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CONTINUOUS CONVERGENCE IN $C(X)$

D. KENT, K. MCKENNON, G. RICHARDSON, AND M. SCHRODER

Let X be a convergence space and $C(X)$ the R-algebra of all continuous real-valued functions on X, equipped with the continuous convergence structure. If the natural map from X into $C(C(X))$ is an embedding, then X is said to be a c-space. With each space X there is associated the c-modification cX which is a c-space with the property $C(X) = C(cX)$. This leads to the following theorems which are valid for any convergence space X: (1) $C(X)$ is a topological space iff cX is locally compact; (2) $C(X)$ is locally compact iff cX is finite.

1. Preliminaries. The continuous convergence structure on the function algebra $C(X)$ of continuous real-valued functions on a space X has been studied extensively by Binz and others during the past decade; see, for instance, [3], [4], [6], and [8]. This function space is typically called $C_c(X)$, but we will use instead the notation $C(X)$, assuming this space to be equipped with continuous convergence unless otherwise indicated. The term "space" will always mean "convergence space".

Let Hom $C(X)$ be the subspace of $C(C(X))$ consisting of all nonzero continuous homomorphisms on $C(X)$. X is *c-embedded* if the function $i_X: X \to \text{Hom}\, C(X)$ (defined by $i_X(x)(f) = f(x)$ for all f in $C(X)$) is a homeomorphism. We will use the term *c-space* in place of "c-embedded space"; this terminology is not only more concise, but avoids conflict with the different usage of the term "c-embedded" which is common in the literature.

Starting with a space X, let λX denote the *topological modification* and πX the *pretopological modification* of X. The symbol ωX will denote the *completely regular modification* of X, that is, the finest completely regular topological space on the same underlying set which is coarser than X. X will be called ω-Hausdorff if ωX is Hausdorff, and ω-regular if $\text{cl}_{\omega X} \mathscr{F} \to x$ whenever $\mathscr{F} \to x$. (Here, "$\mathscr{F} \to x$ in X" means "the filter $\mathscr{F}$ converges to x in the space X"; cl_X designates the closure operator for X.) The following proposition will be useful later in the paper.

PROPOSITION 1.1. *A Hausdorff ω-regular space X is ω-Hausdorff.*

Proof. Let $\mathscr{F}$ converge to x and y in ωX. Since $\dot{y} \geqq \text{cl}_{\omega X} \mathscr{F}$, $\dot{y} \to x$ in ωX, and so $\dot{x} \geqq \text{cl}_{\omega X} \dot{y}$. But $\dot{y} \to y$ in X, and, since X is ω-regular, $\dot{x} \to y$ in X. Therefore, $x = y$.

A space is said to be *pseudo-topological* if $\mathscr{F} \to x$ whenever each ultrafilter finer than $\mathscr{F}$ converges to x. X is *locally compact* if every convergent filter contains a compact set, and *first countable* if every convergent filter contains a filter converging to the same point which has a countable filter base. The notation "$X \leqq Y$" for spaces X and Y will mean that X and Y have the same underlying set, and $\mathscr{F} \to x$ in Y implies $\mathscr{F} \to x$ in X.

A collection $\mathscr{A}$ of subsets of a space X is called a *covering system* if each convergent filter contains a member of $\mathscr{A}$. If $\mathscr{A}$ and $\mathscr{A}_1$ are covering systems such that each member of $\mathscr{A}_1$ is a subset of a member of $\mathscr{A}$, then $\mathscr{A}_1$ is said to be a *refinement* of $\mathscr{A}$.

PROPOSITION 1.2. *A subset A of a space X is compact iff, for each covering system $\mathscr{A}$ of A, there is a refinement $\mathscr{A}_1$ of $\mathscr{A}$ such that a finite subcollection of $\mathscr{A}_1$ covers A.*

A covering system $\mathscr{B}$ for X will be called a *basic covering system* if, whenever $\mathscr{F} \to x$, there is a filter $\mathscr{G} \to x$ such that $\mathscr{G} \leqq \mathscr{F}$ and $\mathscr{G}$ has a filter base consisting of members of $\mathscr{B}$. Thus, if X is ω-regular, the set of all ωX-closed subsets of X forms a basic covering system for X.

The abbreviation "u.f." will be used for "ultrafilter". The symbol R will denote the real number system with its usual topology. The complement of a set A is written "Co A", and the symbol $\dot{x}$ represents the fixed ultrafilter generated by $\{x\}$.

