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# GLOBAL REFLECTION FOR A CLASS OF SIMPLE CLOSED CURVES

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**Global reflection is considered for a class of closed Jordan curves $\Gamma$: $[x(\theta), y(\theta)]$, $0 \leqq \theta < 2\pi$ where $x(\theta)$ and $y(\theta)$ are trigonometric polynomials. Every curve of this form is algebraic and global reflection across it reduces to investigating an algebraic function and its critical points. The reflection function is picked to be that solution of the algebraic equation that maps $\Gamma$: $[x(\theta), y(\theta)]$ pointwise into $[x(\theta), -y(\theta)]$. This function is defined and analytic except on a finite set of points inside $\Gamma$, and at each of these points it is continuous.**

**1. Introduction.** Reflection across an analytic arc which is a generalization of inversion in a circle and reflection across a straight line, goes back to Schwarz. Because of the current interest, see e.g. [1], [3], [4], [5], in reflection of solutions of plane elliptic differential equations across analytic arcs, it seems appropriate to analyze the global reflection across a fairly general class of closed Jordan curves.

We shall investigate the class of Jordan rectifiable curves $\Gamma$ of the form

$$x(\theta) = \sum_{k=0}^{n} a_k \cos k\theta + b_k \sin k\theta \tag{1.1}$$

$$0 \leqq \theta < 2\pi,\ n \geqq m$$

$$y(\theta) = \sum_{k=0}^{m} \alpha_k \cos k\theta + \beta_k \sin k\theta \tag{1.2}$$

with $x'^2(\theta) + y'^2(\theta) \neq 0$, $(a_n, b_n) \neq (0, 0) \neq (\alpha_m, \beta_m)$ and if $m = n$ then either $\alpha_n^2 + \beta_n^2 \neq a_n^2 + b_n^2$ or $\alpha_n a_n + \beta_n b_n \neq 0$.

The investigation will be reduced to analyzing a certain algebraic equation $M[z, \zeta] = 0$ arising from (1.1) and (1.2) (see (2.5)-(2.8)).

Let $R$ be a simply-connected region bounded by a curve $\Gamma$ of the form (1.1) and (1.2). Let $S$ be the finite set of points made up of zeros of the resultant polynomials

$$P(z) = R[M[z, \zeta], M_\zeta[z, \zeta]] = 0$$

and

$$Q(z) = R[M[z, \zeta], M_z[z, \zeta]] = 0\,.$$

Let $L_i$ be a rectifiable Jordan arc in $R$ containing $\{e_1, \cdots, e_s\} = R \cap S$. Then for one of the functions

$$\zeta = G(z)$$

defined by $M[z, \zeta] = 0$ we have

1 $G(z)$ is defined and analytic on $R - \{e_1, \cdots, e_s\}$,

2 $G(z)$ is single-valued on $R\backslash L_i \cup \Gamma$,

3 $G'(z) \neq 0$ on $R\backslash L_i \cup \Gamma$.

4 For $z$ on $\Gamma$, $\bar{z} = G(z)$.

5 $\overline{G[R\backslash L_i]} \cap R\backslash L_i = \varnothing$.

6 About each $e_j$ $(1 \leqq j \leqq s)$ we have either

(i) $G(z)$ is defined, single-valued and analytic on a neighborhood of $e_j$ with

$$G(z) = (z - e_j)^2 G^*(z) ,$$

$G^*(z)$ analytic and thus $G'(e_j) = 0$ or

(ii) on some neighborhood of $e_j$

$$G(z) = \sum_{k=0}^{\infty} f_k[\sqrt[p]{z - e_j}]^k , \qquad 1 \leqq p \leqq 2n \qquad (n \text{ of } (1.1)) ,$$

$f_k$ constant, or

(iii) $G(z)$ is defined, single-valued and analytic on a neighborhood of $e_j$ and $G'(e_j) \neq 0$.

In the event $M[z, \zeta]$ is irreducible 6(iii) is excluded. We shall denote for $z$ in $R\backslash L_i$

$$\hat{z} = \overline{G(z)} .$$

$G(z)$ is the reflection function and $\hat{z}$ is the reflection of $z$ across $\Gamma$.

7 $G(z)$ can be extended to be defined and analytic and single-valued on

$$\{R\backslash L_i\} \cup \Gamma \cup \{\overline{G(R\backslash L_i)}\} = \{R\backslash L_i\} \cup \Gamma \cup \{\widehat{R\backslash L_i}\}$$

with

$$\hat{\hat{z}} = z \quad \text{for } z \text{ in } \{R\backslash L_i\} \cup \Gamma \cup \{\widehat{R\backslash L_i}\} .$$

It is the proof of 6(ii) that gives the most difficulty.

