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THE CENTER OF A POSET

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The definition of neutral element is extended in a natural way from lattices to posets. The centers of posets of varying degrees of generality are then characterized.

The center of a poset. In [1], G. Birkhoff states that the factorization of a poset with universal bounds is best analyzed by considering its center. He characterizes the center of a lattice as its set of complemented, neutral elements and asks (Problem 7, page 78) if the concept of neutral element can be extended to posets. We will generalize the definition of neutral element in such a way as to be able to extend his theorem on the center of a lattice to posets. For example we shall show that an element is in the center of a poset if and only if it is complemented and satisfies a generalized associative and distributive law and that an element is in the center of a multilattice if and only if it is complemented and satisfies a generalized distributive law. Other approaches to the factorization of a poset are possible (cf. [2]). However, our approach gives a direct generalization of Birkhoff's result.

Let P be a poset with universal bounds 0 and 1. If $A \subseteq P$ define

$$\begin{aligned} U(A) &= \{x \in P: x \geqq y \text{ for some } y \in A\}\,, \\ L(A) &= \{x \in P: x \leqq y \text{ for some } y \in A\}\,, \\ M(A) &= \{x \in A: y > x \text{ for no } y \in A\}\,, \\ m(A) &= \{x \in A: y < x \text{ for no } y \in A\}\,. \end{aligned}$$

Notice it follows from these definitions that $U(\phi) = L(\phi) = M(\phi) = m(\phi) = \phi$. Let $S(P)$ be the power set on P. Then $u, l\colon S(P) \times S(P) \rightarrow S(P)$ are defined by $A\,l\,B = M[L(A) \cap L(B)]$ and $A\,u\,B = m[U(A) \cap U(B)]$. For convenience we write x for the singleton set $\{x\}$. Thus if P is a lattice $x\,l\,y = x \wedge y$ and $x\,u\,y = x \vee y$. An element $e \in P$ is *central* or (*in the center*) if $P = X \cdot Y$ where X, Y are posets with 0 and 1 and $e = (0, 1)$ or $(1, 0)$. We denote the center of P by $Z(P)$. An element $e \in P$ is *complemented* if there is an $e' \in P$ such that $e\,u\,e' = 1$ and $e\,l\,e' = 0$.

To extend the concept of neutrality to posets, in a natural way, we must define the distributive laws for posets. Consider the following equations:

(D1) $$e\,u\,(x\,l\,y) = (e\,u\,x)\,l\,(e\,u\,y)\,,$$

(D2) $$x\,u\,(e\,l\,y) = (x\,u\,e)\,l\,(x\,u\,y)\,,$$

and dually

(D3) $$e\,l\,(x\,u\,y) = (e\,l\,x)\,u\,(e\,l\,y)\,,$$

(D4) $$x\,l\,(e\,u\,y) = (x\,l\,e)\,u\,(x\,l\,y)\,.$$

We say that an element $e \in P$ is *weakly distributive* if (D1), (D4) hold whenever, $x\,l\,y \neq \phi$ and (D2), (D3) hold whenever $x\,u\,y \neq \phi$. We say that e is *distributive* if e is weakly distributive and (D2) holds whenever $x\,l\,e = y\,l\,e$ and $x\,u\,e = y\,u\,e$. An element e is *strongly distributive* if (D1)-(D4) hold for all $x, y \in P$. In lattices these three types of distributive elements coalesce to the concept of neutral element.

An element $e \in P$ is *associative* if $x \leqq e$, $y\,l\,e = 0$ imply $s\,u\,(x\,u\,y) = (s\,u\,x)\,u\,y$ for all $s \in P$ and dually $x \geqq e$, $y\,u\,e = 1$ imply $s\,l\,(x\,l\,y) = (s\,l\,x)\,l\,y$ for all $s \in P$. Of course, if P is a lattice all elements are associative. This need not be the case for posets.

Notice if e is complemented and distributive then the complement is unique. Indeed, if e', e'' are complements of e then since $e'\,l\,e = e''\,l\,e$ and $e'\,u\,e = e''\,u\,e$ we can apply (D2) to obtain

$$e' = e'\,u\,(e\,l\,e'') = (e'\,u\,e)\,l\,(e'\,u\,e'') = e'\,u\,e''\,.$$

Thus $e' \geqq e''$ and by symmetry $e' \leqq e''$.