2. *c*-spaces. It is shown in [8] that X is a c-space iff X is Hausdorff, ω-regular, and solid (see [8] for the definition of a *solid space*). We will begin by showing that X is a c-space iff X is Hausdorff, ω-regular, and pseudo-topological.

PROPOSITION 2.1. *A c-space is Hausdorff, ω-regular, and pseudo-topological.*

Proof. A c-space must be Hausdorff in order for the natural function from X into Hom $C(X)$ to be injective. That a c-space is ω-regular is established in [6].

To show that X must be pseudo-topological, it is sufficient to show that $C(X)$ is pseudo-topological, since this property is hereditary. Let Λ be a filter on $C(X)$, and assume $\Omega \to f$ for all u.f.'s $\Omega \geqq \Lambda$. Let $\mathscr{F} \to x$ in X. To show $\Lambda \to f$, it is enough to show that $\Lambda(\mathscr{F}) \geqq \bigcap \{\Omega(\mathscr{F})\colon \Omega \text{ an u.f., } \Omega \geqq \Lambda\}$. We will do this by showing that if $\mathscr{A}$ is an u.f. on R and $\mathscr{A} \geqq \bigwedge(\mathscr{F})$, then $\mathscr{A} \geqq \Omega(\mathscr{F})$ for some u.f. $\Omega \geqq \Lambda$.

Let $\mathscr{X} = \{\Sigma : \Sigma$ a filter on $C(X)$, $\Sigma \geqq \Lambda$, and $\mathscr{A} \geqq \Sigma(\mathscr{F})\}$. A standard Zorn's lemma argument establishes that $\mathscr{X}$ contains a maximal element; call it Ω.

To show that Ω is an u.f., assume that $A \cup B \in \Omega$. If neither A nor B is in Ω, then the filter Ω_A generated by $\{A \cap M: M \in \Omega\}$, and Ω_B generated by $\{B \cap M: M \in \Omega\}$ must both be proper filters on $C(X)$ which are strictly finer than Ω. Thus Ω_A and Ω_B must both fail to be in $\mathscr{X}$, and so there are sets M_1, M_2 in Ω and F_1 and F_2 in $\mathscr{F}$ such that $(M_1 \cap A)(F_1) \notin \mathscr{A}$ and $(M_2 \cap A)(F_2) \notin \mathscr{A}$. Let $M = M_1 \cap M_2$ and $F = F_1 \cap F_2$. But $A \cup B \in \Omega$, and so $(M \cap (A \cup B))(F) \in \mathscr{A}$. However, $(M \cap (A \cup B))(F) \subseteqq (M_1 \cap A)(F_1) \cup (M_2 \cap B)(F_2)$, and this contradicts the fact that $\mathscr{A}$ is an ultrafilter. Thus either A or B must be in Ω.

For any ω-Hausdorff space X, define $\tilde{X}$ to be the space on the same underlying set with convergence defined as follows: $\mathscr{F} \to x$ in $\tilde{X}$ iff, whenever $\Lambda \to f$ in $C(X)$, $\Lambda(\mathscr{F}) \to f(x)$ in R.

PROPOSITION 2.2. *The following statements are valid for any ω-Hausdorff space X:* (a) $\omega X \leqq \tilde{X} \leqq X$; (b) $C(\tilde{X}) = C(X)$; (c) $\tilde{X}$ *is the finest c-space coarser than* X.

Proof. Assertions (a) and (b) are obvious. It is also clear from the definition that $\tilde{X}$ is c-embedded. If X_1 is c-embedded and $X_1 \leqq X$, then the identity map from $X \to X_1$ is continuous. Thus the induced map $C(X_1) \to C(X) = C(\tilde{X})$ is continuous, and so is the induced map from $\operatorname{Hom} C(\tilde{X}) \to \operatorname{Hom} C(X_1)$. But $\tilde{X}$ is homeomorphic to $\operatorname{Hom} C(\tilde{X})$ and $\operatorname{Hom} C(X_1)$ is homeomorphic to X_1, which establishes $X_1 \leqq \tilde{X}$.