**2. Geometrical reflection.** To begin our investigation of reflection across a rectifiable Jordan curve $\Gamma$ of the form

$$x(\theta) = \sum_{k=0}^{n} a_k \cos k\theta + b_k \sin k\theta \tag{2.1}$$

$$0 \leqq \theta < 2\pi,\ n \geqq m$$

$$y(\theta) = \sum_{k=0}^{m} \alpha_k \cos k\theta + \beta_k \sin k\theta \tag{2.2}$$

we let

$$t = e^{i\theta} = \cos\theta + i\sin\theta$$

and express (2.1) and (2.2) in terms of $t$ and $\bar{t}$, then (2.1) and (2.2) become:

$$2x = 2a_0 + \bar{c}_1 t + c_1 \bar{t} + \bar{c}_2 t^2 + c_2 \bar{t}^2 + \cdots + \bar{c}_n t^n + c_n \bar{t}^n \tag{2.3}$$

and

$$2y = 2\alpha_0 + \bar{\gamma}_1 t + \gamma_1 \bar{t} + \bar{\gamma}_2 t^2 + \gamma_2 \bar{t}^2 + \cdots + \bar{\gamma}_m t^m + \gamma_m \bar{t}^m \tag{2.4}$$

with

$$c_k = a_k + ib_k, \qquad \gamma_k = \alpha_k + i\beta_k .$$

If we multiply (2.3) by $\bar{t}^n$ and (2.4) by $\bar{t}^m$ we see that (2.3) and (2.4) are equivalent respectively to:

$$f(t) \equiv \bar{c}_n + \bar{c}_{n-1}\bar{t} + \cdots + \bar{c}_1 \bar{t}^{n-1} + 2(a_0 - x)\bar{t}^n + c_1 \bar{t}^{n+1} + \cdots + c_n \bar{t}^{2n} = 0$$

and

$$g(t) \equiv \bar{\gamma}_m + \bar{\gamma}_{m-1}\bar{t} + \cdots + \bar{\gamma}_1 \bar{t}^{m-1} + 2(\alpha_0 - y)\bar{t}^m + \gamma_1 \bar{t}^{m+1} + \cdots + \gamma_m \bar{t}^{2m} = 0 .$$

Thus the curve $\Gamma$ is given by exactly those $t$ for which $f(t) = 0$ and $g(t) = 0$, i.e., by the common roots of $f(t)$ and $g(t)$. But a necessary and sufficient condition for $f(t)$ and $g(t)$ to have common roots is that Sylvester's determinant $D(f, g)$ of order $(2n + 2m) \times (2n + 2m)$ vanish. If we let

$$\alpha(x) = 2(a_0 - x) , \qquad \beta(y) = 2(\alpha_0 - y)$$

then

$$D(f, g) = \det \left[\begin{array}{l} \bar{c}_n\ \bar{c}_{n-1} \cdots \quad \bar{c}_1\ \alpha\ c_1 \cdots \quad c_{n-1}\ c_n\ 0\ 0\ 0 \cdots 0\ 0 \\ 0\ \bar{c}_n\ \bar{c}_{n-1} \cdots \quad \bar{c}_1\ \alpha\ c_1 \cdots \quad c_{n-1}\ c_n\ 0\ 0 \cdots 0\ 0 \\ 0\ 0\ \bar{c}_n\ \bar{c}_{n-1} \cdots \quad \bar{c}_1\ \alpha\ c_1 \cdots \quad c_{n-1}\ c_n\ 0 \cdots 0\ 0 \\ 0\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad c_{n-1}\ c_n \\ \bar{\gamma}_m\ \bar{\gamma}_{m-1} \cdots \quad \bar{\gamma}_1\ \beta\ \gamma_1 \cdots \quad \gamma_{m-1}\ \gamma_m\ 0\ 0 \cdots 0\ 0 \\ \quad \bar{\gamma}_m\ \bar{\gamma}_{m-1} \cdots \quad \bar{\gamma}_1\ \beta\ \gamma_1 \cdots \quad \gamma_{m-1}\ \gamma_m\ 0 \cdots 0\ 0 \\ \qquad \bar{\gamma}_m\ \bar{\gamma}_{m-1} \cdots \quad \bar{\gamma}_1\ \beta\ \gamma_1 \cdots \quad \gamma_{m-1}\ \gamma_m \cdots 0\ 0 \\ 0\ \ 0\ \ 0 \cdots \bar{\gamma}_m\ \bar{\gamma}_{m-1} \quad \bar{\gamma}_1\ \beta\ \gamma_1 \cdots \quad \gamma_{m-1}\ \gamma_m \end{array}\right] \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} 2m \\ \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} 2n \end{array} \tag{2.5}$$

provided $c_n \neq 0 \neq \gamma_m$. Since $f$ and $g$ are fixed, we define

$$\Delta[\alpha(x), \beta(y)] = D(f, g) .$$

Then $\Gamma$ is given by the algebraic equation:

$$\Delta[\alpha(x), \beta(y)] = 0 , \qquad \alpha = 2(a_0 - x) , \qquad \beta = 2(\alpha_0 - y) .$$

We now consider the algebraic equation of order $2n$ in $\zeta$:

$$
\begin{aligned}
M[z, \zeta] &= \Delta\left[\alpha\left(\frac{z+\zeta}{2}\right), \beta\left(\frac{z-\zeta}{2i}\right)\right] \\
&= g_{2n}(z)\zeta^{2n} + g_{2n-1}(z)\zeta^{2n-1} + \cdots + g_2(z)\zeta^2 + g_1(z)\zeta + g_0(z) = 0
\end{aligned}
$$

and investigate the Riemann surface that this equation defines over the $z = x + iy$ plane.

