sigma)\end{aligned}$$

from which it follows that

$$\varlimsup_{t\to\infty} (2n-1)!\, (t - T^*)^{1-2n} y(t) \leqq D^{2n-1}y(\sigma) .$$

Since σ is arbitrary and $\lim_{t\to\infty} D^{2n-1}y(t)$ exists, it follows that

$$\varlimsup_{t\to\infty} (2n-1)!\, (t - T^*)^{1-2n} y(t) \leqq \lim_{t\to\infty} D^{2n-1}y(t) .$$

Combining this with (3.3) yields the desired result.

THEOREM 3.2. *Equation* (3.1) *has a positive solution* $y(t)$ *satisfying*

$$\lim_{t\to\infty} t^{1-2n}y(t) = k,\ 0 < k < \infty\ , \tag{3.4}$$

if, and only if, for some $C > 0$

$$\int^{\infty} t^{2n-1}f(t, Ct^{2n-1})dt < \infty\ . \tag{3.5}$$

Proof. Suppose (3.5) holds. A lower limit T_0 can be chosen sufficiently large so that

$$\int_{T_0}^{\infty} t^{2n-1}f(t, Ct^{2n-1})dt < (2n-1)! - 1/2\ . \tag{3.5$'$}$$

If $y(t) = y(t, T_0)$ is the solution of (3.1) defined by $D^k y(T_0) = 0$, $k = 0, \cdots, 2n-2$, $D^{2n-1}y(T_0) = (2n-1)!\,C$ and $y(t) = 0$ for $T_0 - T \leqq t \leqq T_1$, then $y(t, T_0)$ is positive on some open interval whose left endpoint is T_0. Let $t = T_1$ be the first zero of $y(t, T_0)$ in (T_0, ∞). By Taylor's theorem we have

$$\begin{aligned}(2n-1)!\,C(t-T_0)^{2n-1} &= (2n-1)!\,y(t, T_0)\\ &\quad + \int_{T_1}^{t}(t-s)^{2n-1}y_\tau(s)f(s, y_\tau(s))ds\ .\end{aligned} \tag{3.6}$$

Since $y(s) > 0$ for $T_0 - T \leqq s \leqq T_1$, $y_\tau(s) > 0$ for $T_0 - T \leqq s - \tau(s) \leqq T_1$, i.e., for $s \geqq T_0 - T + \tau(s)$. Hence $y_\tau(s) > 0$ for $s \geqq T_0$. A similar argument shows that $y_\tau(s) > 0$ for $s \leqq T_1$. Thus

$$y(t) = y(t, T_0) \leqq C(t-T_0)^{2n-1},\ T_0 \leqq t \leqq T_1\ . \tag{3.7}$$

Moreover, letting $t = T_1$ in (3.6) we have

$$\begin{aligned}(2n-1)!\,C(T_1-T_0)^{2n-1} &= \int_{T_0}^{T_1}(T_1-s)^{2n-1}y_\tau(s)f(s, y_\tau(s))ds\\ &\leqq (T_1-T_0)^{2n-1}\int_{T_0}^{T_1}y_\tau(s)f(s, y_\tau(s))ds\ .\end{aligned} \tag{3.8}$$

By condition (iii) and (3.7)

$$y_\tau(s)f(s, y_\tau(s)) \leqq Cs^{2n-1}f(s, Cs^{2n-1})\ ,$$

so that (3.8) yields

$$(2n-1)! \leqq \int_{T_0}^{T_1} s^{2n-1}f(s, Cs^{2n-1})ds\ ,$$

which contradicts (3.5′) and demonstrates the existence of a positive solution satisfying (3.4).