THEOREM. *An element e is in $Z(P)$ if and only if e is associative, complemented and distributive.*

Proof. For necessity, suppose $e \in Z(P)$ and $e = (1, 0)$ with respect to some factorization $P = X \cdot Y$. For $x, y \in P$ let $x = (x_1, x_2)$, $y = (y_1, y_2)$ where $x_1, y_1 \in X$, $x_2, y_2 \in Y$. If $z \in x\,l\,y$, then $z_1 \leqq x_1, y_1$ and $z_2 \leqq x_2, y_2$. If there exists a w_1 such that $z_1 < w_1 \leqq x_1, y_1$ then $(z_1, z_2) < (w_1, z_2) \leqq (x_1, x_2), (y_1, y_2)$ a contradiction. Thus $z_1 \in x_1\,l\,y_1$ and $z_2 \in x_2\,l\,y_2$. The argument reverses so $z \in x\,l\,y$ if and only if $z_1 \in x_1\,l\,y_1$ and $z_2 \in x_2\,l\,y_2$. We now show that e is weakly distributive. If $x\,l\,y \neq \phi$ then

$$\begin{aligned} e\,u\,(x\,l\,y) &= m[U(1, 0) \cap U\{(z_1, z_2): z_i \in x_i\,l\,y_i\}] \\ &= m\{(1, v): v \geqq z_2 \in x_2\,l\,y_2\} \\ &= \{(1, z_2): z_2 \in x_2\,l\,y_2\} = M\{(1, v): v \leqq x_2, y_2\} \\ &= M[L(1, x_2) \cap L(1, y_2)] = M[L(e\,u\,x) \cap L(e\,u\,y)] \\ &= (e\,u\,x)\,l\,(e\,u\,y)\,. \end{aligned}$$

If $x\,u\,y \neq \phi$ then

$$\begin{aligned} x\,u\,(e\,l\,y) &= m[U(x) \cap U\{(z_1, z_2): z_i \in e_i\,l\,y_i\}] \\ &= m[U(x_1, x_2) \cap U(y_1, 0)] = \{(v_1, x_2): v_1 \in x_1\,u\,y_1\} \\ &= M\{(w_1, w_2): w_1 \leqq v_1 \in x_1\,u\,y_1,\ w_2 \leqq x_2\} \\ &= M[L(1, x_2) \cap L\{(v_1, v_2): v_i \in x_i\,u\,y_i\}] \\ &= M[L(x\,u\,e) \cap L(x\,u\,y)] = (x\,u\,e)\,l\,(x\,u\,y)\,. \end{aligned}$$

The dual statements follow in a similar manner so e is weakly distributive. To show e is distributive suppose $x\,l\,e = y\,l\,e$ and $x\,u\,e = y\,u\,e$. It follows that $x = y$ so clearly $x\,u\,y \neq \phi$ and (D2) holds. If $x \leqq e$ and $y\,l\,e = 0$ then $x = (x_1, 0)$ and $y = (0, y_2)$ so

$$\begin{aligned}(s\,u\,x)\,u\,y &= \{(r, s_2)\colon r \in s_1\,u\,x_1\}\,u\,(0, y_2)\\ &= \{(r, t)\colon r \in s_1\,u\,x_1,\ t \in s_2\,u\,y_2\}\\ &= (s_1, s_2)\,u\,(x_1, y_2) = s\,u\,(x\,u\,y)\ .\end{aligned}$$

The dual is similar so e is associative. It is clear that $(0, 1)$ is a complement for e.

We divide the proof of sufficiency into seven steps. Suppose e is associative, complemented and distributive.

(1) We first show that $e \vee x$ and $e \wedge x$ exist for all $x \in P$. Now $e\,u\,x \neq \phi$ since otherwise $e = e\,u\,(x\,l\,0) = (e\,u\,x)\,l\,(e\,u\,0) = \phi\,l\,e = \phi$, contradiction. If $e \vee x$ does not exist there are distinct elements $z, w \in P$ with $z, w \in e\,u\,x$ since otherwise $e\,u\,x$ is a singleton set and if $s \geqq e, x$ we have $(e\,u\,x)\,l\,s = (e\,l\,s)\,u\,(x\,l\,s) = e\,u\,x$ so $e\,u\,x \leqq s$ which would imply $e\,u\,x = e \vee x$. Now

$$z = z\,l\,(e\,u\,x) = (z\,l\,e)\,u\,(z\,l\,x) = e\,u\,x \supseteqq \{z, w\}$$

which is impossible. Thus $e \vee x$ exists and the existence of $e \wedge x$ follows dually.