First we prove:

LEMMA 2.1.

( i ) *If* $n > m$ *and* $c_n \neq 0 \neq \gamma_m$, *then*

$$g_{2n}(z) = \pm\frac{1}{(2n)!}\,|\,c_n\,|^{2m} = \text{constant} \neq 0\ .$$

(ii) *If* $n = m$ *and* $c_n \neq 0 \neq \gamma_n$ *and* $c_n + i\gamma_n \neq 0 \neq \bar{c}_n + i\bar{\gamma}_n$, *then*

$$g_{2n}(z) = \pm\frac{1}{(2n)!}(c_n + i\gamma_n)^n(\bar{c}_n + i\bar{\gamma}_n)^n = \text{constant} \neq 0\ .$$

*Proof.* First we note that

$$g_{2n}(z) = \frac{1}{(2n)!}\,\frac{\partial^{2n}}{\partial\zeta^{2n}}M[z, \zeta]_{\zeta=0}\ .$$

In order to differentiate $M[z, \zeta]$, it will be convenient to introduce the notation:

$$
C = C^{m\times m} = \begin{bmatrix} c_n & c_{n-1} & c_{n-2} & \cdots & c_{n-m} \\ 0 & c_n & c_{n-1} & \cdots & c_{n-m+1} \\ 0 & 0 & c_n & \cdots & c_{n-m+2} \\ & & & & \\ 0 & 0 & 0 & \cdots & c_n \end{bmatrix}, \quad \Gamma = \Gamma^{m\times m} = \begin{bmatrix} \gamma_m & \gamma_{m-1} & \gamma_{m-2} & \cdots & \gamma_1 \\ 0 & \gamma_m & \gamma_{m-1} & \cdots & \gamma_2 \\ 0 & 0 & \gamma_m & \cdots & \gamma_3 \\ & & & & \\ 0 & 0 & 0 & \cdots & \gamma_m \end{bmatrix}.
$$

Then

$$
M[z, \zeta] = \begin{array}{r|c|c|c|c|c|l}
 & m & n-m & 2m & n-m & m & \\ \hline
m & \bar{C} & A_1 & \alpha \quad \alpha & A_2 & 0 & \\ \hline
m & 0 & B_1 & \ddots \quad \alpha & B_2 & C^{\mathsf{T}} & \\ \hline
m & \bar{\Gamma} & \multicolumn{3}{c|}{\beta \quad \beta \quad \ddots} & 0 & 2n-m \\ \hline
2n-m & 0 & \multicolumn{3}{c|}{\ddots \quad \beta} & \Gamma^{\mathsf{T}} & m \\ \hline
 & m & \multicolumn{3}{c|}{2n} & m &
\end{array}
$$

$$\alpha = 2\alpha_0 - (z+\zeta)$$

$$\beta = 2\alpha_0 + i(z-\zeta)$$

where, as indicated, $A_1$, $B_1$, $A_2$, $B_2$ are matrices of size $n - m \times m$. Thus

$$\frac{\partial^{2n}}{\partial \zeta^{2n}} M[z, \zeta] = \left|\begin{array}{c|c|c|c|c} \bar{C} & 0 & \multicolumn{1}{c|}{} & 0 & 0 \\ \cline{1-2}\cline{4-5} 0 & 0 & \multicolumn{1}{c|}{-I} & 0 & C^{\mathrm{T}} \\ \hline \bar{\Gamma} & \multicolumn{3}{c|}{} & 0 \\ \cline{1-1} 0 & \multicolumn{3}{c|}{-iI} & \Gamma^{\mathrm{T}} \end{array}\right|$$

$$\frac{\partial^{2n}}{\partial \zeta^{2n}} M[z, \zeta] = \left|\begin{array}{c|c|c|c|c} \bar{C} & 0 & \multicolumn{1}{c|}{} & 0 & 0 \\ \cline{1-2}\cline{4-5} 0 & 0 & \multicolumn{1}{c|}{I} & 0 & C^{\mathrm{T}} \\ \hline \bar{\Gamma} & \multicolumn{3}{c|}{} & 0 \\ \cline{1-1} 0 & \multicolumn{3}{c|}{iI} & \Gamma^{\mathrm{T}} \end{array}\right| .$$

Next we perform the following set of operations

$(\mathrm{i})_1$ multiply the $m + 1$ column by $i\bar{\gamma}_m$ and add it to the first column

$(\mathrm{ii})_1$ multiply the $m + 1$ column by $i\bar{\gamma}_{m-1}$ and add it to the second column

$(\mathrm{iii})_1$ multiply the $m + 1$ column by $i\bar{\gamma}_{m-2}$ and add it to the third column etc. to $m$