LEMMA 2.3. *Let $\mathscr{B}$ be a basic covering system for X, and let t be the topology on $C(X)$ with subbase $\{(B, W): B \in \mathscr{B}$, W open in $R\}$, where $(B, W) = \{g \in C(X): g(B) \subseteqq W\}$. Then $C(X) \leqq C_t(X)$.*

Proof. Let $\Phi \to h$ in $C_t(X)$ and $\mathscr{G} \to y$ in X; let W be an open neighborhood of $h(y)$ in R. Let $\mathscr{H} \to y$ in X such that $\mathscr{G} \geqq \mathscr{H}$, and $\mathscr{H}$ has a filter base in $\mathscr{B}$. Let $B \in \mathscr{B}$ be a basic set in $\mathscr{H}$ such that $h(B) \subseteqq W$. Then (B, W) is t-open, and hence $(B, W) \in \Phi$. But $(B, W)(B) \subseteqq W$, and it follows that $\Phi(\mathscr{H}) \to h(y)$ in R. Thus $\Phi(\mathscr{G}) \to h(y)$ in R, and so $\Phi \to h$ in $C(X)$.

THEOREM 2.4. *X is a c-space iff X is Hausdorff, ω-regular, and pseudo-topological.*

Proof. Assume the three conditions. By Proposition 1.1, X is ω-Hausdorff. Thus $\tilde{X}$ exists, and it is sufficient to show that $X = \tilde{X}$. Since X and $\tilde{X}$ are both pseudo-topological, and $\tilde{X} \leqq X$, we can complete the proof by showing that each u.f. which $\tilde{X}$-converges to x also X-converges to x.

Suppose $\mathscr{G}$ is an u.f., $\mathscr{G} \to x$ in $\tilde{X}$, and $\mathscr{G} \nrightarrow x$ in X. If $\mathscr{H}$ is an X-convergent filter, then $\mathscr{G} \ngeq \mathrm{cl}_{\omega X} \mathscr{H}$, and so there is an ωX-closed set H in $\mathscr{H}$ such that $\mathrm{Co}\, H \in \mathscr{G}$. The set of all such H's forms a covering system $\mathscr{A}$ for X, and the set $\mathscr{B}$ of all ωX-closed subsets of members of $\mathscr{A}$ is a basic covering system for X consisting entirely of sets whose complements are in $\mathscr{G}$. Let $C_t(X)$ be the topological space derived from $\mathscr{B}$ as in Lemma 2.3. Then, by the same lemma and Proposition 2.2, $C_t(X) \geqq C(X) = C(\tilde{X})$.

Let f in $C(X)$ be defined by $f(x) = 0$, all x in X. Let Λ be the t-neighborhood filter at f. Then $\Lambda \to f$ in $C_t(X)$, which implies $\Lambda \to f$ in $C(\tilde{X})$, and so $\Lambda(\mathscr{G}) \to 0$ in R. Let W be a neighborhood of 0 not containing 1. Then there are sets G in $\mathscr{G}$ and L in Λ such that $L(G) \subseteqq W$. L contains a set of the form $(A_1, V_1) \cap \cdots \cap (A_n, V_n)$, where each A_i is in $\mathscr{B}$ and each V_i is an open neighborhood of 0 in R. Since $\mathrm{Co}\, A_i \in \mathscr{G}$ for $i = 1, \cdots, n$, we can choose $G_1 \subseteqq G$ such that G_1 is in $\mathscr{G}$ and $G_1 \cap (\bigcup A_i)$ is the empty set. Let z be any element of G_1, and let g in $C(X)$ be constructed such that $g(z) = 1$ and $g(\cup A_i) = 0$. Then g is in (A_i, V_i) for all i, but $g(G)$ is not a subset of W. This contradiction establishes that $\mathscr{G} \to x$ in X, and the proof is complete.

For any space X, let $cX = \mathrm{Hom}\, C(X)$. Then cX is a c-space, and we will refer to it as the *c-modification of X*. The next result can be easily verified.

PROPOSITION 2.5. (a) *$C(X)$ and $C(cX)$ are homeomorphic.*
(b) *If X is ω-Hausdorff, then $\tilde{X}$ and cX are homeomorphic.*

When X is ω-Hausdorff, it is convenient to think of cX as coinciding with $\tilde{X}$. In general, the underlying set for cX can be thought of as consisting of equivalence classes relative to the following equivalence relation on X: $x \sim y$ iff $f(x) = f(y)$, for all f in $C(X)$.

For the purpose of studying $C(X)$, X can be replaced by the c-space cX. If one wishes to determine what properties of $C(X)$ are induced by given properties of X, it would naturally be of interest to know when a given property of X extends to cX. Two such properties are "Lindelof" and "second countable"; for definitions of these concepts in a convergence space setting, the reader is referred to [6].

PROPOSITION 2.6. *If X is Lindelof* (second countable), *then cX is Lindelof* (second countable).