(2) We now show that $e\,l\,A = M\{e \wedge a\colon a \in A\}$. By definition $e\,l\,A = M[L(e) \cap L(A)]$. If $x \in e\,l\,A$ then $x \leqq e$ and $x \leqq a$ for some $a \in A$. If there exists $y \leqq e, a$ and $x < y$ this would contradict the maximality of x in $L(e) \cap L(A)$, so $x \in e\,l\,a$ and $x = e \wedge a$. If $z = e \wedge a_1$ for some $a_1 \in A$ then $z \in L(e) \cap L(A)$ so $z \not> x$. Hence $x \in M\{e \wedge a\colon a \in A\}$. Conversely, if $x \in M\{e \wedge a\colon a \in A\}$ then $x \in L(e) \cap L(A)$. Suppose $y \in L(e) \cap L(A)$ and $y > x$. Then $y \leqq e, a$ for some $a \in A$ so $y \leqq e \wedge a$ which implies $x < e \wedge a$, a contradiction. Hence $x \in M[L(e) \cap L(A)] = e\,l\,A$. That $e\,u\,A = m\{e \vee a\colon a \in A\}$ follows dually.

Let $\phi_e\colon P \rightarrow L(e) \cdot U(e)$ be defined by $\phi_e(x) = (x \wedge e, x \vee e)$.

(3) To show ϕ_e is injective, suppose $\phi_e(x) = \phi_e(y)$. Then $x \wedge e = y \wedge e$, $x \vee e = y \vee e$. It follows from (D2) that

$$\begin{aligned}y &= (y \wedge e)\,u\,y = (x \wedge e)\,u\,y = (x\,u\,y)\,l\,(e \vee y)\\ &= (x\,u\,y)\,l\,(e \vee x) = (y \wedge e)\,u\,x = (x \wedge e)\,u\,x = x\ .\end{aligned}$$

(4) If $x \leqq y$ then $x \wedge e \leqq y \wedge e$ and $x \vee e \leqq y \vee e$ so ϕ_e is order preserving.

(5) We now show that $d \wedge e'$ exists if $d \geqq e$. By associativity, $o\,l\,(d\,l\,e') = (o\,l\,d)\,l\,e' = 0$ so $d\,l\,e' \neq \phi$. Suppose $s, t \in d\,l\,e'$. By (2) $m\{e \vee w\colon w \in d\,l\,e'\} = e\,u\,(d\,l\,e') = e\,u\,d = d$. Since $d \geqq e, x, t$ we have

$d \geqq e \vee s, e \vee t$. Hence $d = e \vee s = e \vee t$. Since $s, t \leqq e'$ we have $e \wedge s, e \wedge t \leqq e \wedge e' = 0$ so $e \wedge s = e \wedge t = 0$. Hence $\phi_e(s) = \phi_e(t)$ so by (3) $s = t$. Thus $d\,l\,e'$ is a singleton set. If $z \leqq d, e'$ then by associativity $z\,l\,(d\,l\,e') = (z\,l\,d)\,l\,e' = z$ so $z \leqq d\,l\,e'$ and $d\,l\,e' = d \wedge e'$.

(6) We now show that ϕ_e is bijective. Let $(c, d) \in L(e) \cdot U(e)$. We shall prove that $x \equiv c\,u\,(d \wedge e') = c \vee (d \wedge e')$ and that $e \wedge x = c$, $e \vee x = d$. Since $c \leqq e$ and $(d \wedge e')\,l\,e = 0$, applying associativity $1\,u\,[c\,u\,(d\,l\,e')] = (1\,u\,c)\,u\,(d\,l\,e') = 1$ so $c\,u\,(d\,l\,e') \neq \phi$. If $s, t \in c\,u\,(d \wedge e')$ then $s, t \geqq c$ and since $e \geqq c$ we have $e \wedge s, e \wedge t \geqq c$. Applying (2) we conclude that $e \wedge s = e \wedge t = c$. Now $s, t \geqq d \wedge e'$ and hence $s \vee e, t \vee e \geqq (d \wedge e') \vee e = (d \vee e) \wedge (e' \vee e) = d \vee e = d$. Now