$(\mathrm{i})_2$ multiply the $m + 2$ column by $i\bar{\gamma}_m$ and add it to the second column

$(\mathrm{ii})_2$ multiply the $m + 2$ column by $i\bar{\gamma}_{m-1}$ and add it to the third column etc. to $m - 1$

$\vdots$

$(\mathrm{i})_m$ multiply the $2m$ column by $i\bar{\gamma}_m$ and add it to the $m$th column. This yields the following result:

$$\frac{\partial^{2n}}{\partial \zeta^{2n}} M[z, \zeta] = \left|\begin{array}{c|c|c|c|c} \tilde{C} & 0 & \multicolumn{1}{c|}{} & 0 & 0 \\ \cline{1-2}\cline{4-5} 0 & 0 & \multicolumn{1}{c|}{I} & 0 & C^{\mathrm{T}} \\ \hline 0 & \multicolumn{3}{c|}{} & 0 \\ \cline{1-1} 0 & \multicolumn{3}{c|}{iI} & \Gamma^{\mathrm{T}} \end{array}\right|$$

where if $n - m \leqq m$, i.e., $n \leqq 2m$ then

$$\tilde{C} = \bar{C} + i \begin{array}{c} \begin{array}{cc} \overbrace{\qquad}^{n-m} & \qquad\qquad\qquad m \times m \end{array} \\ \left(\begin{array}{c|cccccc} & \bar{\gamma}_m & \bar{\gamma}_{m-1} & \bar{\gamma}_{m-2} & \cdots & \\ & 0 & \bar{\gamma}_m & \bar{\gamma}_{m-1} & \cdots & \\ & 0 & 0 & \bar{\gamma}_m & \cdots & \\ 0 & & & & & \\ & 0 & 0 & 0 & \cdots & 0 \\ & 0 & 0 & 0 & \cdots & 0 \end{array}\right) \end{array}$$

and if $n - m > m$

$$\tilde{C} = \bar{C} \,.$$

Next we perform the following set of operations

$(\mathrm{i})_1$ multiply column $2n + m$ by $i\gamma_m$ and add it to column $2n + 2m$

$(\mathrm{ii})_1$ multiply column $2n + m$ by $i\gamma_{m-1}$ and add it to column $2n + 2m - 1$

$(\mathrm{iii})_1$ multiply column $2n + m$ by $i\gamma_{m-2}$ and add it to column $2n + 2m - 2$ etc. to $m$

$(\mathrm{i})_2$ multiply column $2n + 2m - 1$ by $i\gamma_m$ and add it to column $2n + 2m - 1$

$(\mathrm{ii})_2$ multiply column $2n + m - 1$ by $i\gamma_{m-1}$ and add it to column $2n + 2m - 2$ etc. to $m - 1$

$\vdots$

$(\mathrm{i})_m$ multiply column $2n + 1$ by $i\gamma_m$ and add it to column $2n + m + 1$. This yields the following result:

$$\frac{\partial^{2n}}{\partial \zeta^{2n}} M[z, \zeta] = \begin{array}{|c|c|c|c|c|} \hline \tilde{C} & 0 & & 0 & 0 \\ \cline{1-2}\cline{4-5} 0 & 0 & \smash{I} & 0 & C^* \\ \hline 0 & & & & 0 \\ \cline{1-1}\cline{5-5} 0 & & iI & & 0 \\ \hline \end{array}$$

where if $n - m \leqq m$, i.e., $n \leqq 2m$ then

$$C^* = C^{\mathsf{T}} + i \begin{array}{c} \begin{array}{cc} \qquad\qquad\qquad\qquad & \overbrace{\qquad}^{n-m} \;\; m \times m \end{array} \\ \left(\begin{array}{cccccc|c} 0 & 0 & 0 & \cdot\;\cdot\;\cdot & & 0 & \\ 0 & 0 & 0 & \cdot\;\cdot\;\cdot & & 0 & \\ & & & & & & 0 \\ \cdot\;\cdot\;\cdot & \gamma_m & 0 & 0 & & & \\ \cdot\;\cdot\;\cdot & \gamma_{m-1} & \gamma_m & 0 & & & \\ \cdot\;\cdot\;\cdot & \gamma_{m-1} & \gamma_{m-1} & \gamma_m & & & \end{array}\right) \end{array}$$

and if $n - m > m$

$$C^* = C^{\top} .$$

The next set of operations is as follows:

( i ) multiply row $n + m + 1$ by $i$ and add to row 1

(ii) multiply row $n + m + 2$ by $i$ and add to row 2

(iii) multiply row $n + m + 3$ by $i$ and add to row 3

($2m$) multiply row $n + 2m$ by $i$ and add to row $2m$.