To prove necessity let $y(t)$ be a positive solution of (3.1) satisfying (3.4). It follows from Lemma 3.1 that

$$(3.9) \qquad \int_{T_1}^{\infty} y_\tau(s) f(s, y_\tau(s)) ds = D^{2n-1} y(T_1) - (2n - 1)!\, k\ .$$

(3.4) shows that for any $\varepsilon > 0$ there is a $T_* \geqq T_1$ such that $y(t) \geqq (k - \varepsilon)t^{2n-1}$ for all $t \geqq T_*$ so that $(k - \varepsilon)(t - T)^{2n-1} \leqq y_\tau(t)$ for $t > T^* = T_* + T$. Also, it follows from (ii) that

$$f(s, y_\tau(s)) \geqq f[s, (k - \varepsilon)(s - T)^{2n-1}]\ .$$

Since (3.9) is valid with T_1 replaced by T^*,

$$\begin{aligned} &D^{2n-1} y(T^*) - (2n - 1)!\, k \\ &\quad \geqq (k - \varepsilon) \int_{T^*}^{\infty} (s - T)^{2n-1} f[s, (k - \varepsilon)(s - T)^{2n-1}] ds\ . \end{aligned}$$

For so $s \geqq 2T$,

$$\begin{aligned} &\int^{\infty} s^{2n-1} f(s, Cs^{2n-1}) ds < 2^{2n-1} \\ &\quad \times \int^{\infty} (s - T)^{2n-1} f[s, 2^{2n-1} C (s - T)^{2n-1}] ds \leqq N_1\ , \end{aligned}$$

where $N_1 = 2^{2n-1}(k - \varepsilon)^{-1}[D^{2n-1} y(T^*) - (2n - 1)!k]$ and $C = (k - \varepsilon)/2^{2n-1}$. The conclusion follows.

For the linear equation (2.5) with $\lambda = 0$, $n = 2$, $r(t) \equiv 1$, and $\tau(t) \equiv 0$, this result reduces to that of Leighton and Nehari [6]. As in the discussion following Corollary 2.8, we observe that when $\lambda = 0$ and $r(t) \equiv 1$, (2.5) is not a special case of (3.1). However, Theorem 3.2 remains valid and the proof given above may be easily modified to yield the result. The details of this are omitted for brevity. For the nonlinear equation (3.1) with $\tau(t) \equiv 0$ and $n = 2$, see also Wong [10]. The case of (3.1) with $\tau(t) \not\equiv 0$ and $n = 2$ is treated by Terry and Wong [8].

4. In this section we shall apply the direct method of Lyapunov to obtain nonoscillation criteria for (1.1). For convenience, we first introduce some notation which will be used in the section. Let $R_a = [a, \infty)$, $a \geqq 0$, $R^* = (0, \infty)$, $R_* = (-\infty, 0)$ and $R^1 = R = (-\infty, \infty)$. We shall abbreviate the cartesian products of these intervals as follows:

$$\begin{aligned} R^{p*} &= R^* \times R^* \times \cdots \times R^*,\ p \text{ times} \\ R_{p*} &= R_* \times R_* \times \cdots \times R_*,\ p \text{ times} \\ R_*{}^* &= R_* \times R^*;\ R^*{}_* = R^* \times R_*\ . \end{aligned}$$

Other products may be defined in terms of these, e.g.,

$$R_a^{p*} = R_a \times R^{p*}\ .$$

To begin with we shall consider an arbitrary $2n$th order equation of the form

$$(4.1)\qquad D^n[r(t)D^n y(t)] + F(t, \sigma(t), \sigma_\tau(t)) = 0 ,$$

where

$$\sigma(t) = (y_0(t), \cdots, y_{2n-1}(t)) \text{ and } \sigma_\tau(t) = \sigma(t - \tau(t)) .$$

A real-valued function $V(t, \sigma)$ will be called a Lyapunov function if $V(t, \sigma)$ is continuous in its domain and locally Lipschitzian in σ. Following Yoshizawa [11, 12], we define the trajectory derivative $\dot{V}_{(1)}$ of V along solutions $\sigma(t)$ of (4.1) by

$$(4.2)\qquad \dot{V}_{(1)}(t, \sigma) = \overline{\lim_{h\downarrow 0}}\frac{1}{h}\,[V(t + h, \sigma(t + h)) - V(t, \sigma(t))] .$$

The first result of this section is an extension of Yoshizawa's theorem to (4.1) and the proof is based on his.