Proof. In Theorem 1, [6], Feldman shows that $C(X)$ is first countable whenever X is Lindelof, and that X is Lindelof whenever X is a c-space and $C(X)$ is first countable. The assertion for "Lindelof" follows immediately from these results. The assertion for "second countable" can be proved similarly with the help of Theorem 2 of [6].

In the next section, we will show that cX is locally compact whenever X is locally compact. We conclude this section with a simple example which shows that X can be first countable when cX is not first countable.

EXAMPLE 2.7. Let Y be the interval $[0, 1]$ with the usual topology. Let X be the space with the same underlying set whose convergence to nonzero points is discrete, and with convergence to 0 defined as follows: $\mathscr{F} \to 0$ in X iff there is a free filter $\mathscr{G}$ and a point y in Y such that: (1) $\mathscr{G}$ is finer than the Y-neighborhood filter at y; (2) $\mathscr{F} \geqq \mathscr{G} \cap \dot{0}$. In other words, $\mathscr{F} \to 0$ in X means that $\mathscr{F}$ is finer than the Y-neighborhood filter at 0, or else $\mathscr{F}$ is finer than $\mathscr{N}(y)$ for some y in X, where $\mathscr{N}(y)$ is generated by sets of the form $(V - \{y\}) \cup \{0\}$ and V is a Y-neighborhood of y.

The space X is clearly compact and first countable. But cX (which turns out to be finest pseudo-topological space coarser than X) is homeomorphic to the one-point compactification of the interval $(0, 1]$ with the discrete topology, and so is not first countable.

3. Local compactness. In this section, we examine the consequences of assuming that either X or $C(X)$ is locally compact. Arens, [1], proved for a completely regular topological space X that $C(X)$ is a topological space iff X is locally compact. In Theorem 3.6, we show that Arens' theorem is valid in the larger class of ω-regular convergence spaces. We also show that when $C(X)$ is a topology, then $C(X)$ has the compact-open topology relative to cX, but not, in general, relative to X.

LEMMA 3.1. *Let X be a space, $\mathscr{F} \to x$ in X, and $\Phi \to f$ in $C(X)$. Let W be an open neighborhood of $f(x)$ in R.*

(a) *If F is a compact set in $\mathscr{F}$ such that $f(F) \subseteqq W$, then there is a set A in Φ such that $A(F) \subseteqq W$.*

(b) *If A is a compact set in Φ and $A(x) \subseteqq W$, then there is a set F in $\mathscr{F}$ such that $A(F) \subseteqq W$.*

Proof. The proofs of (a) and (b) are essentially the same, so we will prove only (b).

Let $\{\Lambda_i: i \in I\}$ be the collection of all filters on $C(X)$ which converge in $C(X)$ to a point in A. Let $\Lambda_i \to g$ in A; then there are sets L_i in Λ_i and F_i in $\mathscr{F}$ such that $L_i(F_i) \subseteqq W$. The set $\{L_i: i \in I\}$ is a covering system for A which, by Proposition 1.2, reduces to a finite subcover $L_1, \cdots, L_n$ of A. Let $F_1, \cdots, F_n$ be the corresponding members of $\mathscr{F}$, and let $F = \bigcap F_k$. Then $(\bigcup L_k)(F) \subseteqq W$, and so $A(F) \subseteqq W$.

THEOREM 3.2. *If X is a locally compact space, then $C(X)$ is a topological space.*

Proof. $C(X)$ is known to be a convergence group, and it is also known (see [7], Theorem 5, § 3) that a pretopological convergence group is topological. Thus we can complete the proof by showing that, whenever f is in $C(X)$, $\mathscr{F} \to x$ in X, and W is an open neighborhood of $f(x)$ in R, there is a neighborhood V of f in $C(X)$ and F in $\mathscr{F}$ such that $V(F) \subseteqq W$. Assume that W_1 is a closed neighborhood of $f(x)$ contained in W and F_0 a compact set in $\mathscr{F}$. Let $F_1 = F_0 \cap f^{-1}(W_1)$. Then F_1 is compact, F_1 is in $\mathscr{F}$, and $f(F_1) \subseteqq W$, so that Lemma 3.1(a) can be applied to obtain, for each filter $\Lambda \to f$ in $C(X)$, a set $L_\Lambda \in \Lambda$ such that $L_\Lambda(F_1) \subseteqq W_1$. If V is the union of these L_Λ's, over all Λ's converging to f in $C(X)$, then V is a neighborhood of f, and $V(F_1) \subseteqq W$.