This yields:

$$\frac{\partial^{2n}}{\partial\zeta^{2n}}M[z, \zeta] = \left|\begin{array}{c|c|c|c|c} \tilde{C} & 0 & & 0 & 0 \\ \cline{1-2}\cline{4-5} 0 & 0 & 0 & 0 & C^* \\ \hline 0 & & & & 0 \\ \cline{1-1}\cline{5-5} 0 & & iI & & 0 \end{array}\right| .$$

Since $c_n \neq 0$ and if $n = m$, $\bar{c}_n + i\gamma_n \neq 0 \neq c_n + i\gamma_n$ the above determinant, with appropriate column operations, is also given by:

$$\frac{\partial^{2n}}{\partial\zeta^{2n}}M[z, \zeta] = (i)^{2n}\left|\begin{array}{c|c|c|c|c} \tilde{D} & 0 & & 0 & 0 \\ \cline{1-2}\cline{4-5} 0 & 0 & 0 & 0 & D^* \\ \hline 0 & & & & 0 \\ \cline{1-1}\cline{5-5} 0 & & I & & 0 \end{array}\right|$$

where

$$\tilde{D} = \begin{cases} \bar{c}_n I^{m\times m} & \text{if} \quad n > m \\ (\bar{c}_n + i\gamma_n) I^{n\times n} & \text{if} \quad n = m \end{cases}$$

$$D^* = \begin{cases} c_n I^{m\times m} & \text{if} \quad n > m \\ (c_n + i\gamma_n) I^{n\times n} & \text{if} \quad n = m . \end{cases}$$

Thus

$$\begin{aligned} \frac{\partial^{2n}}{\partial\zeta^{2n}}M[z, \zeta] &= \pm(\bar{c}_n)^m (c_n)^m \quad \text{if} \quad n > m \\ &= \pm(\bar{c}_n + i\gamma_n)^n (c_n + i\gamma_n)^n \quad \text{if} \quad n = m \end{aligned}$$

which proves the lemma.

If we write

$$M[z, \zeta] = \Delta\left[\alpha\left(\frac{z+\zeta}{2}\right), \beta\left(\frac{z-\zeta}{2i}\right)\right]$$
$$= h_{2n}(\zeta)z^{2n} + h_{2n-1}(\zeta)z^{2n-1} + \cdots + h_1(\zeta)z + h_0(\zeta) = 0\,,$$

then we also have:

LEMMA 2.2.
(i) *If* $n > m$ *and* $c_n \neq 0 \neq \gamma_m$ *then*

$$h_{2n}(\zeta) = \pm\frac{1}{(2n)!}\,|\,c_n\,|^{2m} = \text{constant} \neq 0\,.$$

(ii) *If* $n = m$ *and* $c_n \neq 0 \neq \gamma_n$ *and* $c_n - i\gamma \neq 0 \neq \bar{c}_n - i\bar{\gamma}_n$ *then*

$$h_{2n}(\zeta) = \pm\frac{1}{(2n)!}(c_n - i\gamma_n)^n(\bar{c}_n - i\bar{\gamma}_n)^n\,.$$

*Proof.* As in Lemma 2.1

$$h_{2n}(\zeta) = \frac{1}{(2n)!}\frac{\partial^{2n}}{\partial z^{2n}}M[z, \zeta]_{z=0}$$

and the proof proceeds as in Lemma 2.1.

We now recall some well-known facts from the theory of algebraic functions and Riemann surfaces, see e.g., [2].

We restrict ourselves to the case where $M[z, \zeta]$ is irreducible.

A point $z_0$ is called a critical point of $M[z, \zeta]$ if either
(i) $g_{2n}(z_0) = 0$ or
(ii) $M[z_0, \zeta] = 0$ has multiple roots.

THEOREM (i). *If* $z_0$ *is a point such that* $g_{2n}(z_0) \neq 0$ *and* $\zeta_0$ *is a root of* $M[z_0, \zeta]$ *of multiplicity* $l$, $1 \leqq l \leqq 2n$, *then there exists an* $\varepsilon > 0$ *and* $\delta(\varepsilon) > 0$ *such that if* $z_1 \neq z_0$ *lies in the disc* $D(z_0, \delta)$ *of radius* $\delta$ *about* $z_0$ *then* $M(z_1, w) = 0$ *has exactly* $l$ *distinct roots in* $D(\zeta_0, \varepsilon)$. [1], p. 122.

If $z_0$ is a point such that $g_{2n}(z_0) \neq 0$ and $M[z_0, \zeta]$ has no multiple roots, $M[z_0, \zeta_0] = 0$, then from the above theorem $l = 1$ and for every $z = z_1$ in $D(z_0, \delta)$, there is exactly one root of $M[z_1, \zeta] = 0$. Thus on $D(z_0, \delta)$ $M[z, \zeta] = 0$ defines a single-valued continuous function $\zeta = g_1(z)$ for which $\zeta_0 = g_1(z_0)$.

THEOREM (ii). $g_1(z)$ *is an analytic function of* $z$ *on* $D(z_0, \delta)$.

In the case $g_{2n}(z) = \text{constant} \neq 0$ then there are at most a finite number of critical points.

Let $e_1, e_2, \cdots, e_r$ be the critical points and join these and $\infty$ by $L_0$, any nonintersecting smooth arc and half line. Cut the plane along $L_0$. Then on the cut plane, $M[z, \zeta] = 0$ defines $2n$ single-valued analytic functions $G_1(z), \cdots, G_{2n}(z)$.