$$\begin{aligned} c\,u\,(d \wedge e') = (c \wedge e)\,u\,(d \wedge e') &= [c\,u\,(d \wedge e')]\,l\,[e\,u\,(d \wedge e')] \\ &= [c\,u\,(d \wedge e')]\,l\,d\,, \end{aligned}$$

and hence $s, t \leqq d$. Since $e \leqq d$ it follows that $s \vee e, t \vee e \leqq d$ and hence $s \vee e = t \vee e = d$. Applying (3), $s = t = x$ and $x \wedge e = c$, $x \vee e = d$ so ϕ_e is bijective. If $z \geqq c, d \wedge e'$, then by associativity $z\,u\,[c\,u\,(d \wedge e')] = (z\,u\,c)\,u\,(d \wedge e') = z$ so $z \geqq c\,u\,(d \wedge e')$ and $c\,u\,(d \wedge e') = c \vee (d \wedge e')$.

(7) To show ϕ_e^{-1} is order preserving suppose $(a, b), (c, d) \in L(e) \cdot U(e)$ and $(a, b) \leqq (c, d)$. Then by (6) $\phi_e^{-1}(a, b) = a \vee (b \wedge e')$ and $\phi_e^{-1}(c, d) = c \vee (d \wedge e')$. But clearly $a \vee (b \wedge e') \leqq c \vee (d \wedge e')$.

It follows that P is isomorphic to $L(e) \cdot U(e)$ and the proof is complete.

COROLLARY. *If e is associative, complemented and strongly distributive, then $e \in Z(P)$.*

The converse of the corollary does not hold. Specifically, if $e \in Z(P)$ then although e must be associative, complemented and distributive, e need not be strongly distributive. For example, let $X = \{0, 1\}$ and let Y be a poset with $0, 1$ and two elements x, y satisfying $xly = \phi$. If $P = Y \cdot X$ then $e = (1, 0) \in Z(P)$. However,

$$e\,u\,[(x, 0)\,l\,(y, 0)] = e\,u\,\phi = \phi \neq e = e\,l\,e = [e\,u\,(x, 0)]\,l\,[e\,u\,(y, 0)]$$

so e is not strongly distributive. This difficulty can not be eliminated by making the convention $U(\phi) = L(\phi) = P$. Indeed, in this case if we let $P = X \cdot Y$ then $e = (1, 0) \in Z(P)$, but

$$e\,u\,[(0, x)\,l\,(0, y)] = e\,u\,\phi = e \neq \phi = (1, x)\,l\,(1, y) = [e\,u\,(0, x)]\,l\,[e\,u(0, y)]$$

so again e is not strongly distributive.

For the posets in the next corollary the three types of distributivity are the same.

COROLLARY. *Let P satisfy $x u y, x l y \neq \phi$ for all $x, y \in P$. Then $e \in Z(P)$ if and only if e is associative, complemented and strongly distributive.*

A *multilattice* is a poset for which $s \leqq x, y$ implies there exist $z \in x l y$ such that $s \leqq z$ and $t \geqq x, y$ implies there is a $w \in x u y$ such that $t \geqq w$. Multilattices are still vast generalizations of lattices; in particular, any poset with no infinite chains is a multilattice. In a multilattice the associative laws always hold so every element is associative. Indeed, if $z \in s u (x u y)$ then $z \geqq s$, $z \geqq z_1 \in x u y$ and z is minimal. Then $z \geqq s, x, y$ so $z \geqq z_2 \in s u x$. Suppose there is $z_3 \geqq y$, $z_3 \geqq z_4 \in s u x$, with $z > z_3$. Then $z_3 \geqq s$, $z_3 \geqq z_5 \in x u y$ which contradicts the minimality of z. Thus $z \in (s u x) u y$. By symmetry $s u (x u y) = (s u x) u y$ and the other associative law holds similarly. Our next result gives the most direct generalization of Birkhoff's theorem [1, page 69].

COROLLARY. *If P is a multilattice with $0, 1$ then $e \in Z(P)$ if and only if e is complemented and strongly distributive.*

Of course, in this case the three types of distributivity are the same.

REFERENCES

1. G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloq. Publ. XXV, Providence, R. I., 1967.
2. V. Novak and J. Schmidt, *Direct decompositions of partially ordered sets with zero*, J. Reine Angew. Math., **239/240** (1969), 97–108.

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