The preceding proof made use of Lemma 3.1(a) to show that $C(X) = \pi C(X)$ when X is locally compact. An analogous argument, based on Lemma 3.1(b), establishes the following result.

LEMMA 3.3. *Let X be any space such that $C(X)$ is locally compact. Then $C(X) = C(\pi X)$.*

PROPOSITION 3.4. *If $C(X)$ is a locally compact space, then $C(X) = C(\pi X) = C(\lambda X)$.*

Proof. In view of Lemma 3.3, it is sufficient to assume that X is pretopological and show that $C(X) = C(\lambda X)$. Let $\Phi \to f$ in $C(X)$ and $x \in X$. Let W be an open neighborhood of $f(x)$ in R. Select a compact set A in Φ and a neighborhood B of x such that $A(B) \subseteqq W$. Given z in B, we can use Lemma 3.1 to find a neighborhood B_z of z such that $A(B_z) \subseteqq W$. Let $B_1 = \bigcup \{B_z: z \text{ in } B\}$; then $A(B_1) \subseteqq W$. Next, given z in B_1, use Lemma 3.1 again to find a neighborhood C_z of z such that $A(C_z) \subseteqq W$. Let $B_2 = \bigcup \{C_z: z \text{ in } B_1\}$; then $A(B_2) \subseteqq W$.

Continue in this way to obtain B_n such that $A(B_n) \subseteqq W$ for all natural numbers n. Let $U = \bigcup \{B_n: n \text{ a natural number}\}$. Then U is a λX-neighborhood of x and $A(U) \subseteqq W$. It follows that $\Phi \to f$ in $C(\lambda X)$, and the proof is complete.

COROLLARY 3.5. (a) *If X is locally compact, then $C(X) = \lambda C(X)$.* (b) *If $C(X)$ is locally compact, then $C(X) = C(\lambda X)$.*

THEOREM 3.6. *If X is an ω-regular space, then $C(X)$ is a topological space iff X is locally compact.*

Proof. Assume that $C(X)$ is a topological space, and let f in $C(X)$ be the constant map $f(x) = 0$, all x in X. Let W be any open neighborhood of 0 in R not containing 1. Let $\mathscr{F} \to x$ in X; since X is ω-regular, there is an ωX-closed set F_0 in $\mathscr{F}$ such that, for some neighborhood U_0 of f in $C(X)$, $U_0(F_0) \subseteqq W$. We will complete the proof by showing that F_0 is compact.

Let $\mathscr{A}$ be a covering system for F_0. Let $\mathscr{A}_1 = \mathscr{A} \cup \{\text{Co}\, F_0\} \cup \{B \cup \text{Co}\, F_0: B \in \mathscr{A}\}$; then $\mathscr{A}_1$ is a covering system for X. By Lemma 2, [6], we can replace $\mathscr{A}_1$ by a refinement $\mathscr{A}_2$ composed of ωX-closed sets. Let $\mathscr{B}$ be the basic covering system for X obtained by adding to the collection $\mathscr{A}_2$ all ωX-closed subsets of members of $\mathscr{A}_2$. If t is the topology on $C(X)$ defined from $\mathscr{B}$ as in Lemma 2.3, then $C(X) \leqq C_t(X)$ follows from the same lemma. Thus there is a t-neighborhood U_1 of f, with $U_1 = \bigcap \{(F_k, W_k): k = 1, \cdots, n\}$, such that $U_1 \subseteqq U_0$, where the sets $F_1, \cdots, F_n$ are ωX-closed members of $\mathscr{B}$. To show that $F_0 \subseteqq \bigcup F_k$, assume the contrary, and let $z \in F_0 - (\bigcup F_k)$. Then there is h in $C(X)$ such that $h(z) = 1$ and $h(\cup F_k) = 0$. This yields a contradiction, since h is in U_1, a subset of U_0, but $h(z) = 1$ implies $U_0(F_0) \not\subseteqq W$.

To conclude that F_0 is compact, let $G_i = F_i \cap F_0$, $i = 1, \cdots, n$, and let $\mathscr{A}^* = \{B \cap F_0: B \in \mathscr{A}_2\}$. Then $\mathscr{A}^*$ is an ωX-closed refinement of $\mathscr{A}$, and each G_i is a subset of some member of $\mathscr{A}^*$. Since $F_0 \subseteqq \bigcup G_k$, the compactness of F_0 follows from Proposition 1.2, and the proof is complete.

Even in the class of topological spaces, there are ω-regular spaces which are not completely regular; an example of such a space can be found in [5], page 85, Ex. 4.