THEOREM (iii). *About each critical point $e$ we have the following expansion in a neighborhood of $e$*

$$G(z) = \sum_{m=-\infty}^{\infty} f_m(\sqrt[p]{z-e})^m$$

*where*

(1) $1 \leqq p \leqq 2n$

(2) *there are at most a finite number of negative powers*

(3) *if $g_{2n}(e) \neq 0$ there are no negative powers.*

We shall also need

LEMMA 2.3. *Let*

(1) $x'^2(\theta) + y'^2(\theta) \neq 0$

(2) $(a_n, b_n) \neq 0 \neq (\alpha_n, \beta_n)$

(3) *either* $\alpha_n^2 + \beta_n^2 \neq b_n^2 + a_n^2$ *or* $\alpha_n a_n + b_n\beta_n \neq 0$

*then for no point $z$ of $\Gamma$* (i) *is $\zeta = \bar{z}$ a multiple root of $M[z, \zeta]$*; (ii) *is $g_{2n}(z) = 0$.*

*Proof.* Since

$$(\gamma_n - ic_n) = (\alpha_n + i\beta_n) - i(a_n + ib_n) = \alpha_n + b_n + i(\beta_n - a_n)$$
$$(\bar{\gamma}_n - i\bar{c}_n) = (\alpha_n - i\beta_n) - i(a_n - ib_n) = \alpha_n - b_n - i(\beta_n + a_n)$$

then

$$(\gamma_n - ic_n)(\bar{\gamma}_n - i\bar{c}_n) = \alpha_n^2 + \beta_n^2 - (a_n^2 + b_n^2) - 2i[\alpha_n a_n + b_n\beta_n] \neq 0 \text{ by (2)}\,.$$

Thus we see by Lemma 2.1 that $g_{2n}(z_0)$ is never zero. For $z$ on $\Gamma$, i.e., for $z = x(\theta) + iy(\theta)$, $0 \leqq \theta < 2\pi$, we have

$$M[x(\theta) + iy(\theta), x(\theta) - iy(\theta)] \equiv 0\,, \qquad 0 \leqq \theta < 2\pi\,.$$

Thus if we differentiate with respect to $\theta$, we see since (1) holds, that

$$\frac{(x' + iy')^2}{x'^2 + y'^2} = \frac{M_\zeta[z, \bar{z}]}{M_z[z, \bar{z}]} \neq 0 \quad \text{on} \quad \Gamma\,.$$

Thus $M_\zeta[z_0, \zeta] \neq 0$ for $\zeta$ on $\Gamma$, i.e., for $\zeta = \bar{z}_0$ and the proof is complete.

LEMMA 2.4. *Let*

(1) $(a_n, b_n) \neq 0 \neq (\alpha_n, \beta_n)$ *and*

(2) *either* $\alpha_n^2 + \beta_n^2 \neq a_n^2 + b_n^2$ *or* $\alpha_n a_n + \beta_n b_n \neq 0$ *then there are at most a finite number of* $z$ *for which*

$$M[z, \zeta] = 0 \quad and \quad M_z[z, \zeta] = 0 .$$

*Proof.* $z$ is a point at which the result of the lemma holds if and only if for $\zeta$ fixed

$$M_1[z] = M[z, \zeta]$$

has a multiple root. Thus

$$\begin{aligned} M_1[z] &= h_{2n}(\zeta)z^{2n} + h_{2n-1}(\zeta)z^{2n-1} + \cdots + h_0(\zeta) \\ M_1'[z] &= 2nh_{2n}(\zeta)z^{2n-1} + \cdots + h_1(\zeta) \end{aligned}$$

and from Lemma 2.2 and the assumption of the lemma

$$h_{2n}(\zeta) = \text{constant} \neq 0 .$$

But a necessary and sufficient condition for $M_1[z]$ to have a multiple root is that the resultant

$$R[M_1, M_1'] = 0 .$$

As this is a polynomial in $\zeta$, the conclusion of the lemma follows.

LEMMA 2.5. *Let the hypotheses of Lemma* 2.3 *hold, then for no point* $z$ *of* $\Gamma$ *do we have for* $\zeta = \bar{z}$

$$M[z, \zeta] = 0 \quad and \quad M_z[z, \zeta] = 0 .$$

*Proof.* From the proof of Lemma 2.3

$$\frac{M_z[z, \bar{z}]}{M_\zeta[z, \bar{z}]} = \frac{(x' - iy')^2}{x'^2 + y'^2} \neq 0$$

which is the conclusion.

We shall assume that for $\Gamma$ we have

(1) $x'^2(\theta) + y'^2(\theta) \neq 0$

(2) $(a_n, b_n) \neq 0 \neq (\alpha_n, \beta_n)$

(3) either $\alpha_n^2 + \beta_n^2 \neq (b_n^2 + a_n^2)$ or $\alpha_n a_n + b_n\beta_n \neq 0$.