THEOREM 4.1. *Let V be a real-valued continuous function defined on $R_a^{(2j+2)*} \times (R_*{}^*)^{n-j-1}$ for some $a > 0$ such that*

(i) *V tends to infinity uniformly for $\sigma \in R^{(2j+2)*} \times (R_*{}^*)^{n-j-1}$ as t tends to infinity; and*

(ii) *for each solution $y(t)$ of* (4.1) *such that $(t, \sigma(t)) \in R_b^{(2j+2)*} \times (R_*{}^*)^{n-j-1}$ for some $b \geqq a$, $\dot{V}_{(1)}(t, \sigma(t)) \leqq 0$.*

Then (4.1) *has no solution of type B_j.*

Proof. Let $y(t)$ be a solution of *type* B_j. There is a positive a for which $\sigma(t) \in R_a^{(2j+2)*} \times (R_*{}^*)^{n-j-1}$ if $t \geqq a$. By (ii) for t sufficiently large, i.e., for $t \geqq b \geqq a$,

$$V(t, \sigma(t)) \leqq V(b, \sigma(b)) .$$

On the other hand, (i) implies that there is a $c \geqq b$ for which

$$V(t, \sigma(t)) > V(b, \sigma(b))$$

if $t > c$, which is a contradiction.

As in other applications of the direct method of Lyapunov, the key to applying this result is the construction of suitable Lyapunov functions V having the requisite properties. In this case of (1.1) Theorem 2.5 may be regarded as a special case of this result, for if $y(t)$ is a B_j-solution there are constants $C > 0$ and $K > 0$ such that for sufficiently large t, $y_\tau(t) \geqq NKt^{2j}y_{2j}(t) \geqq Ct^{2j}$. It follows that by taking a suitable T^* the function

$$V(t, \sigma(t)) = y_{2n-1}(t)[y_{2j}(t)]^{-1} + NK\int_{T^*}^{t} s^{2j}f(s, Cs^{2j})ds$$

will be a Lyapunov function satisfying the conditions of Theorem 4.1 provided (2.4) holds.

Another application of Theorem 4.1 to equation (4.1) is given by the following result.

THEOREM 4.2. *Suppose there exist functions* $p(t)$ *and* $\rho(u)$ *of class* $C(R_a)$ *and* $C^1(R)$ *respectively such that*

(i) $\rho'(u) \geqq 0$; $u\rho(u) > 0$, $u \neq 0$;

(ii) $F(t, \sigma(t)) \geqq p(t)\rho(y)$, $y \geqq 0$;

(iii) $\int^{\infty} p(t)dt = +\infty$.

Then (4.1) *has no solutions of type* B_j.

Proof. Suppose $y(t)$ is a solution of (4.1) of type B_j. Then $(t, \sigma(t)) \in R_b^{(2j+2)*} \times (R_*{}^*)^{n-j-1}$ for some $b \geqq a$. Let $V(t, \sigma)$ be the function defined by

$$V(t, \sigma(t)) = y_{2n-1}(t)[\rho(y(t))]^{-1} + \int_b^t p(s)ds .$$

In view of hypotheses (i), (ii), and (iii), V will clearly satisfy condition (i) of Theorem 4.1. Moreover, since both $y'(t)$ and $y_{2n-1}(t)$ are positive for large t, a simple calculation with the help of (i) and (ii) shows that

$$\begin{aligned}\dot{V}_{(1)} &= \frac{Dy_{2n-1}(t)}{\rho[y(t)]} + p(t) - \frac{\rho'[y(t)]}{\rho^2[y(t)]} y_{2n-1}(t)y'(t) \\ &\leqq p(t) - \frac{F(t, \sigma(t))}{\rho[y(t)]} \leqq 0 .\end{aligned}$$

Theorem 4.1 is thus applicable and we conclude that (4.1) cannot have any solution of type B_j.

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