COROLLARY 3.7. *If X is locally compact, then cX is locally compact.*

Proof. This follows because $C(X)$ is a topology (Theorem 3.2),

cX is ω-regular (Proposition 2.1), and $C(cX)$ is homeomorphic to $C(X)$ (Proposition 2.5).

An immediate consequence of Theorem 2.3 and Corollary 3.7 is

COROLLARY 3.8. *For any space X, $C(X)$ is a topological space iff cX is locally compact.*

$C(X)$ is said to have the compact-open topology relative to X if $C_t(X) = C(X)$, where $C_t(X)$ is the topological space derived, as in Lemma 2.3, from the collection $\mathscr{B}$ of all compact subsets of X. When X is a completely regular topological space, then it is known that $C(X)$ has the compact-open topology relative to X whenever $C(X)$ is a topology. The situation for convergence spaces can be summarized as follows.

THEOREM 3.9. *Let $C(X)$ be a topological space.*

(a) *If X is ω-regular, then $C(X)$ has the compact-open topology relative to X.*

(b) *$C(X)$ always has the compact-open topology relative to cX.*

Proof. Both assertions follow from the fact that $C(X)$ has the compact-open topology relative to X whenever X is locally compact. Assume that X is locally compact, and let $\mathscr{B}$ be the collection of all compact subsets of X; let t be the compact-open topology on $C(X)$. $C(X) \leqq C_t(X)$ follows from Lemma 2.3. If $\Phi \rightarrow f$ in $C(X)$ and $f \in (K, W)$, where $K \in \mathscr{B}$ and W is open in R, then the argument used in proving Lemma 3.1 can be applied to obtain a set A in Φ such that $A(K) \subseteqq W$. Thus $A \subseteqq (K, W)$, and $C(X) = C_t(X)$ is established.

It is not generally true that $C(X)$ has the compact-open topology relative to X whenever $C(X)$ is topological. One can obtain a counter-example by taking X to be the space of Theorem 6.21, [2].

THEOREM 3.10. *$C(X)$ is locally compact iff cX is finite.*

Proof. If cX is finite, then $C(X) = C(cX)$ is a finite dimensional topological linear space, and hence locally compact.

Conversely, assume that $C(X)$ is locally compact; for convenience, let $Y = cX$. Then Y is a completely regular topological space, since Y is a subspace of $C(C(X))$, which has the compact-open topology by Theorem 3.9.

Let $A = \{f \in C(X): |f(x)| \leqq 1, \text{ for all } x \text{ in } X\}$. A is evidently closed in $C(X)$. Note that the filter on $C(X)$ generated by $\{(1/n)A: n = 1, 2, \cdots\}$ converges to the zero function; since $C(X)$ is locally compact, some set of the form $(1/n)A$ is compact, and it follows that A is compact in $C(X)$. We can also regard A as a subset of the product R^Y; A will then be compact relative to the product topology on R^Y.

Assume that Y is infinite. If Y were discrete, then $C(Y) = R^Y$, and $C(Y)$ would not be locally compact. Thus some element y in Y has a neighborhood filter distinct from $\dot{y}$. For each open set V in the neighborhood filter at y, choose y_V in $V - \{y\}$ and f_V in A such that $f_V(y) = 1$ and $f_V(Y - V) = 0$. Since A is compact in R^Y, the net (f_V) has a pointwise-convergent subnet $(f_{V_\alpha})_{\alpha \in \Delta}$. Let f be the pointwise-convergent limit of this subnet. Then $(f_{V_\alpha}(y))_{\alpha \in \Delta} \rightarrow f(y) = 1$, and $(f_{V_\alpha}(y_{V_\beta}))_{\alpha \in \Delta} \rightarrow f(y_{V_\beta}) = 0$ for each $\beta \in \Delta$. But the net $(y_\beta)_{\beta \in \Delta} \rightarrow y$, and so f is not in $C(Y)$. But $f \in A \subseteqq C(Y)$, since A is compact, a contradiction. It follows that $Y = cX$ is finite.

The preceding theorem and Corollary 3.8 imply that $C(X)$ is topological whenever $C(X)$ is locally compact. Combining this result with Proposition 3.4, we obtain the following.

COROLLARY 3.11. *If $C(X)$ is locally compact, then* $C(X) = \lambda C(X) = C(\lambda X)$.

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WASHINGTON STATE UNIVERSITY
EAST CAROLINA UNIVERSITY
AND
UNIVERSITY OF WAIKATO