Assume, moreover, that $M[z, \zeta]$ is irreducible. Let $e_1, e_2, \cdots, e_r$ be the set of critical points of $M[z, \zeta] = 0$ and let $e_{r+1}, e_{r+2}, \cdots, e_{s_0}$ (by Lemma 2.4) be the set of $z$ for which

$$M[e_j, \zeta] = 0 \quad \text{and} \quad M_z[e_j, \zeta] = 0$$

and let

$$S = \{e_j : 1 \leqq j \leqq s_0\} .$$

Let $z_0$ be a point of $\Gamma$ and $M[z_0, \bar{z}_0] = 0$. By Lemma 2.3, $\zeta = \bar{z}_0$ is not a multiple root of $M[z_0, \zeta]$ and thus $M[z_0, \zeta] = 0$ defines a single-valued function of $z$, $\zeta = G(z)$ in some neighborhood $N_0$ of $z_0$ with $G(z_0) = \bar{z}_0$. Moreover, for each point $z$ on $\Gamma \cap N_0$ we have

$$\bar{z} = G(z)$$

by Lemma 2.3. Analytically continuing $G(z)$ around $\Gamma$ we return to $G(z)$. Since if we arrive at $G_1(z)$ where $G(z)$ and $G_1(z)$ are defined on a common neighborhood of $z_0$, then for $z$ on $\Gamma$, $z(\theta) = x(\theta) + iy(\theta)$ and $\zeta(\theta) = \overline{z(\theta)}$ is a periodic function and thus

$$G(z) = G_1(z) \quad \text{on} \quad \Gamma .$$

Therefore, they agree on the common neighborhood. From this it follows that $G(z)$ is single-valued and analytic on a neighborhood of $\Gamma$.

Let $S_i = \{e_1, \cdots, e_s\}$ be that subset of $S$ (renumber if necessary) that is contained in the interior of $\Gamma$ and let $S_e = S \backslash S_i$, then $G(z)$ is analytic on $R \backslash S_i$. Moreover, if we join each $e_j$ of $S_i$ by a Jordan arc $L_i$ then $G(z)$ is analytic and single-valued on

$$R_0 = R \backslash L_i \cup \quad \text{neighborhood of} \quad \Gamma .$$

Since on $R_0$ we have $G(z)$ is single-valued and analytic then

$$G'(z) = -\frac{M_z[z, \zeta]}{M_\zeta[z, \zeta]}\bigg|_{\zeta = G(z)}$$

which is $\neq 0$ for $z$ on $\Gamma$ and $\zeta = \bar{z}$ on $\Gamma$ by Lemma 2.5, and also $\neq 0$ for $z$ on $R \backslash L_i$ since all of the points $M_z[z, \zeta]|_{\zeta = G(z)} = 0$ lie on $L_i$ by construction. Thus we have partially proved the

THEOREM. *$M[z, \zeta] = 0$ defines a function*

$$\zeta = G(z)$$

*which is determined by having $z_0$ on $\Gamma$ correspond to $\bar{z}_0 = G(z_0)$. For this $G(z)$ we have*

(1) *$G(z)$ is defined and analytic on $R - \{e_1, \cdots e_s\} \cup$ neighborhood of $\Gamma$.*

(2) *$G(z)$ is single-valued on $R \backslash L_i \cup$ neighborhood of $\Gamma$.*

(3) *$G'(z) \neq 0$ on $R \backslash L_i \cup$ neighborhood of $\Gamma$.*

(4) *For $z$ on $\Gamma$, $\bar{z} = G(z)$.*

(5) *$\overline{G[R \backslash L_i]} \cap R \backslash L_i = \varnothing$.*

(6) *About each $e_j$ $(1 \leqq j \leqq s)$ we have either*

(i) *$G(z)$ is defined, single-valued and analytic on a neighborhood of $e_j$ with*

$$G(z) = (z - e_j)^2 G^*(z)$$

*$G^*(z)$ analytic and thus $G'(e_j) = 0$ or*

(ii) *on some neighborhood of $e_j$*

$$G(z) = \sum_{k=0}^{\infty} f_k[\sqrt[p]{z - e_j}]^k\,, \qquad 1 \leqq p \leqq 2n\,,$$

*$f_k$ constant, or*

(7) *If we let $\hat{z} = G(z)$ then $G(z)$ can be extended to be defined analytic and single-valued on*

(i) $\{R\backslash L_i\} \cup \Gamma \cup \{\widehat{R\backslash L_i}\}$

*with*

(ii) $\hat{\hat{z}} = z$ *there.*

*Proof.* (1)–(4) have been proved.

(5) Since $\overline{G(\Gamma)} = \Gamma$ and since $G$ is continuous on $R\backslash L_i$ we know either $\overline{G[R\backslash L_i]} \subset R$ or $\overline{G[R\backslash L_i]} \cap R\backslash L_i = \emptyset$. We shall have proved the result if we can show that for one point $z \in R\backslash L_i$, $\overline{G(z)} \notin R$.

Since $x(\theta)$ and $y(\theta)$ are analytic functions for real $\theta$ with $x'^2(\theta) + y'^2(\theta) \neq 0$ they can be continued as analytic functions $x(\tau)$ and $y(\tau)$ of the complex variable $\tau = \theta + i\eta$ on some circle $|\tau| < \rho$ for which, on $|\tau| < \rho$, $x'(\tau) + iy'(\tau) \neq 0$. Then

$$g(\tau) = x(\tau) + iy(\tau) \qquad |\tau| < \rho$$

maps $\tau = \theta + i\eta$, $\eta = 0$ onto a subarc $\Gamma_0$ of $\Gamma$ and thus maps $|\tau| < \rho$ 1–1 onto a neighborhood of $\Gamma_0$. Consider

$$H(\tau) = \overline{G[g(\tau)]}$$

for $\tau$ such that $g(\tau) \subset$ domain of $G$. Since $G(z)$ is defined and $G'(z) \neq 0$ on a neighborhood of $\Gamma$ then $H(\tau)$ is defined on a neighborhood $N$ of $|\tau| < \rho$, $\eta = 0$ with $\eta = 0$ mapping onto $\Gamma_0$ and $H(\tau)$ establishes a 1–1 correspondence between points of $N$ and $H(N)$. Thus that portion $N$ of $N$ for which $\eta < 0$ maps onto the region $R_-$ of one side of $\Gamma_0$ and $N_+$, that portion of $N$ for which $\eta > 0$ maps onto the other side $R_+$ of $\Gamma_0$. Without loss of generality let $R_- \cap R\backslash L_i \neq \emptyset$, $R_+ \cap R\backslash L_i = \emptyset$ and let $z_0 \in R_- \cap R\backslash L_i$ be such that if $g(\tau_0) = z_0$ then $\bar{\tau}_0 \in N_+$. Then $g(\bar{\tau}_0) \in R_+$. Note that on that neighborhood of $\eta = 0$ where everything is defined

$$\overline{g^{-1}[\overline{G[g(\tau)]}]}\,,$$

is analytic with

$$\overline{g^{-1}[\overline{G[g(\tau)]}]} = \overline{g^{-1}[g(\tau)]} = \bar{\tau} = \tau$$

for $\tau$ on $\eta = 0$ and thus is the identity map. Hence

$$g(\bar{\tau}_0) = \overline{G[g(\tau_0)]} = \overline{G[z_0]}$$

and $\overline{G(z_0)} \in R_+$ where as $z_0 \in R\backslash L_i$. This completes the proof of (5).

(6) (i) and (ii) follow immediately from Theorem (iii) and Lemma 2.1.

(7) (i) follows from (3) and (5). (ii) follows from the fact that

$$\hat{\hat{z}} = \overline{G[\overline{G(z)}]}$$

is an analytic function on $\{R\backslash L_i\} \cup \widehat{\Gamma \cup \{R\backslash L_i\}}$ with

$$\hat{\hat{z}} = z \qquad \text{on } \Gamma$$

and thus

$$\hat{\hat{z}} = z \quad \text{on} \quad \{R\backslash L_i\} \cup \Gamma \cup \{\widehat{R\backslash L_i}\}$$

and the theorem is proved.

In the event $M[z, \zeta]$ is not irreducible then the analysis and the theorem will hold provided we decompose $M[z, \zeta]$ into its irreducible factors and (1) study that factor which determines $\Gamma$ and (2) prove that for this factor we have the coefficient of the highest order term in $\zeta$ is constant and the coefficient of the highest order term in $z$ is constant. We shall be possibly excluding an unnecessary number of points $e_j$ where $G(z)$ may be analytic single-valued and $G'(z) \neq 0$. To see that the coefficient of the highest order term of $\zeta$ and $z$ are constants we let

$$M[z, \zeta] \equiv Q_1(z, \zeta)Q_2(z, \zeta) \cdots Q_r(z, \zeta)$$

where the $Q_j(z, \zeta)$ are irreducible. Then if

$$Q_j(z, \zeta) = q_{js_j}(z)\zeta^{sj} + q_{js_j-1}(z)\zeta^{sj-1} + \cdots + q_{j0}(z)$$

we have $s_1 + s_2 + \cdots + s_r = 2n$. Moreover,

$$q_{js_j}(z) = \text{constant} \neq 0 \quad \text{for all} \quad j = 1, 2, \cdots, r\,,$$

since

$$q_{1s_1}(z) \cdot q_{2s_2}(z) \cdots q_{rs_r}(z) \equiv g_{2n}(z) = \text{constant}\,.$$

Similarly if we write

$$Q_j(z, \zeta) \equiv p_{js_j}(\zeta)z^{sj} + p_{js_j-1}(\zeta)z^{sj-1} + \cdots + p_{j0}(\zeta)$$

we see that

$$p_{js_j}(\zeta) = \text{constant} \neq 0 \quad \text{for all} \quad j = 1, 2, \cdots, r\,.$$

It would be of interest to find conditions on the $c_k$ and $\gamma_k$ so that $M[z, \zeta]$ is irreducible. This would eliminate the calculation of an unnecessary number of points.

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